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WEAKLY μ-COMPACT SPACES

Abstract. We introduce and study weakly μ-compact μ-space, i.e. μ-space (X, μ) in which every cover of X by μ-open sets has a finite subfamily the union of the μ-closures of whose members covers X.

1. Introduction and preliminaries

A generalized topology (breifly GT) [1] μ on a nonempty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. Elements of μ will be called μ-open sets, and a subset A of (X, μ) will be called μ-closed if $X \backslash A$ is μ-open. Clearly, a subset A of (X, μ) is μ-open if and only if for each $x \in A$, there exists $U_x \in \mu$ such that $x \in U_x \subset A$, or equivalently, A is the union of μ-open sets. The pair (X, μ) will be called generalized topological space (breifly GTS). By a space X or (X, μ), we will always mean a GTS. A space (X, μ) is called a μ-space [9] if $X \in \mu$. (X, μ) is called a quasi-topological space [3] if μ is closed under finite intersections. Clearly, every topological space is a quasi-topological space, every quasi-topological space is a GTS, and a space (X, μ) is a topological space if and only if (X, μ) is both μ-space and quasi-topological space.

If A is a subset of a space (X, μ), then the μ-closure of A [2], $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the μ-interior of A [2], $i_\mu(A)$, is the union of all μ-open sets contained in A. It was pointed out in [2] that each of the operators c_μ and i_μ are monotonic [4], i.e. if $A \subset B \subset X$, then $c_\mu(A) \subset c_\mu(B)$ and $i_\mu(A) \subset i_\mu(B)$, idempotent [4], i.e. if $A \subset X$, then $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$, c_μ is enlarging [4], i.e. if $A \subset X$, then $c_\mu(A) \supset A$, i_μ is restricting [4], i.e. if $A \subset X$, then $i_\mu(A) \subset A$, A is μ-open if and only if $A = i_\mu(A)$, and $c_\mu(A) = X \backslash i_\mu(X \backslash A)$.

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Clearly, A is μ-closed if and only if $A = c_\mu(A)$, $c_\mu(A)$ is the smallest μ-closed set containing A, $i_\mu(A)$ is the largest μ-open set contained in A, $x \in c_\mu(A)$ if and only if any μ-open set containing x intersects A, and $x \in i_\mu(A)$ if and only if there exists a μ-open set U such that $x \in U \subset A$. For the concepts and terminology not defined here, we refer the reader to [6].

Concluding this section, we recall the following definitions and facts for their importance in the material of our paper.

Definition 1.1. [2] Let A be a subset of a space (X, μ). Then A is called

(i) μ-semi-open if $A \subset c_\mu(i_\mu(A))$,
(ii) μ-preopen if $A \subset i_\mu(c_\mu(A))$,
(iii) μ-β-open if $A \subset c_\mu(i_\mu(c_\mu(A)))$,
(iv) μ-α-open if $A \subset i_\mu(c_\mu(i_\mu(A)))$.

Proposition 1.2. [2] *Let A be a subset of a space (X, μ). Then*

(i) *if A is μ-open, then A is μ-α-open,*
(ii) *A is μ-α-open if and only if A is both μ-semi-open and μ-preopen,*
(iii) *if A is μ-semi-open, then A is μ-β-open,*
(iv) *if A is μ-preopen, then A is μ-β-open.*

Definition 1.3. [12] A function $f : (X, \mu) \to (Y, \kappa)$ is called (μ, κ)-continuous if the inverse image of each κ-open set is μ-open.

Definition 1.4. [13] Let A be a nonempty subset of a space (X, μ). The generalized subspace topology on A is the collection $\{U \cap A : U \in \mu\}$, and will be denoted by μ_A. The generalized subspace A is the generalized topological space (A, μ_A).

Remark 1.5. [13] Let A be a nonempty subset of a μ-space (X, μ). Then (A, μ_A) is a μ_A-space.

