DEMONSTRATIO MATHEMATICA
Vol. XLV No 4 2012

10.1515/dema-2013-0410

Hajime Sato, Keizo Yamaguchi

LIE TENSOR PRODUCT MANIFOLDS

Abstract. We study the geometric structures of parabolic geometries. A parabolic geometry is defined by a parabolic subgroup of a simple Lie group corresponding to a subset of the positive simple roots. We say that a parabolic geometry is fundamental if it is defined by a subset corresponding to a single simple root. In this paper we will be mainly concerned with such fundamental parabolic geometries.

Fundamental geometries for the Lie algebra of A_n type are Grassmann structures. For B_n, C_n, D_n types, we investigate the geometric feature of the fundamental geometries modeled after the quotients of the real simple groups of split type by the parabolic subgroups. We name such geometries Lie tensor product structures. Especially, we call Lie tensor metric structure for B_n or D_n type and Lie tensor symplectic structure for C_n type. For each manifold with a Lie tensor product structure, we give a unique normal Cartan connection by the method due to Tanaka. Invariants of the structure are the curvatures of the connection.

1. Introduction

The **Parabolic Geometry** is a geometry modeled after the homogeneous space G/P, where G is a simple Lie group and P is a parabolic subgroup of G. Precisely, in this paper, we mean, by a parabolic geometry, the geometry associated with the simple graded Lie algebra in the sense of N. Tanaka [Ta]. The geometric structures we will consider in this paper are those parabolic geometries which are given by maximal parabolic subgroups. Let G be a complex simple Lie group of rank n and X_n be its Dynkin diagram. Then a parabolic subgroup can be described by fixing the subset Δ_1 of the simple root systems Δ_n, which form the vertices of X_n, by suitable choices of a Cartan subalgebra and a simple root system Δ_n of the Lie algebra $\mathfrak{g}$ of G (see §2, cf. [Ya1], §3.3). Maximal parabolic subgroups correspond to the cases when Δ_1 consists of a single simple root, which is usually described by the Dynkin diagram with one node marked black. We call such parabolic geometries to be fundamental. Fundamental geometries

2000 *Mathematics Subject Classification*: 53C10, 53C15, 53D10.
Key words and phrases: Lie tensor product, Cartan connection.

for the Lie algebra of the A_n type is called Grassmann structure [MS] (see §3 for other nominations).

In this paper, we further restrict ourselves to the case when we mark a single node of the Dynkin diagram of classical simple Lie algebras of type B_n, C_n, D_n. A notable difference from the case of A_n type is, in most cases, the appearance of differential systems as the underlying geometries. We will investigate the geometric feature of the fundamental geometries modeled after the quotients of the real simple groups of split type by the parabolic subgroups. We call **Lie tensor metric structures** for B_n and D_n types and **Lie tensor symplectic structures** for C_n type. Most of them turn out to be geometries of regular differential systems of some types.

In §2, we will recall the basic materials in simple Lie algebras and parabolic geometries. Especially, to describe a parabolic subalgebra $\mathfrak{p}$ of the complex simple Lie algebra $\mathfrak{g}$, the natural gradation of $\mathfrak{g}$ associated with $\mathfrak{p}$ will be explained. Previous studies for A_n type will be mentioned in §3. In §4, for B_n and D_n types, we will describe, explicitly in matrices form, the gradations associated with fundamental parabolic subalgebras. Here, a little generally, we will describe the gradations of real simple Lie algebras other than a split real form. Utilizing these matrices description, we will describe the symbol algebras of underlying differential systems. In view of the study in §4, we will introduce the notion of **Lie tensor metric structure** and give a basic structure theorem for these structures by virtue of Tanaka theory in §5. In §6, for C_n type, we will describe, explicitly in matrices form, the gradations associated with fundamental parabolic subalgebras. Utilizing these matrices description, we will describe the symbol algebras of underlying differential systems. In view of the study in §6, we will introduce the notion of **Lie tensor symplectic structure** and give a basic structure theorem for these structures by virtue of Tanaka theory in §7. Finally in §8, we will give several examples of Lie tensor metric structures. Especially we will show that the set of singular D-curves of a flat maximally nondegenerate distribution of rank n has the structure of $(3, n-1)$ Lie tensor metric manifold with signature $(2, 1)$.

The authors would like to thank Professors T. Mizutani and T. Inaba for their valuable comments.

2. Parabolic geometries

Let $\mathfrak{g}$ be a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and G a Lie group whose Lie algebra is $\mathfrak{g}$. Choose a Cartan subalgebra $\mathfrak{h}$ and fix a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{g}$. Then the Dynkin diagram of $\mathfrak{g}$ is a graph made of white nodes corresponding to each simple roots with edges (or directed multi-edges) connecting some nodes.

We choose a subset $\Delta_1 \subset \Delta$ and indicate Δ_1 by marking the corresponding nodes black in the Dynkin diagram.

A choice of Δ_1 defines a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ as follows [Ya1, p. 444].

Let $\Phi = \Phi^+ \cup \Phi^-$ be the set of positive and negative roots. An element $\alpha \in \Phi^+$ is written as $\alpha = \sum_{i=1}^n n_i(\alpha)\alpha_i,\ n_i(\alpha) \geq 0$. We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi^+} \big(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\big).$$

Associated with Δ_1, for $k \geq 0$, put

$$\Phi_k^+ = \Big\{\alpha = \sum_{i=1}^n n_i(\alpha)\alpha_i \in \Phi^+ \ \Big|\ \sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k\Big\}.$$

Put $\mu = \max\{k \mid \Phi_k^+ \neq \emptyset\}$ and

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_0^+} \big(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\big),$$

$$\mathfrak{g}_k = \sum_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \sum_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \quad \text{for} \quad 1 \leq k \leq \mu.$$

Further we put

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \sum_{k=1}^{\mu} \mathfrak{g}_k, \qquad \mathfrak{m} = \sum_{k=1}^{\mu} \mathfrak{g}_{-k}.$$

Then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ and $\mathfrak{p}$ is a parabolic subalgebra, that is, $\mathfrak{p}$ contains the Borel (=maximally solvable) subalgebra.

