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Katarzyna Troczka-Pawelec

CONTINUITY OF SUBQUADRATIC SET-VALUED FUNCTIONS

Abstract. Let $X = (X, +)$ be an arbitrary topological group. The aim of the paper is to prove a regularity theorem for set-valued subquadratic functions, that is solutions of the inclusion

$$F(s+t) + F(s-t) \subset 2F(s) + 2F(t), \qquad s, t \in X,$$

with values in a topological vector space.

1. Introduction

In the present paper subquadratic set-valued functions, defined on a topological group X, that is solutions of the inclusion

$$(1) \qquad F(s+t) + F(s-t) \subset 2F(s) + 2F(t), \qquad s, t \in X,$$

with non-empty compact values in a topological vector space are studied. If the sign of the inclusion in (1) is replaced by " $\supset$ " then F is called superquadratic set-valued function and if we have " $=$ " instead of " $\subset$ " in (1) then we say that F is quadratic set-valued function. We investigate a regularity theorem for subquadratic set-valued functions. It is proved here that upper semi-continuity at a point zero with condition $F(0) = \{0\}$ implies the continuity of subquadratic set-valued function F everywhere in X. This theorem generalizes some earlier results of this type obtained by D. Henney [1], K. Nikodem [2] and W. Smajdor [5] for quadratic set-valued functions. We start our consideration from basic properties for functions of this type, which play a crucial role in the proof of this theorem, which is presented in the third part. At the end of the second part of this paper we also present some examples of subquadratic set-valued functions.

Let us start with the notation used in this paper. Throughout this paper $\mathbb{R}$ stands for the set of reals. All vector spaces considered in this paper are

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real. Let Y be a topological vector space. Let $n(Y)$ denotes the family of all non-empty subsets of Y, $c(Y)$-the family of all compact members of $n(Y)$, $cl(Y)$-the family of all closed members of $n(Y)$ and $Bcl(Y)$-the family of all bounded and closed sets from $n(Y)$. The term set-valued function will be abbreviated in the form s.v.f.

Now we present here some definitions for the sake of completeness.

Definition 1.1. (cf. [5]) A s.v.f. $F\colon X \to n(Y)$ is said to be upper semi-continuous (abbreviated u.s.c.) at $x \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x+t) \subset F(x) + V$$

for every $t \in U$.

Definition 1.2. (cf. [5]) A s.v.f. $F\colon X \to n(Y)$ is said to be lower semi-continuous (abbreviated l.s.c.) at $x \in X$ iff for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$F(x) \subset F(x+t) + V$$

for every $t \in U$.

Definition 1.3. (cf. [5]) A s.v.f. $F\colon X \to n(Y)$ is said to be continuous at $x \in X$ iff it is both u.s.c. and l.s.c. at x. It is said to be continuous iff it is continuous at every point $x \in X$.

We adopt the following two definitions.

Definition 1.4. Let X be a topological group. A set $A \subset X$ is bounded in X iff for every neighbourhood U of zero in X there exists an $n \in \mathbb{N} \cup \{0\}$ such that

$$A \subset 2^n U.$$

Definition 1.5. Let X be a topological group and Y be a vector space. A s.v.f. $F\colon X \to n(Y)$ has the property (O) iff for every bounded set A in X the set $F(A)$ is bounded in Y.

We will use frequently the following well known lemma.

Lemma 1.6. (see [4]) *Let Y be a topological vector space. Let A, B, C be subsets of Y such that $A + C \subset B + C$. If B is closed and convex and C is bounded then $A \subset B$.*

In our proofs we will often use three known lemmas (see Lemma 1.1, Lemma 1.3 and Lemma 1.6 in [2]). The first lemma says that for a convex subset A of an arbitrary real vector space Y the equality $(s+t)A = sA + tA$ holds for every $s, t \geq 0$ (or $s, t \leq 0$). The second lemma says that for two convex subsets $A, B \subset Y$ the set $A + B$ is also convex and the last lemma says that if $A \subset Y$ is a closed set and $B \subset Y$ is a compact set then the set $A + B$ is closed.

2. Basic properties

In this section we present some basic properties of subquadratic set-valued functions.