Definition 1.6. [13] Let (X_α, μ_α) be a generalized topological space for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets. We define the collection μ of subsets of $\bigcup X_\alpha$ as follows:

$$\mu = \left\{ U \subset \bigcup X_\alpha : U \cap X_\alpha \in \mu_\alpha, \forall \alpha \in \Lambda \right\}.$$

Proposition 1.7. [13] *Let (X_α, μ_α) be a generalized topological space for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets, and let μ be as in Definition* 1.6. *Then μ is a generalized topology on $\bigcup X_\alpha$. The generalized topological space $(\bigcup X_\alpha, \mu)$ will be called the generalized topological sum of $X_\alpha, \alpha \in \Lambda$, and will be denoted by $\oplus X_\alpha$.*

Remark 1.8. [13] Let (X_α, μ_α) be a μ_α-space for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α), $\alpha \in \Lambda$. Then $(\oplus X_\alpha, \mu)$ is a μ-space.

Proposition 1.9. [13] *Let (X_α, μ_α) be a generalized topological space for each $\alpha \in \Lambda$, and let $(\oplus X_\alpha, \mu)$ be the generalized topological sum of (X_α, μ_α), $\alpha \in \Lambda$. Then*
(i) $\bigcup \mu_\alpha \subset \mu$,
(ii) $\mu_{X_\alpha} = \mu_\alpha$ *for each* $\alpha \in \Lambda$.

Proposition 1.10. [13] *Let (X, μ) and (Y, κ) be generalized topological spaces, and let $\mathcal{U} = \{U \times V : U \in \mu, V \in \kappa\}$. Then $\mathcal{U}$ generates a generalized topology σ on $X \times Y$, called the generalized product topology on $X \times Y$, that is,*

$$\sigma = \{\textit{all possible unions of members of } \mathcal{U}\}.$$

Remark 1.11. [13] Let (X, μ) be a μ-space, (Y, κ) be a κ-space, and σ be the generalized product topology on $X \times Y$. Then $(X \times Y, \sigma)$ is a σ-space.

Proposition 1.12. [13] *Let (X, μ) be a μ-space, (Y, κ) be a κ-space, and σ be the generalized product topology on $X \times Y$. Then the projection function $P_X : (X \times Y, \sigma) \to (X, \mu)$ (resp. $P_Y : (X \times Y, \sigma) \to (Y, \kappa)$) is (σ, μ)-continuous (resp. (σ, κ)-continuous).*

Definition 1.13. [13] A subset A of a μ-space (X, μ) is called μ-compact if any cover of A by μ-open subsets of X has a finite subcover of A.

Definition 1.14. [13] A μ-space (X, μ) is called μ-compact if any μ-open cover of X has a finite subcover.

2. Weakly μ-compact spaces

Definition 2.1. A μ-space (X, μ) is called weakly μ-compact (briefly $w\mu$-compact) if any cover of X by μ-open sets has a finite subfamily, the union of the μ-closures of whose members covers X.

It is clear that every μ-compact space (X, μ) is $w\mu$-compact. However, the converse is not true as shown by the following example.

Example 2.2. Let $\kappa\mathbb{N}$ be the Katetov extension of the set of natural numbers $\mathbb{N}$ (see e.g. [10]). It was pointed out in Example 2.5 (i) of [5], that if μ is the set of all preopen subsets of $\kappa\mathbb{N}$ (i.e. sets that are contained in the interior of its closure, see [8]), then $(\kappa\mathbb{N}, \mu)$ is $w\mu$-compact but not μ-compact.

Definition 2.3. Let A be a subset of a space (X, μ). Then A is called
(i) μ-regular closed if $A = c_\mu(i_\mu(A))$,
(ii) μ-regular open if $X \backslash A$ is μ-regular closed,
(iii) μ-semi-closed if $X \backslash A$ is μ-semi-open,
(iv) μ-preclosed if $X \backslash A$ is μ-preopen,
(v) μ-β-closed if $X \backslash A$ is μ-β-open,

(vi) μ-α-closed if $X\backslash A$ is μ-α-open.

The proofs of the following two lemmas are straightforward and thus omitted.