Let $P \subset G$ be the parabolic Lie subgroup whose Lie algebra is equal to $\mathfrak{p}$. The tangent space at the base point of the homogeneous space $M = G/P$ is isomorphic to the graded vector space $\mathfrak{m} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1}$. Such homogeneous manifold $M = G/P$ is called a generalized flag manifold (or R-space). Since

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \qquad -\mu \leq i, j \leq \mu,$$

$\mathfrak{g}$ and $\mathfrak{m}$ have the graded Lie algebra structure.

As for Lie algebras over $\mathbb{R}$, the above argument is valid when $\mathfrak{g}$ is a noncompact simple Lie algebra which is a split, or normal, real form. The split real Lie algebra of the classical types is one of the following,

$$\mathfrak{sl}(n+1, \mathbb{R}),\ \mathfrak{so}(n+1, n),\ \mathfrak{sp}(n, \mathbb{R}),\ \mathfrak{so}(n, n).$$

This corresponds to the types A_n, B_n, C_n, D_n in this order. Corresponding to the split Lie algebras, we choose Lie groups G to be $\mathrm{SL}(n+1, \mathbb{R})$, $\mathrm{SO}(n+1, n)$, $\mathrm{Sp}(n, \mathbb{R})$ or $\mathrm{SO}(n, n)$. Especially, if we set all nodes black, then $M =$

G/B where B is the maximal solvable subgroup (Borel subgroup) and M is diffeomorphic to the maximal compact subgroup K of G. If we leave all nodes white, then $\mathfrak{p} = \mathfrak{g}$ and $M = \{\mathrm{pt}\}$. Here we note that, for the model space $M_{\mathfrak{g}} = G/P$ of Tanaka theory for the parabolic geometry associated to $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$, i.e., the geometry associated to the simple graded Lie algebra $\mathfrak{g} = \sum_{p=-\mu}^{\mu} \mathfrak{g}_p$, G is defined as the adjoint group of $\mathfrak{g}$ in [Ta]. Precisely our G/P is a covering space over $M_{\mathfrak{g}}$.

In the following, we study the geometric structure of the real generalized flag manifold G/P such that the Lie algebra $\mathfrak{g}$ of G is a split real form of semisimple Lie algebra of the classical types.

Let X_n be one of A_n, B_n, C_n, D_n. We denote by X_n^k the generalized flag manifold G/P defined by setting one k-th node of Dynkin diagram black. We follow [Bo] for the numbering of simple roots.

3. A_n^k type

Let $G = SL(n+1, \mathbb{R})$ and put $A_n^k = G/P$. Then the diagram for A_n^k is given by

$$\circ - \circ - \cdots - \overset{k}{\bullet} - \cdots - \circ$$

We have the diffeomorphism

$$A_n^k \cong SO(n+1)/S\left(O(k) \times O(n+1-k)\right)$$

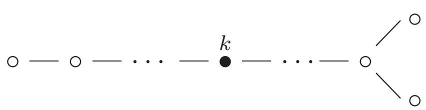
which is a Grassmann manifold. The geometric structures modeled after Grassmann manifolds are studied in [MS] under the correspondence in twistor diagrams. The structures have been studied by many people with different namings; almost Grassmannian [Mikhailov 78], [Akivis, Goldberg 96], Grassmannian spinor [Manin 88], tensor product [Hangan 66], [Ishihara 70], paraconformal [Bailey, Eastwood 91], generalized conformal [Goncharov 87], ($k = 2$) Segré [McKay 05], etc.

4. B_n^k and D_n^k types

Let $G = O(n+1, n)$ and put $B_n^k = G/P$. The parabolic group P is represented by

$$\circ - \circ - \cdots - \overset{k}{\bullet} - \cdots - \circ \Longrightarrow \circ \qquad (n \geq 3).$$

And let $G = O(n, n)$ and put $D_n^k = G/P$. The parabolic group P is represented by



$(n \geq 4)$.

We represent the Lie algebra $\mathfrak{g} = \mathfrak{o}(n+1,n)$ or $\mathfrak{o}(n,n)$ as follows. Let

$$K = K_\ell = \begin{pmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{pmatrix}$$

be the anti-diagonal unit $\ell \times \ell$-matrix. The eigenvalues of K_ℓ are 1 with multiplicity $n+1$ (respectively n) and -1 with multiplicity n when $\ell = 2n+1$ (respectively when $\ell = 2n$).

We put

$$\mathfrak{g} = \{\, X \in \mathfrak{gl}(\ell, \mathbb{R}) \mid {}^tXK + KX = 0\}.$$

Then $\mathfrak{g} = \mathfrak{o}(n+1,n)$ when $\ell = 2n+1$ and $\mathfrak{g} = \mathfrak{o}(n,n)$ when $\ell = 2n$.

For an $r \times s$ matrix $Y \in M(r,s)$, write $Y' = K_s\, {}^tY\, K_r \in M(s,r)$. Then, when $r = s$, Y' is the "transposed" matrix with respect to the anti-diagonal line.