Lemma 2.1. *Let X be a group and Y be a topological vector space. If a s.v.f. $F\colon X \to cl(Y)$ is subquadratic and $F(0) = \{0\}$, then F has convex values.*

Proof. Putting $t = 0$ in (1) and using the condition $F(0) = \{0\}$, we get $F(s) + F(s) \subset 2F(s)$, for every $s \in X$. Since $F(s)$ is closed, this implies that it is convex. ∎

Lemma 2.2. *Let X be a group and Y be a topological vector space. If $F\colon X \to n(Y)$ with bounded, closed and convex values is a subquadratic s.v.f., then*

$$\{0\} \subset F(0)$$

and

$$F(nx) \subset n^2 F(x)$$

for every $x \in X$ and $n \in \mathbb{N}$.

Proof. Setting $s = t = 0$ in (1) by Lemma 1.1 in [2], we infer that

$$2F(0) \subset 2F(0) + 2F(0).$$

Hence

$$\{0\} + 2F(0) \subset 2F(0) + 2F(0).$$

According to Lemma 1.6, we get

$$\{0\} \subset F(0). \tag{2}$$

Putting $s = t$ in (1), by Lemma 1.1 in [2], we obtain $F(2t) + F(0) \subset 4F(t)$, whence, by (2), we get

$$F(2t) \subset 4F(t), \qquad t \in X.$$

Assume now that

$$F(kx) \subset k^2 F(x), \qquad x \in X, k \in \{1, 2, \ldots, n\} \tag{3}$$

for some positive integer n.

Consider two cases. If n is even then $n = 2k$ for some positive integer $k \geq 1$. Therefore

$$(1 + n)x = (1 + k)x + kx \quad \text{and} \quad x = (1 + k)x - kx.$$

According to (1) and (3) we get

$$F((1+n)x) + F(x) \subset 2(1+k)^2 F(x) + 2k^2 F(x).$$

Since F is subquadratic s.v.f. with convex values, then by Lemma 1.1 in [2]

$$\begin{aligned} 2(1+k)^2 F(x) + 2k^2 F(x) &= 2(1+k)^2 F(x) + (2k^2-1)F(x) + F(x) \\ &= (1+n)^2 F(x) + F(x). \end{aligned}$$

Finally, we obtain

$$F((1+n)x) + F(x) \subset (1+n)^2 F(x) + F(x).$$

If n is odd, then $n+1$ is even. Therefore, again by (1) and (2)

$$\begin{aligned} F((n+1)x) + \{0\} &\subset F\left(\frac{n+1}{2}x + \frac{n+1}{2}x\right) + F\left(\frac{n+1}{2}x - \frac{n+1}{2}x\right) \\ &\subset 2F\left(\frac{n+1}{2}x\right) + 2F\left(\frac{n+1}{2}x\right). \end{aligned}$$

Using again Lemma 1.1 in [2] and (3), we get

$$2F\left(\frac{n+1}{2}x\right) + 2F\left(\frac{n+1}{2}x\right) \subset 4\left(\frac{n+1}{2}\right)^2 F(x) = (n+1)^2 F(x).$$

Finally,

$$F((n+1)x) \subset (n+1)^2 F(x).$$

This ends the proof of Lemma 2.2. ■

Lemma 2.3. *Let X be a topological group and Y be a topological vector space. If a subquadratic s.v.f. $F\colon X \to Bcl(Y)$ is u.s.c. at zero and $F(0) = \{0\}$, then F has the property (O).*

Proof. Let V be an arbitrary neighbourhood of zero in Y. We may choose a neighbourhood U of zero in X such that

$$F(U) \subset V. \tag{4}$$

Let $A \subset X$ be a bounded set. There exists an $n \in \mathbb{N} \cup \{0\}$ such that

$$A \subset 2^n U. \tag{5}$$

According to Lemma 2.1, F has convex values. By Lemma 2.2 and (5), we have

$$F(A) \subset F(2^n U) \subset 4^n F(U). \tag{6}$$

Hence, by (4) and (6)

$$\frac{1}{4^n} F(A) \subset V.$$

The proof is completed. ■

Now we present some examples of subquadratic set-valued functions.