Lemma 2.4. *Let A be a subset of a space (X, μ). Then*
(i) *A is μ-semi-closed if and only if $i_\mu(c_\mu(A)) \subset A$,*
(ii) *A is μ-preclosed if and only if $c_\mu(i_\mu(A)) \subset A$,*
(iii) *A is μ-regular open if and only if $A = i_\mu(c_\mu(A))$,*
(iv) *A is μ-β-closed if and only if $i_\mu(c_\mu(i_\mu(A))) \subset A$,*
(v) *A is μ-α-closed if and only if $c_\mu(i_\mu(c_\mu(A))) \subset A$,*
(vi) *A is μ-regular open if and only if $A = i_\mu(B)$ for some μ-closed set B,*
(vii) *A is μ-regular closed if and only if $A = c_\mu(B)$ for some μ-open set B.*

Lemma 2.5. *For a subset A of a space (X, μ), the following are equivalent:*
(i) *A is μ-regular open,*
(ii) *A is μ-open and μ-semi-closed,*
(iii) *A is μ-open and μ-β-closed,*
(iv) *A is μ-α-open and μ-β-closed,*
(v) *A is μ-α-open and μ-semi-closed,*
(vi) *A is μ-preopen and μ-semi-closed.*

Corollary 2.6. *For a subset A of a space (X, μ), the following are equivalent:*
(i) *A is μ-regular closed,*
(ii) *A is μ-closed and μ-semi-open,*
(iii) *A is μ-closed and μ-β-open,*
(iv) *A is μ-α-closed and μ-β-open,*
(v) *A is μ-α-closed and μ-semi-open,*
(vi) *A is μ-preclosed and μ-semi-open.*

Proposition 2.7. *A μ-space (X, μ) is $w\mu$-compact if and only if any cover of X by μ-regular open sets has a finite subfamily, the union of the μ-closures of whose members covers X.*

Proof. The necessity is clear. Suppose that $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a cover of X by μ-open sets. Then by Lemma 2.4 (vi), $\mathcal{V} = \{i_\mu(c_\mu(U_\alpha)) : \alpha \in \Lambda\}$ is a cover of X by μ-regular open sets. Thus by assumption, there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ such that $X = \bigcup_{i=1}^n c_\mu(i_\mu(c_\mu(U_{\alpha_i})))$. By Lemma 2.4 (vii), $c_\mu(U_{\alpha_i})$ is regular closed for each i, and thus, $X = \bigcup_{i=1}^n c_\mu(U_{\alpha_i})$. Hence, (X, μ) is $w\mu$-compact. ∎

The proof of the following result is straightforward and thus omitted.

Proposition 2.8. *For a μ-space (X, μ), the following are equivalent:*

(i) *X is $w\mu$-compact,*

(ii) *for any family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $\bigcap \{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\bigcap \{i_\mu (U_\alpha) : \alpha \in \Lambda_0\} = \emptyset$,*

(iii) *for any family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of μ-regular closed subsets of X such that $\bigcap \{U_\alpha : \alpha \in \Lambda\} = \emptyset$, there exists a finite subset Λ_0 of Λ such that $\bigcap \{i_\mu (U_\alpha) : \alpha \in \Lambda_0\} = \emptyset$.*

Definition 2.9. Let A be a subset of a μ-space (X, μ). A point $x \in X$ is said to be a θ_μ-accumulation point of A if $c_\mu (U) \cap A \neq \emptyset$ for every μ-open subset U of X that contains x. The set of all θ_μ-accumulation points of A is called the θ_μ-closure of A and is denoted by $(c_\mu)_\theta (A)$. A is said to be μ_θ-closed if $(c_\mu)_\theta (A) = A$. The complement of a μ_θ-closed set is called μ_θ-open.

It is clear that A is μ_θ-open if and only if for each $x \in A$, there exists a μ-open set U such that $x \in U \subset c_\mu (U) \subset A$.

Definition 2.10. A μ-space (X, μ) is called μ-regular if for each μ-open subset U of X and for each $x \in U$, there exists a μ-open subset V of X and a μ-closed subset F of X such that $x \in V \subset F \subset U$.

The following lemma can be easily established.