We will introduce the gradation of $\mathfrak{g}$ by subdividing $X \in \mathfrak{g}$ as follows:

$$\begin{array}{c c} & \begin{array}{ccc} k & p & k \end{array} \\ \begin{array}{c} k \\ p \\ k \end{array} & \begin{pmatrix} A & -F' & D \\ B & G & F \\ C & -B' & -A' \end{pmatrix} \end{array}, \tag{1}$$

where $C = -C'$, $D = -D'$ and $G = -G'$. Then, when $k \geq 2$, the Lie algebra $\mathfrak{g}$ has the gradation

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 \tag{2}$$

where

$$\mathfrak{g}_{-2} = \langle C \rangle,\ \mathfrak{g}_{-1} = \langle B \rangle,\ \mathfrak{g}_0 = \langle A \rangle \oplus \langle G \rangle,\ \mathfrak{g}_1 = \langle F \rangle,\ \mathfrak{g}_2 = \langle D \rangle.$$

Put $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then P is the Lie subgroup of G corresponding to $\mathfrak{p} \subset \mathfrak{g}$. Hence, corresponding to $\mathfrak{g}_{-1}$, the model space $B_n^k = G/P$ or $D_n^k = G/P$ has the G-invariant differential system $D_{\mathfrak{g}}$ and the tangent space $T_0(M)$ at the base point of $M = G/P$ is identified with $\mathfrak{m}$.

Thus, from (1) and (2), we have

$$\dim \mathfrak{g}_{-2} = \frac{k(k-1)}{2}, \dim \mathfrak{g}_{-1} = pk,$$

where $\ell = p + 2k$, $p = 2(n-k)+1$ in case of type B_n^k and $p = 2(n-k)$ in case of type D_n^k. We put $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{m}^1(p,k)$.

Here we first notice the following: In case $p = 1$, or equivalently in case $k = n$ and $\ell = 2n+1$, i.e., in case of type B_n^n, under the assumption $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, the Lie algebra structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, such that

$\dim \mathfrak{g}_{-1} = n$ and $\dim \mathfrak{g}_{-2} = \frac{n(n-1)}{2}$, is unique and can be described as;

$$\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \quad \mathfrak{g}_{-2} = \wedge^2 V \quad \text{and} \quad \mathfrak{g}_{-1} = V,$$

i.e., $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the universal fundamental graded Lie algebra of second kind [Ta]. In this case, when $n \geq 3$, $\mathfrak{g} = \mathfrak{o}(n+1, n)$ is the prolongation of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and $\mathfrak{g}_0 \cong \mathfrak{gl}(V)$ (cf. §5.3 in [Ya1]).

In the rest of this section, we assume $k = 2$ and $p \geq 3$ or $k \geq 3$ and $p \geq 2$. We also notice that, in case $p = 2$, or equivalently in case $k = n-1$ and $\ell = 2n$ ($n \geq 4$), the above gradation (2) is that of type $D_n^{n-1,n}$ (see §4.4 in [Ya1]).

On $\hat{U} = \mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n}$, we give an inner product $(\,,\,)$ by $(\boldsymbol{x}, \boldsymbol{y}) = {}^t\boldsymbol{x}\, K \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^\ell$. We write these inner product spaces by $\mathbb{R}^{n+1,n}$ and $\mathbb{R}^{n,n}$ since the signature of the inner product are $(n+1, n)$ and (n, n) respectively.

Here, a little generally, we will consider other real forms of $\mathfrak{o}(\ell, \mathbb{C})$. Put $\hat{S} = \begin{pmatrix} 0 & 0 & K \\ 0 & S & 0 \\ K & 0 & 0 \end{pmatrix}$, where $K = K_k$ is the anti-diagonal unit $k \times k$ matrix and $S = \begin{pmatrix} E_r & 0 \\ 0 & -E_s \end{pmatrix}$, where E_r is the unit $r \times r$ matrix and $p = r + s$. On $\hat{U} = \mathbb{R}^\ell$, we give an inner product $(\,,\,)$ by $(\boldsymbol{x}, \boldsymbol{y}) = {}^t\boldsymbol{x}\, \hat{S} \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^\ell$. Then the signature of $(\hat{U}, (\,,\,))$ is $(k+r, k+s)$. Moreover, on $U = \mathbb{R}^p$, we give an inner product $(\,,\,)$ by $(\mathbf{a}, \mathbf{b}) = {}^t\mathbf{a}\, S \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$. Then the signature of $(U, (\,,\,))$ is (r, s).

We put

$$\mathfrak{g} = \mathfrak{o}(\hat{U}) = \mathfrak{o}(k+r, k+s) = \{\, X \in \mathfrak{gl}(\ell, \mathbb{R}) \mid {}^t X \hat{S} + \hat{S} X = 0 \}.$$

We will introduce the gradation of $\mathfrak{g} = \mathfrak{o}(\hat{U})$ again by subdividing $X \in \mathfrak{g}$ as follows:

(3)
$$\begin{array}{c} \\ k \\ p \\ k \end{array} \!\!\begin{array}{c} \begin{array}{ccc} k & p & k \end{array} \\ \begin{pmatrix} A & -\hat{F} & D \\ B & G & F \\ C & -\hat{B} & -A' \end{pmatrix} \end{array},$$

where $C = -C'$, $D = -D'$, $G \in \mathfrak{o}(r, s)$ $\hat{B} = K {}^t B S$ and $\hat{F} = K {}^t F S$. Then, when $k \geq 2$, the Lie algebra $\mathfrak{g}$ has the gradation

(4)
$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

where

$$\mathfrak{g}_{-2} = \langle C \rangle,\ \mathfrak{g}_{-1} = \langle B \rangle,\ \mathfrak{g}_0 = \langle A \rangle \oplus \langle G \rangle,\ \mathfrak{g}_1 = \langle F \rangle,\ \mathfrak{g}_2 = \langle D \rangle.$$

Put $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then P is the Lie subgroup of $G = O(\hat{U})$ corresponding to $\mathfrak{p} \subset \mathfrak{g}$. Hence, corresponding to $\mathfrak{g}_{-1}$, the model space $O(\hat{U})/P$ has the G-invariant differential system $D_{\mathfrak{g}}$ and the tangent space $T_0(M)$ at the base point of $M = O(\hat{U})/P$ is identified with $\mathfrak{m}$.