Example 2.4. The s.v. function $F\colon \mathbb{R} \to n(\mathbb{R})$

$$F(x) = \mid x \mid \cdot [0,1], \qquad x \in \mathbb{R},$$

is subquadratic.

Example 2.5. The s.v. function $F\colon \mathbb{R} \to n(\mathbb{R})$

$$F(x) = (cx^2 + b)[0,1],$$

where $c \in \mathbb{R}$ and $b \geq 0$, is subquadratic.

Example 2.6. Let X be a group. The s.v. function $F\colon X \to n(\mathbb{R})$

$$F(x) = [g(x), f(x)],$$

where $f, g\colon X \to \mathbb{R}$ are subquadratic and superquadratic s.v.f., respectively, is also subquadratic.

3. The main result

Now we shall prove the main theorem of this paper. Let us start with definition.

Definition 3.1. A topological group X is said to be locally bounded group iff there exists in it a bounded neighbourhood of zero.

The idea of the proof of the next theorem is due to W. Smajdor (Theorem 4.3 in [5]).

Theorem 3.2. *Let X be a 2-divisible locally bounded topological group and Y be a locally convex topological space. If a subquadratic s.v.f. $F\colon X \to c(Y)$ is u.s.c. at zero and $F(0) = \{0\}$, then it is continuous everywhere in X.*

Proof. It suffices to prove that s.v.f. F is u.s.c. and l.s.c. in X. Suppose that s.v.f. is not l.s.c. at $z \in X$. Then there exists a neighbourhood V of zero in Y such that for every neighbourhood U of zero in X there exists $x_u \in U$ such that

$$F(z) \not\subseteq F(z + x_u) + V.$$

There exists a convex balanced neighbourhood W of zero in Y such that $\overline{W} \subset V$. Then

$$F(z) \not\subseteq F(z + x_u) + \overline{W}. \tag{7}$$

We shall show by induction that

$$F(z) + 2^k(2^k - 1)F(x_u) \not\subseteq F(z + 2^k x_u) + 2^k\overline{W} \tag{8}$$

for $k = 1, 2, \ldots$. For $k = 0$ (8) holds by (7). We assume that (8) holds for some positive integer $k \geq 0$. By Lemma 2.1 F has convex values. By Lemma

2.2 and (1), we have

$$(9)\qquad \begin{aligned} &F(z+2^{k+1}x_u)+F(z)+2^{k+1}\overline{W}\\ &\quad=F(z+2^kx_u+2^kx_u)+F(z+2^kx_u-2^kx_u)+2^{k+1}\overline{W}\\ &\quad\subset 2F(z+2^kx_u)+2F(2^kx_u)+2^{k+1}\overline{W}\\ &\quad\subset 2[F(z+2^kx_u)+2^k\overline{W}]+2^{2k+1}F(x_u). \end{aligned}$$

Since the sum of convex sets is convex and the sum of compact set and closed set is also closed (Lemma 1.3 and 1.6 in [2]), then according to Lemma 1.6 and (8), we obtain

$$(10)\qquad \begin{aligned} &2[F(z)+2^k(2^k-1)F(x_u)]+2^{2k+1}F(x_u)\\ &\qquad\not\subseteq 2[F(z+2^kx_u)+2^k\overline{W}]+2^{2k+1}F(x_u). \end{aligned}$$

By (9) and (10) we have

$$2F(z)+2^{k+1}(2^k-1)F(x_u)+2^{2k+1}F(x_u)\not\subseteq F(z+2^{k+1}x_u)+2^{k+1}\overline{W}+F(z).$$

By Lemma 1.1 in [2], we have

$$2F(z)+[2^{k+1}(2^k-1)+2^{2k+1}]F(x_u)\not\subseteq F(z+2^{k+1}x_u)+2^{k+1}\overline{W}+F(z).$$

Using the equality

$$2^{k+1}(2^k-1)+2^{2k+1}=2^{k+1}(2^{k+1}-1),$$

we obtain

$$2F(z)+2^{k+1}(2^{k+1}-1)F(x_u)\not\subseteq F(z+2^{k+1}x_u)+2^{k+1}\overline{W}+F(z).$$

By convexity of the set $F(z)$, we get

$$F(z)+F(z)+2^{k+1}(2^{k+1}-1)F(x_u)\not\subseteq F(z+2^{k+1}x_u)+2^{k+1}\overline{W}+F(z)$$

and finally

$$F(z)+2^{k+1}(2^{k+1}-1)F(x_u)\not\subseteq F(z+2^{k+1}x_u)+2^{k+1}\overline{W}.$$

Thus (8) is generally valid for all integer $k\geq 0$.