Lemma 2.11. *Let A be a subset of a μ-space (X, μ). Then*

(i) *if A is μ_θ-open, then A is the union of μ-regular open sets,*

(ii) *(X, μ) is μ-regular if and only if every μ-open subset of X is μ_θ-open,*

(iii) *if A is μ-clopen, i.e. μ-open and μ-closed, then A is μ_θ-closed,*

(iv) *$c_\mu (A) \subset (c_\mu)_\theta (A)$,*

(v) *if A is μ-open, then $c_\mu (A) = (c_\mu)_\theta (A)$.*

Proposition 2.12. *If a μ-space (X, μ) is $w\mu$-compact, then every cover of X by μ_θ-open sets has a finite subcover.*

Proof. Suppose that (X, μ) is $w\mu$-compact and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by μ_θ-open sets. Then for each $x \in X$, there exists $\alpha_x \in \Lambda$ such that $x \in U_{\alpha_x}$. Since U_{α_x} is μ_θ-open, there exists a μ-open set V_x such that $x \in V_x \subset c_\mu (V_x) \subset U_{\alpha_x}$, but X is $w\mu$-compact, so there exist $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^n c_\mu (V_{x_i}) = \bigcup_{i=1}^n U_{\alpha_{x_i}}$. ∎

The following example shows that the converse of Proposition 2.12 is not true.

Example 2.13. Let $\mathbb{Z}$ be the Khalimsky line [7] (= the digital line), i.e. the set of integers eqquiped with the topology τ having for a subbase $\mathcal{S} = \{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$. It was shown in [11], that if μ is the set of all open (preopen) subsets of $\mathbb{Z}$, then every cover of $\mathbb{Z}$ by μ_θ-open sets has a finite subcover, but $(\mathbb{Z}, \mu)$ is not $w\mu$-compact.

Corollary 2.14. *Let (X, μ) be a μ-regular space. Then (X, μ) is $w\mu$-compact if and only if (X, μ) is μ-compact.*

Proof. Follows from Lemma 2.11 (ii) and Proposition 2.12. ∎

Definition 2.15. A filter base $\mathcal{F}$ on a μ-space (X, μ) is said to θ_μ-converge to a point $x \in X$ if for each μ-open subset U of X such that $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subset c_\mu(U)$. $\mathcal{F}$ is said to θ_μ-accumulate at $x \in X$ if $(c_\mu(U)) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and for every μ-open subset U of X such that $x \in U$.

Observe that if a filter base $\mathcal{F}$ θ_μ-converges to a point $x \in X$, then $\mathcal{F}$ θ_μ-accumulates at x. On the other hand, it is easy to see that a maximal filter base $\mathcal{F}$ θ_μ-converges to a point $x \in X$ if and only if $\mathcal{F}$ θ_μ-accumulates at x.

Proposition 2.16. *For a μ-space (X, μ), the following are equivalent:*
(i) *X is $w\mu$-compact,*
(ii) *every maximal filter base on X θ_μ-converges to some point of X,*
(iii) *every filter base on X θ_μ-accumulates at some point of X.*

Proof. (i)→(ii): Let $\mathcal{F}$ be a maximal filter base on X such that $\mathcal{F}$ does not θ_μ-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not θ_μ-accumulate at any point of X. Thus, for each $x \in X$, there exists $F_x \in \mathcal{F}$ and a μ-open subset U_x of X such that $x \in U_x$ and $(c_\mu(U_x)) \cap F_x = \emptyset$, but X is $w\mu$-compact, so there exist $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^n c_\mu(U_{x_i})$. Since $\mathcal{F}$ is a filter base on X, there exists $F \in \mathcal{F}$ such that $F \subset \bigcap_{i=1}^n F_{x_i}$, but $(c_\mu(U_{x_i})) \cap F_{x_i} = \emptyset$ for each $i \in \{1, 2, ..., n\}$, so $(c_\mu(U_{x_i})) \cap F = \emptyset$ for each $i \in \{1, 2, ..., n\}$, i.e. $(\bigcup_{i=1}^n c_\mu(U_{x_i})) \cap F = X \cap F = F = \emptyset$, a contradiction.

(ii)→(iii): Let $\mathcal{F}$ be a filter base on X. Then $\mathcal{F}$ is contained in a maximal filter base $\mathcal{H}$ on X. By (ii), $\mathcal{H}$ θ_μ-converges to some point x of X, thus $\mathcal{H}$ θ_μ-accumulates at x, but $\mathcal{F} \subset \mathcal{H}$, so $\mathcal{F}$ θ_μ-accumulates at x.