Thus, from (3) and (4), we have

$$\dim \mathfrak{g}_{-2} = \frac{k(k-1)}{2}, \dim \mathfrak{g}_{-1} = pk$$

where $\ell = p + 2k$, $p = r + s$. We put $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{m}^1(\hat{U})$.

Now we will describe the Lie algebra structure of $\mathfrak{m}$ as in the following: By utilizing the matrices description (1) (resp. (3)), we identify $\mathfrak{g}_{-1}$ with $M(p,k)$ and $\mathfrak{g}_{-2}$ with $T = \{C \in M(k,k) \mid C = -C'\}$, and we calculate

$$[B_1, B_2] = -B_1' B_2 + B_2' B_1 \text{ (resp. } = -\hat{B}_1 B_2 + \hat{B}_2 B_1) \in \mathfrak{g}_{-2} = T,$$

for $B_1, B_2 \in \mathfrak{g}_{-1} = M(p,k)$. Moreover, identifying T with $\mathfrak{o}(k)$ by $T \ni C \mapsto -KC \in \mathfrak{o}(k)$, we get

$$[B_1, B_2] = {}^t B_1 K B_2 - {}^t B_2 K B_1 \text{ (resp. } = {}^t B_1 S B_2 - {}^t B_2 S B_1) \in \mathfrak{o}(k).$$

Now, identifying $M(p,k)$ with $\mathbb{R}^p \otimes \mathbb{R}^k$, we obtain

$$[\mathbf{a} \otimes \mathbf{e_i}, \mathbf{b} \otimes \mathbf{e_j}] = (\mathbf{a}, \mathbf{b})(E_{ij} - E_{ji}) \in \mathfrak{o}(k),$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p = M(p,1)$, $\{\mathbf{e_1}, \ldots, \mathbf{e_k}\}$ is the natural basis of $\mathbb{R}^k = M(1,k)$, $(\mathbf{a}, \mathbf{b}) = {}^t\mathbf{a} K \mathbf{b}$ (resp. $= {}^t\mathbf{a} S \mathbf{b}$) is the inner product in $U = \mathbb{R}^p$ and E_{ij} denotes the matrix whose (i,j)-component is 1 and all of whose other components are 0. Here $\mathbf{a} \otimes \mathbf{e_i}$ corresponds to the $p \times k$ matrix whose i-th row is $\mathbf{a}$ and all of other rows are $\mathbf{0}$. Thus, finally, identifying $\mathfrak{o}(k)$ with $\wedge^2 V$, $V = \mathbb{R}^k$, we obtain the following description of $\mathfrak{m} = \mathfrak{m}^1(p,k)$ or $\mathfrak{m}^1(\hat{U})$ in general

$$\mathfrak{m}^1(U, V) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},\ \mathfrak{g}_{-2} = \wedge^2 V,\ \mathfrak{g}_{-1} = U \otimes V,$$

where U is a vector space with the inner product $(\,,)$ of dimension p and V is a vector space of dimension k. The bracket is defined by

$$[u_1 \otimes v_1, u_2 \otimes v_2] = (u_1, u_2) v_1 \wedge v_2, \qquad \text{for} \quad u_1, u_2 \in U, v_1, v_2 \in V.$$

Thus if $2 \le k < n$, $\dim \mathfrak{g}_{-2} = \frac{k(k-1)}{2} > 0$ and $D_{\mathfrak{g}}$ is defined by $\frac{k(k-1)}{2}$ Pfaffian forms. We have the following coordinate description of the standard differential system of type $\mathfrak{m}^1(p,k)$;

$$\big\{ x_{ij} (1 \le i < j \le k),\ y_{\alpha j}\ (1 \le \alpha \le p,\ 1 \le j \le k) \big\}$$

with 1-forms θ_{ij} $(1 \le i < j \le k)$ such that $D = \mathrm{Ker}\{\theta_{ij}\}$ and

$$\theta_{ij} = dx_{ij} + \frac{1}{2} \sum_{\alpha=1}^{p} \big(y_{p+1-\alpha, j} dy_{\alpha i} - y_{\alpha i} dy_{p+1-\alpha, j} \big).$$

In fact we have

$$d\theta_{ij} = \sum_{\alpha=1}^{p} dy_{p+1-\alpha,j} \wedge dy_{\alpha i}.$$

Taking the dual frame $\{\frac{\partial}{\partial x_{ij}}(1 \leq i < j \leq k), Y_{\alpha i}(1 \leq \alpha \leq p, 1 \leq i \leq k)\}$ to the coframe $\{\theta_{ij}(1 \leq i < j \leq k), dy_{\alpha i}(1 \leq \alpha \leq p, 1 \leq i \leq k)\}$, we have

$$Y_{\alpha i} = \frac{\partial}{\partial y_{\alpha i}} - \frac{1}{2}\sum_{j=1}^{k} y_{p+1-\alpha,j}\frac{\partial}{\partial x_{ij}}.$$

Here we put $x_{ij} = -x_{ji}$ for $1 \leq j < i \leq k$. Then we have

$$[Y_{\alpha i}, Y_{\beta j}] = \delta_{\alpha,p+1-\beta}\frac{\partial}{\partial x_{ij}}.$$

Thus we obtain the coordinate description of the standard differential system of type $\mathfrak{m}^1(p,k)$.