There exists a bounded set U_0 of zero in X. According to Lemma 2.3, F has the property (O). There exists $\lambda>0$ such that

$$(11)\qquad \lambda F(z+x)\subset W,\qquad x\in U_0.$$

Now we choose a $k\in\mathbb{N}$ so large that the inequality

$$(12)\qquad 2^k>\frac{3}{\lambda}$$

holds. Since F is u.s.c. at zero and $F(0)=\{0\}$, there exists a neighbourhood U of zero in X such that

$$F(t)\subset\frac{1}{\lambda 2^k(2^k-1)}W,\qquad t\in U$$

and

$$U \subset \frac{1}{2^k} U_0.$$

There exists $x_u \in U$ such that the condition (8) holds. Moreover

$$2^k x_u \in U_0 \tag{13}$$

and

$$2^k(2^k - 1)F(x_u) \subset \frac{1}{\lambda} W. \tag{14}$$

Let $a \in F(z + 2^k x_u)$, $b \in F(z)$ i $c \in F(x_u)$. Then by $(11) - (14)$, we obtain

$$\begin{aligned} b + 2^k(2^k - 1)c - a \in F(z) + 2^k(2^k - 1)F(x_u) - F(z + 2^k x_u) \\ \subset \frac{1}{\lambda} W + \frac{1}{\lambda} W + \frac{1}{\lambda} W \subset 2^k W. \end{aligned}$$

Therefore

$$b + 2^k(2^k - 1)c \in a + 2^k W$$

and

$$F(z) + 2^k(2^k - 1)F(x_u) \subset F(z + 2^k x_u) + 2^k \overline{W}$$

in spite of (8).

Now we show that F is u.s.c. in X. Let $x_0 \in X$ and V_0 be a neighbourhood of zero in Y. We choose convex neighbourhood V of zero in Y such that $3\overline{V} \subset V_0$. Since F is u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood U of zero in X such that

$$F(t) \subset V, \qquad t \in U. \tag{15}$$

In the first part of this proof we have proved that F is l.s.c. in X. There exists a symmetric neighbourhood $\widetilde{U}$ of zero in X such that

$$F(x_0) \subset F(x_0 + t) + V, \qquad t \in \widetilde{U} \tag{16}$$

and

$$F(x_0) \subset F(x_0 - t) + V, \qquad t \in \widetilde{U}. \tag{17}$$

Let $t \in U_1 \subset U \cap \widetilde{U}$, where U_1 is a symmetric neighbourhood of zero in X. Since F has convex values, by Lemma 1.1 in [2], (1), (15) and (17), we obtain

$$\begin{aligned} F(x_0 + t) + F(x_0 - t) &\subset 2F(x_0) + 2F(t) = F(x_0) + F(x_0) + 2F(t) \\ &\subset F(x_0 - t) + \overline{V} + F(x_0) + 2\overline{V} \\ &\subset F(x_0 - t) + F(x_0) + 3\overline{V}. \end{aligned}$$

Since the sum of convex sets is convex and the sum of compact set and closed set is also closed (Lemma 1.3 and 1.6 in [2]), then according to Lemma 1.6

we have proved that

$$F(x_0 + t) \subset F(x_0) + 3\overline{V} \subset F(x_0) + V_0, \qquad t \in U_1.$$

The proof is completed. ∎

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INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
JAN DŁUGOSZ UNIVERSITY OF CZĘSTOCHOWA
Al. Armii Krajowej 13/15
42-200 CZĘSTOCHOWA, POLAND
E-mail: k.troczka@ajd.czest.pl

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