(iii)→(i): Suppose that X is not $w\mu$-compact. Then by Proposition 2.8, there exists a cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X by μ-open sets such that for any finite subset Λ_0 of Λ, $\bigcap \{i_\mu(X \backslash U_\alpha) : \alpha \in \Lambda_0\} \neq \emptyset$. For each finite subset Λ_0 of Λ, let $F_{\Lambda_0} = \bigcap \{i_\mu(X \backslash U_\alpha) : \alpha \in \Lambda_0\}$. Then $\mathcal{F} = \{F_{\Lambda_0} : \Lambda_0$ is a finite subset of $\Lambda\}$ is a filter base on X. Thus by (iii), $\mathcal{F}$ θ_μ-accumulates at some point x of X. Since $\mathcal{U}$ is a cover of X, there exists $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$, but $\mathcal{F}$ θ_μ-accumulates at x, so $(c_\mu(U_{\alpha_0})) \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Let $F = i_\mu(X \backslash U_{\alpha_0})$. Then $F \in \mathcal{F}$ and thus $(c_\mu(U_{\alpha_0})) \cap (i_\mu(X \backslash U_{\alpha_0})) \neq \emptyset$, a contradiction. ∎

3. Weakly μ-compact subsets

Definition 3.1. A subset A of a μ-space (X, μ) is called weakly μ-compact (briefly $w\mu$-compact) if any cover of A by μ-open subsets of X has a finite subfamily, the union of the μ-closures of whose members covers A.

We observe that every μ-compact subset of a μ-space (X, μ) is $w\mu$-compact.

Proposition 3.2. *A subset A of a μ-space (X, μ) is $w\mu$-compact if and only if any cover of A by μ-regular open subsets of X has a finite subfamily, the union of the μ-closures of whose members covers A.*

Proposition 3.3. *For a subset A of a μ-space (X, μ), the following are equivalent:*

(i) *A is $w\mu$-compact,*

(ii) *for any family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of μ-closed subsets of X such that $[\bigcap \{U_\alpha : \alpha \in \Lambda\}] \cap A = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap \{i_\mu (U_\alpha) : \alpha \in \Lambda_0\}] \cap A = \emptyset$,*

(iii) *for any family $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of μ-regular closed subsets of X such that $[\bigcap \{U_\alpha : \alpha \in \Lambda\}] \cap A = \emptyset$, there exists a finite subset Λ_0 of Λ such that $[\bigcap \{i_\mu (U_\alpha) : \alpha \in \Lambda_0\}] \cap A = \emptyset$.*

Proposition 3.4. *Let A be a $w\mu$-compact subset of a μ-space (X, μ). Then every cover of A by μ_θ-open subsets of X has a finite subcover of A.*

Corollary 3.5. *Let (X, μ) be a μ-regular μ-space. Then a subset A of (X, μ) is $w\mu$-compact if and only if A is μ-compact.*

Proposition 3.6. *For a subset A of a μ-space (X, μ), the following are equivalent:*

(i) *A is $w\mu$-compact,*

(ii) *every maximal filter base on X, each of whose members meets A, θ_μ-converges to some point of A,*

(iii) *every filter base on X, each of whose members meets A, θ_μ-accumulates at some point of A.*

Proposition 3.7. *Let A, B be subsets of a μ-space (X, μ). If A is μ_θ-closed and B is $w\mu$-compact, then $A \cap B$ is $w\mu$-compact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of $A \cap B$ by μ-open sets. Then $\mathcal{U} \cup \{X \backslash A\}$ is a cover of B. Since $X \backslash A$ is μ_θ-open, for each $x \notin A$, there exists a μ-open set U_x such that $x \in U_x \subset c_\mu (U_x) \subset X \backslash A$. Thus $\mathcal{U} \cup \{U_x : x \in X \backslash A\}$ is a cover of B by μ-open sets, but B is $w\mu$-compact, so there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ and there exist $x_1, x_2, ..., x_m \in X \backslash A$ such that $B \subset (\bigcup_{i=1}^n c_\mu (U_{\alpha_i})) \cup (\bigcup_{i=1}^m c_\mu (U_{x_i}))$, but $c_\mu (U_{x_i}) \subset X \backslash A$, so $A \cap B \subset \bigcup_{i=1}^n c_\mu (U_{\alpha_i})$. Hence, $A \cap B$ is $w\mu$-compact. ∎

Corollary 3.8. *Let A be a μ_θ-closed subset of a $w\mu$-compact space (X, μ). Then A is $w\mu$-compact.*

The following example shows that if every proper μ_θ-closed subset A of a μ-space (X, μ) is $w\mu$-compact, then (X, μ) is not necessarily $w\mu$-compact.