One interpretation is that we regard B_n^k as the set of all null k-planes in the $2n+1$ dimensional Euclidean space $\mathbb{R}^{n+1,n}$ with the signature $(n+1,n)$. Thus we have

$$B_n^k \cong (SO(n+1) \times SO(n))\,/S\,(O(n-k+1) \times O(n-k) \times \Delta O(k))\,,$$
$$\dim B_n^k = \frac{k}{2}(4n-3k+1).$$

For any $k(1 \leq k \leq n-2)$, we can regard D_n^k as the set of all null k-planes in $\mathbb{R}^{n,n}$ (for $k = n-1$, see a little carefully §4.4 in [Ya1]). We have

$$D_n^k \cong (SO(n) \times SO(n))\,/S\,(O(n-k) \times O(n-k) \times \Delta O(k))\,.$$

For $k = 1$, we have $\mathfrak{g}_{-2} = \mathfrak{g}_2 = 0$. Hence we have $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. The universal covering $\widetilde{M}$ of $M = G/P$ is diffeomorphic to $S^n \times S^{n-1}$ which is the conformal compactification of $\mathbb{R}^{n-1,n}$ in case of B_n^1 type and to $S^{n-1} \times S^{n-1}$ which is a conformal compactification of $\mathbb{R}^{n-1,n-1}$ in case of D_n^1 type.

For $k \geq n-1$, we have $D_n^{n-1} = D_n^n$. See a little carefully §4.4 in [Ya1]. Corresponding to $D_n^{n-1} = D_n^n$, the Lie algebra $\mathfrak{g}$ has the gradation $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ such that

$$\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1 = \frac{n(n-1)}{2}, \quad \dim \mathfrak{g}_0 = n^2.$$

5. Lie tensor metric structure

To investigate the structure of B_n^k, D_n^k or $O(\hat{U})/P$, we introduce the notion of Lie tensor metric structure.

First, we study vector spaces with tensor product structures.

A pk-dimensional vector space W is a (p,k) **tensor product space** if there exists an isomorphism $h : U \otimes V \to W$, with $\dim U = p$, $\dim V = k$.

The product group $GL(U) \times GL(V)$ acts naturally on the (p,k) tensor product space W.

A pk-dimensional vector space W is a (p,k) **metric tensor product space with signature** (r,s) $(p = r+s)$ if there exists an isomorphism $h : U \otimes V \to W$, where U is a p-dimensional vector space with a non-degenerate inner product $(,)$ of signature (r,s) and V is a q-dimensional vector space.

The product group $O(U) \times GL(V)$ acts naturally on the (p,k) metric tensor product space W.

A linear transformation ϕ of a (p,q) metric tensor product space W is called a **metric tensor product transformation** if it is expressed as an action of an element in $O(U) \times GL(V)$.

We have the exterior product representation

$$\lambda : GL(V) \to GL(\wedge^2 V).$$

Define a homomorphism $\rho : O(U) \times GL(V) \to GL(\mathfrak{m}^1(U,V))$ by

$$\rho(A,B) = \begin{pmatrix} \lambda(B) & 0 \\ 0 & A \otimes B \end{pmatrix}, \quad A \in O(U),\ B \in GL(V),$$

where $\mathfrak{m}^1(U,V) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, $\mathfrak{g}_{-2} = \wedge^2 V$, $\mathfrak{g}_{-1} = U \otimes V$. Then we see that, through ρ, $O(U) \times GL(V)$ acts as a graded Lie algebra automorphism group of $\mathfrak{m}^1(U,V)$.

Let $G_0 \subset O(\hat{U})$ be the Lie subgroup whose Lie algebra is equal to $\mathfrak{g}_0$ under the decomposition (4) of $\mathfrak{g} = \mathfrak{o}(\hat{U})$. Under the identification of $\mathfrak{m}^1(U,V)$ with the tangent space at the base point of $O(\hat{U})/P$, the group $Im(\rho)$ coincides with the image of the isotropy representation of $G_0 \subset P$ at the base point of $O(\hat{U})/P$.

Define a subgroup $G_0^\sharp$ of $GL(\mathfrak{m}^1(U,V))$ by

$$G_0^\sharp = \left\{ \begin{pmatrix} \lambda(B) & 0 \\ C & A \otimes B \end{pmatrix} \middle| A \in O(U), B \in GL(V), C \in End(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}) \right\}.$$

A (p,k) matrix F and $B \in GL(V)$ defines a linear map $\mu(B,F) : \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ by the matrix multiplication: $\mathfrak{g}_{-2} \ni C \mapsto FCB^{-1} \in \mathfrak{g}_{-1}$, where we identify $\mathfrak{g}_{-2} = T$ and $\mathfrak{g}_{-1} = M(p,k)$ as in §4. We define a closed subgroup $\widetilde{G}$ of $G_0^\sharp$ by

$$\widetilde{G} = \left\{ \begin{pmatrix} \lambda(B) & 0 \\ \mu(B,F) & A \otimes B \end{pmatrix} \middle| A \in O(U), B \in GL(V), F \in M(p,k) \right\}.$$

By calculating the adjoint representation of P on $\mathfrak{g} = \mathfrak{o}(\hat{U})$, we see that the group $\widetilde{G}$ is the linear isotropy representation of P at the base point of

$O(\hat{U})/P$. If $k \geq 3$, then $\widetilde{G} \subsetneq G_0^\sharp$.

Lie tensor metric structure

Let M be a $\left(\dfrac{k(k-1)}{2} + pk\right)$-dimensional manifold M.

By a $G_0^\sharp$-structure on M, we mean a reduction P of the frame bundle $F(M)$ to the group $G_0^\sharp$. Then, on the frame bundle P, the value of the canonical 1-form θ lies in $\mathfrak{m}^1(U,V) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. So that M admits a distribution D, which is defined on P by $\theta_{-2} = 0$, where θ_i $(i = -1, -2)$ is the $\mathfrak{g}_i$-component of the canonical 1-form θ.

A $\left(\dfrac{k(k-1)}{2} + pk\right)$-dimensional manifold M with a $G_0^\sharp$-structure is called a (p,k) **Lie tensor metric manifold** with signature (r,s) if the canonical 1-form θ satisfies

$$(C) \qquad d\theta_{-2} + \tfrac{1}{2}\,[\theta_{-1}, \theta_{-1}] \equiv 0 \pmod{\theta_{-2}}.$$

Here the bracket $[,]$ is that of $\mathfrak{m}^1(U,V)$. This condition (C) is equivalent to the condition that D is a regular differential system of type $\mathfrak{m}^1(U,V)$, i.e., the symbol algebra $\mathfrak{m}(x)$ of (M,D) is isomorphic to $\mathfrak{m}^1(U,V)$ at each $x \in M$ [Ta].

Example 5.1. If $k = 2$, then $(p,2)$ Lie tensor metric manifold M with signature $(p,0)$ is a $2p+1$-dimensional **Lie contact manifold** defined in [SY1].

Example 5.2. The space B_3^2 of the singular D-curve of B_3^3 has the structure of $(3,2)$ Lie tensor contact metric manifold with signature $(2,1)$ (cf. §8).

We have

Proposition 5.1. *For $n \geq 3,\ k \geq 2,\quad B_n^k$ has a structure of $(2(n-k)+1, k)$ Lie tensor metric manifold with signature $(n-k+1, n-k)$.*

For $n \geq 4,\ 2 \leq k \leq n-2,\quad D_n^k$ has a structure of $(2(n-k), k)$ Lie tensor metric manifold with signature $(n-k, n-k)$.

Moreover $O(\hat{U})/P$ has a structure of (p,k) Lie tensor metric manifold with signature (r,s).

Tanaka theory

We review the Tanaka theory [Ta] which solves completely the equivalence problem of parabolic geometries. N. Tanaka constructs the normal Cartan connection uniquely whose curvature provides the all invariants of each structure.

Definition of Cartan connection

Let G be a simple Lie group and let G/P be a generalized flag manifold. Let Q be a P-principal bundle over a manifold M with $\dim M = \dim G/P$. A $\mathfrak{g}$-valued 1-form θ on Q is a **Cartan connection** if

(C1) $X \in T(Q),\ \theta(X) = 0 \Rightarrow X = 0,$
(C2) $\theta(A^*) = A,\quad A^*$ is the vector field generated by $A \in \mathfrak{h}$,
(C3) $R_a{}^*\theta = Ad(a^{-1})\theta,\quad a \in P.$

The curvature K of a Cartan connection is given by

$$K = d\theta + [\theta, \theta],$$

which is a $\mathfrak{g} \otimes \Lambda^2(\mathfrak{m}^*)$ valued function on Q. Put

$$K = \sum_p K^p, \quad K^p = \sum_{\ell<-1} \mathfrak{g}_{\ell+p+1} \otimes \Lambda^2_\ell(\mathfrak{m}^*)$$

where $\Lambda^2_\ell(\mathfrak{m}^*) = \sum_{s+t=\ell, s<0, t<0} \mathfrak{g}_s^* \wedge \mathfrak{g}_t^*$. We have the boundary operators

$$\partial : \mathfrak{g} \otimes \Lambda^q(\mathfrak{m}^*) \to \mathfrak{g} \otimes \Lambda^{q+1}(\mathfrak{m}^*), \qquad \partial^* : \mathfrak{g} \otimes \Lambda^{q+1}(\mathfrak{m}^*) \to \mathfrak{g} \otimes \Lambda^q(\mathfrak{m}^*).$$

A Cartan connection θ is normal if

(NC1) $K^p = 0 \quad (p < 0),$
(NC2) $\partial^* K^p = 0 \quad (p \geq 0).$

A normal Cartan connection is called **Tanaka normal connection**.

In Theorem 5.3 [Ya1], relevant Spencer cohomology groups were calculated, which induces the prolongation condition necessary for the applications of Tanaka theory [Ta]. Precisely $\mathfrak{g} = \mathfrak{o}(\hat{U})$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$, when $q = 2$ and is the prolongation of $\mathfrak{m}$ when $q \geq 3$. The last statement implies that $\rho(O(U) \times GL(V))$ coincides with the full group of graded Lie algebra automorphisms of $\mathfrak{m}^1(U, V)$ when $q \geq 3$. Hence a regular differential system (M, D) of type $\mathfrak{m}^1(U, V)$ admits the (p, q) Lie tensor metric structure when $q \geq 3$. Thus the parabolic geometry modeled after $O(\hat{U})/P$, when $q \geq 3$, is the geometry of regular differential systems of type $\mathfrak{m}^1(\hat{U})$.

Tanaka theory implies that every $G_0^\sharp$-structure further reduces to a $\widetilde{G}$-structure.

By using Tanaka theory [Ta] and Proposition 5.5 [Ya1] (see also Proposition 6.2 [Ya2]), we obtain the following.

Theorem 5.2. *Let M be a (p, q) Lie tensor metric manifold with signature (r, s), $p = r+s$. Then there exists a principal P-bundle Q over M and unique normal Cartan connection θ on Q.*

(1) *$p = 1$ and $q = n \geq 3$, the curvature K has one component.*
(2) *$q = 2$ and $p \geq 3$, the curvature K has 3 components if $p = 4$ and 2 components otherwise.*
(3) *$q = 3$ and $p = 2$, the curvature K has 2 components.*
(4) *$q = 3$ and $p \geq 3$, the curvature K has one component.*
(5) *$q \geq 4$ and $p \geq 2$, the curvature always vanishes.*

The vanishing of the curvature is the condition that M is Lie metric equivalent to the flat model in case (2) *and* (M, D) *is equivalent to the flat model* $(O(\hat{U})/P, D_{\mathfrak{g}})$ *otherwise.*

6. C_n^k type

Let $G = Sp(n, \mathbb{R})$ and put $C_n^k = G/P$. The parabolic group P is represented by

$$\circ - \circ - \cdots - \overset{k}{\bullet} - \cdots - \circ \Longleftarrow \circ \qquad (n \geq 2).$$