Example 3.9. Let $\mathbb{Z}$ be the Khalimsky line. It was also shown in [11], that if μ is the set of all open (preopen) subsets of $\mathbb{Z}$, then every proper μ_θ-closed subset of $(\mathbb{Z}, \mu)$ is $w\mu$-compact, but $(\mathbb{Z}, \mu)$ is not $w\mu$-compact.

Corollary 3.10. *Let A be a μ-clopen subset of a $w\mu$-compact space (X, μ). Then A is $w\mu$-compact.*

Proof. Follows from Lemma 2.11 (iii) and Corollary 3.8. ∎

The proof of the following lemma is straightforward and thus omitted.

Lemma 3.11. *Let A and B be subsets of a space (X, μ) such that $A \subset B$. Then*

$$c_{\mu_B}(A) = c_\mu(A) \cap B.$$

Proposition 3.12. *Let A and B be subsets of a μ-space (X, μ) such that $A \subset B$. If A is $w\mu_B$-compact, then A is $w\mu$-compact.*

Proof. Suppose that A is $w\mu_B$-compact and let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by μ-open sets. Then $\mathcal{U}^B = \{U_\alpha \cap B : \alpha \in \Lambda\}$ is a cover of A by μ_B-open sets, but A is $w\mu_B$-compact, so there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ such that $A \subset \bigcup_{i=1}^n c_{\mu_B}(U_{\alpha_i} \cap B)$. By Lemma 3.11, $c_{\mu_B}(U_{\alpha_i} \cap B) = (c_\mu(U_{\alpha_i} \cap B)) \cap B \subset c_\mu(U_{\alpha_i})$. Hence, A is $w\mu$-compact. ∎

Corollary 3.13. *Let A be a subset of a μ-space (X, μ). If A is $w\mu_A$-compact, then A is $w\mu$-compact.*

The proof of the following proposition is straightforward and thus omitted.

Proposition 3.14. *The finite union of subsets of a μ-space (X, μ), each of which is $w\mu$-compact, is $w\mu$-compact.*

Corollary 3.15. *If a μ-space (X, μ) is the finite union of subsets A_n, each of which is $w\mu_{A_n}$-compact, then X is $w\mu$-compact.*

Proof. Follows from Corollary 3.13 and Proposition 3.14. ∎

Definition 3.16. A μ-space (X, μ) is called μ-connected if X can not be expressed as the union of two disjoint nonempty μ-open sets. In the opposite case, (X, μ) is called μ-disconnected, or equivalently, (X, μ) has a proper nonempty μ-clopen set.

Proposition 3.17. *Let (X, μ) be a μ-disconnected μ-space. Then (X, μ) is $w\mu$-compact if and only if every μ-clopen set is $w\mu$-compact.*

Proof. Necessity. Follows from Corollary 3.10.

Sufficiency. Since (X,μ) is μ-disconnected, X has a partition $\{A,B\}$ such that A is μ-clopen and B is μ-clopen. By assumption, A and B are $w\mu$-compact. Thus by Proposition 3.14, (X,μ) is $w\mu$-compact. ∎

Corollary 3.18. *Let $(\oplus X_i,\mu)$ be the finite generalized topological sum of $w\mu_i$-compact spaces (X_i,μ_i), $i=1,\ 2,\ ...,\ n$. Then $(\oplus X_i,\mu)$ is $w\mu$-compact.*

Proof. Observe first by Remark 1.8 that since (X_i,μ_i) is a μ_i-space, then $(\oplus X_i,\mu)$ is a μ-space. The result follows from Proposition 1.9 (ii) and Corollary 3.15. ∎

Proposition 3.19. *If every proper μ-regular closed subset of a μ-space (X,μ) is $w\mu$-compact, then (X,μ) is $w\mu$-compact.*