We represent the Lie algebra $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ as follows. Let $K = K_n$ be the anti-diagonal unit $n \times n$-matrix and put $J = \begin{pmatrix} O & K \\ -K & 0 \end{pmatrix}$. Then $J^2 = -1$. We have

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R}) = \{\ X \in \mathfrak{gl}(2n, \mathbb{R}) \mid {}^t XJ + JX = 0\}.$$

Explicitly we can write

$$\mathfrak{g} = \left\{ \begin{matrix} & n & n \\ n & A & D \\ n & C & -A' \end{matrix}, \quad D = D',\ C = C' \right\}.$$

The simple roots of the Lie algebra $\mathfrak{g}$ of type C_n are $e_1 - e_2,\ e_2 - e_3, \cdots, e_{n-1} - e_n,\ 2e_n$. The generalized flag manifold C_n^k corresponds to the following block decomposition of $\mathfrak{g}$;

$$(5) \qquad \begin{matrix} & k & 2p & k \\ k & A & \hat{F} & D \\ 2p & B & G & F \\ k & C & \hat{B} & -A' \end{matrix},$$

where $C = C',\ D = D',\ \hat{B} = ({B_2}', -{B_1}')$ for $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $\hat{F} = (-{F_2}', {F_1}')$ for $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, and $G = \begin{pmatrix} A_2 & D_2 \\ C_2 & -{A_2}' \end{pmatrix} \in \mathfrak{sp}(p, \mathbb{R})$ for $p = n - k$. Then corresponding to C_n^k, when $1 \leq k < n$, the Lie algebra $\mathfrak{g}$ has the gradation

$$(6) \qquad \mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,$$

where

$$\mathfrak{g}_{-2} = \langle C \rangle, \quad \mathfrak{g}_{-1} = \langle B \rangle, \quad \mathfrak{g}_0 = \langle A \rangle \oplus \langle G \rangle, \quad \mathfrak{g}_1 = \langle F \rangle, \quad \mathfrak{g}_2 = \langle D \rangle.$$

Put $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then P is the Lie subgroup of $G = Sp(n, \mathbb{R})$ corresponding to $\mathfrak{p} \subset \mathfrak{g}$. Hence, corresponding to $\mathfrak{g}_{-1}$, the model space $C_n^k = G/P$ has the G-invariant differential system $D_{\mathfrak{g}}$ and the tangent space $T_0(M)$ at the base point of $M = C_n^k$ is identified with $\mathfrak{m}$.

Thus, from (5) and (6), we have

$$\dim \mathfrak{g}_{-2} = \frac{k(k+1)}{2}, \qquad \dim \mathfrak{g}_{-1} = 2pk,$$

where $p = n - k$. We put $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{m}^2(p,k)$.

Now, we will describe the Lie algebra structure of $\mathfrak{m} = \mathfrak{m}^2(p,k)$ as in the following: By utilizing the matrices description (5), we identify $\mathfrak{g}_{-1}$ with $M(2p,k)$ and $\mathfrak{g}_{-2}$ with $S = \{C \in M(k,k) \mid C = C'\}$, and we calculate

$$[B_1, B_2] = \hat{B}_1 B_2 - \hat{B}_2 B_1 \in \mathfrak{g}_{-2} = S \quad \text{for } B_1, B_2 \in \mathfrak{g}_{-1} = M(2p,k).$$

Moreover, identifying S with $Sym(k)$ by $S \ni C \mapsto -KC \in Sym(k)$, we get

$$[B_1, B_2] = {}^tB_1 J B_2 - {}^tB_2 J B_1 \in Sym(k).$$

In fact, we calculate

$$\begin{aligned} -K\hat{B}_1B_2 + K\hat{B}_2B_1 &= (-{}^tC_1K, {}^tA_1K)\begin{pmatrix} A_2 \\ C_2 \end{pmatrix} + ({}^tC_2K, -{}^tA_2K)\begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \\ &= ({}^tA_1, {}^tC_1)\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}\begin{pmatrix} A_2 \\ C_2 \end{pmatrix} - ({}^tA_2, {}^tC_2)\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}\begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \end{aligned}$$

for $B_1 = \begin{pmatrix} A_1 \\ C_1 \end{pmatrix}$, $B_2 = \begin{pmatrix} A_2 \\ C_2 \end{pmatrix} \in \mathfrak{g}_{-1}$. Now, identifying $M(2p,k)$ with $\mathbb{R}^{2p} \otimes \mathbb{R}^k$, we obtain

$$[\mathbf{a} \otimes \mathbf{e_i}, \mathbf{b} \otimes \mathbf{e_j}] = \langle \mathbf{a}, \mathbf{b} \rangle (E_{ij} + E_{ji}) \in \mathfrak{o}(k),$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{2p} = M(2p,1)$, $\{\mathbf{e_1}, \ldots, \mathbf{e_k}\}$ is the natural basis of $\mathbb{R}^k = M(1,k)$ and $\langle \mathbf{a}, \mathbf{b} \rangle = {}^t\mathbf{a}J\mathbf{b}$ is the symplectic product in $U = \mathbb{R}^{2p}$. Here $\mathbf{a} \otimes \mathbf{e_i}$ corresponds to the $2p \times k$ matrix whose i-th row is $\mathbf{a}$ and all of other rows are $\mathbf{0}$. Thus, finally, identifying $Sym(k)$ with the symmetric product $S^2(V)$ of $V = \mathbb{R}^k$, we obtain the following description of $\mathfrak{m}^2(p,k) = \mathfrak{m}^2(U,V)$

$$\mathfrak{m}^2(U,V) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1},\ \mathfrak{g}_{-2} = S^2(V),\ \mathfrak{g}_{-1} = U \otimes V,$$

where U is a symplectic vector space with the symplectic product $\langle , \rangle$ of dimension $2p$ and V is a vector space of dimension k. The bracket is defined by

$$[u_1 \otimes v_1, u_2 \otimes v_2] = \langle u_1, u_2 \rangle v_1 \circledcirc v_2, \qquad \text{for} \quad u_1, u_2 \in U, v_1, v_2 \in V.$$