Proof. Suppose that $\mathcal{U}=\{U_\alpha:\alpha\in\Lambda\}$ is a cover of X by μ-open sets. Pick $\alpha_0\in\Lambda$ such that $U_{\alpha_0}\neq\emptyset$. Then by Lemma 2.4 (vi), $X\backslash i_\mu\left(c_\mu\left(U_{\alpha_0}\right)\right)$ is a proper μ-regular closed set. Thus by assumption, there exist $\alpha_1,\alpha_2,...,\alpha_n\in\Lambda$ such that $X\backslash i_\mu\left(c_\mu\left(U_{\alpha_0}\right)\right)\subset\bigcup_{i=1}^{n}c_\mu\left(U_{\alpha_i}\right)$. Therefore,

$$\begin{aligned}X&=\left(\bigcup_{i=1}^{n}c_\mu\left(U_{\alpha_i}\right)\right)\cup i_\mu\left(c_\mu\left(U_{\alpha_0}\right)\right)\\&=\left(\bigcup_{i=1}^{n}c_\mu\left(U_{\alpha_i}\right)\right)\cup\left(c_\mu\left(U_{\alpha_0}\right)\right)=\bigcup_{i=0}^{n}c_\mu\left(U_{\alpha_i}\right).\end{aligned}$$

Hence, X is $w\mu$-compact. ∎

The proof of the following lemma is straightforward and thus omitted.

Lemma 3.20. *Let $f:(X,\mu)\to(Y,\kappa)$ be a function. Then the following are equivalent:*

(i) *f is (μ,κ)-continuous,*
(ii) *for every $x\in X$ and for every κ-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U)\subset V$,*
(iii) *$f\left(c_\mu(A)\right)\subset c_\kappa\left(f(A)\right)$ for every subset A of X,*
(iv) *$c_\mu\left(f^{-1}(B)\right)\subset f^{-1}\left(c_\kappa(B)\right)$ for every subset B of Y.*

Proposition 3.21. *Let $f:(X,\mu)\to(Y,\kappa)$ be a (μ,κ)-continuous function, where (X,μ) is a μ-space and (Y,κ) is a κ-space. If A is a $w\mu$-compact subset of X, then $f(A)$ is $w\kappa$-compact.*

Proof. Let $\mathcal{U}=\{U_\alpha:\alpha\in\Lambda\}$ be a cover of $f(A)$ by κ-open sets. Since f is (μ,κ)-continuous, $\mathcal{V}=\left\{f^{-1}\left(U_\alpha\right):\alpha\in\Lambda\right\}$ is a cover of A by μ-open sets, but A is $w\mu$-compact, so there exist $\alpha_1,\alpha_2,...,\alpha_n\in\Lambda$ such that $A\subset\bigcup_{i=1}^{n}c_\mu\left(f^{-1}\left(U_{\alpha_i}\right)\right)$. Thus $f(A)\subset\bigcup_{i=1}^{n}f\left(c_\mu\left(f^{-1}\left(U_{\alpha_i}\right)\right)\right)$. Since f is (μ,κ)-continuous, it follows from Lemma 3.20 that

$$f\left(c_\mu\left(f^{-1}\left(U_{\alpha_i}\right)\right)\right)\subset c_\kappa\left(f\left(f^{-1}\left(U_{\alpha_i}\right)\right)\right)\subset c_\kappa\left(U_{\alpha_i}\right).$$

Hence, $f(A)$ is $w\kappa$-compact. ∎

Corollary 3.22. *Let $f : (X, \mu) \to (Y, \kappa)$ be a (μ, κ)-continuous surjection, where (X, μ) is a μ-space and (Y, κ) is a κ-space. If X is $w\mu$-compact, then Y is $w\kappa$-compact.*

Corollary 3.23. *Let (X, μ) be a μ-space, (Y, κ) be a κ-space, and σ be the generalized product topology on $X \times Y$. If $X \times Y$ is $w\sigma$-compact, then (X, μ) is $w\mu$-compact and (Y, κ) is $w\kappa$-compact.*

Proof. Observe first by Remark 1.11 that since (X, μ) is a μ-space and (Y, κ) is a κ-space, then $(X \times Y, \sigma)$ is a σ-space. The result follows from Proposition 1.12 and Corollary 3.22. ■

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