Thus if $1 \leq k < n$, $\dim \mathfrak{g}_2 = \frac{k(k+1)}{2} > 0$ and $D_{\mathfrak{g}}$ is defined by $\frac{k(k+1)}{2}$ Pfaffian forms. We have the following coordinate description of the standard differential system of type $\mathfrak{m}^2(p,k)$;

$$\{x_{ij}(1 \le i \le j \le k),\ y_{\alpha j}\ (1 \le \alpha \le 2p,\ 1 \le j \le k)\}$$

with symplectic forms ω_j for $1 \le j \le k$ on $\{y_{\alpha j},\ 1 \le \alpha \le 2p\}$;

$$\omega_j = 2\sum_{\alpha=1}^{p} \big(dy_{2p+1-\alpha,j} \wedge dy_{\alpha j}\big),$$

and 1-forms θ_{ij} $(1 \le i \le j \le k)$ such that $D = \mathrm{Ker}\{\theta_{ij}\}$ and

$$\theta_{ij} = dx_{ij} + \frac{1}{2}\sum_{\alpha=1}^{p} \big(y_{2p+1-\alpha,i}dy_{\alpha j} - y_{\alpha i}dy_{2p+1-\alpha,j} + y_{2p+1-\alpha,j}dy_{\alpha i} - y_{\alpha j}dy_{2p+1-\alpha,i}\big).$$

In fact we have

$$d\theta_{ij} = \sum_{\alpha=1}^{p} \big(dy_{2p+1-\alpha,i} \wedge dy_{\alpha j} + dy_{2p+1-\alpha,j} \wedge dy_{\alpha i}\big).$$

Taking the dual frame $\{\frac{\partial}{\partial x_{ij}}(1 \le i \le j \le k), Y_{\alpha i}, Y_{2p+1-\alpha,i}(1 \le \alpha \le p, 1 \le i \le k)\}$ to the coframe $\{\theta_{ij}(1 \le i \le j \le k), dy_{\alpha i}, dy_{2p+1-\alpha,i}(1 \le \alpha \le p, 1 \le i \le k)\}$, we have

$$Y_{\alpha i} = \frac{\partial}{\partial y_{\alpha i}} - \frac{1}{2}\sum_{j=1}^{k} y_{2p+1-\alpha,j}\frac{\partial}{\partial x_{ij}} - \frac{1}{2}y_{2p+1-\alpha,i}\frac{\partial}{\partial x_{ii}},$$

$$Y_{2p+1-\alpha,i} = \frac{\partial}{\partial y_{2p+1-\alpha,i}} + \frac{1}{2}\sum_{j=1}^{k} y_{\alpha j}\frac{\partial}{\partial x_{ij}} + \frac{1}{2}y_{\alpha i}\frac{\partial}{\partial x_{ii}},$$

for $1 \le \alpha \le p$. Here we put $x_{ij} = x_{ji}$ for $1 \le j < i \le k$. Then we have

$$[Y_{\alpha i}, Y_{\beta j}] = [Y_{2p+1-\alpha,i}, Y_{2p+1-\beta,j}] = 0,$$

$$[Y_{\alpha i}, Y_{2p+1-\beta,i}] = 2\delta_{\alpha\beta}\frac{\partial}{\partial x_{ii}}, \quad [Y_{\alpha i}, Y_{2p+1-\beta,j}] = \delta_{\alpha\beta}\frac{\partial}{\partial x_{ij}} \quad (i \ne j),$$

for $1 \le \alpha \le p$ and $1 \le i,j \le k$. Thus we obtain the coordinate description of the standard differential system of type $\mathfrak{m}^2(p,k)$.

We may regard C_n^k as the set of all isotropic k-planes in symplectic $\mathbb{R}^{2n}$. We have

$$C_n^k \cong U(n)/\left(U(n-k) \times SO(k)\right).$$

If $k = n$, then $\mathfrak{g}_{-1} = \mathfrak{g}_1 = 0$ and $\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_0 + \mathfrak{g}_2$. We have a diffeomorphism $C_n^n \cong U(n)/SO(n)$ which is the set of Lagrangian subspaces of the symplectic $\mathbb{R}^{2n}$.

7. Lie tensor symplectic structure

A $2pq$-dimensional vector space W is a (p,q) **symplectic tensor product space** if there exist a $2p$-dimensional vector space U with a symplectic form ω, a q-dimensional vector space V and an isomorphism $h: U\otimes V \to W$.

Let $Sp(U) \cong Sp(p,\mathbb{R})$ be the set of linear transformations of U which preserve the symplectic form ω invariant. The product group $Sp(U)\times GL(V)$ acts naturally on the (p,q) symplectic tensor product space W.

A linear transformation ϕ of a (p,q) symplectic tensor product space W is called a **symplectic tensor product transformation** if it is expressed as an action of an element in $Sp(U)\times GL(V)$.

Let $S^2(V) \subset V\otimes V$ be the symmetric products of V. We have the symmetric product representation

$$\sigma : GL(V) \to GL(S^2(V)).$$

Define a homomorphism $\rho : Sp(U)\times GL(V) \to GL(\mathfrak{m}^2(U,V))$ by

$$\rho(A,B) = \begin{pmatrix} \sigma(B) & 0 \\ 0 & A\otimes B \end{pmatrix}, \quad A \in Sp(U),\ B\in GL(V),$$

where $\mathfrak{m}^2(U,V) = \mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}$, $\mathfrak{g}_{-2} = S^2(V)$, $\mathfrak{g}_{-1} = U\otimes V$. Then we see that, through ρ, $Sp(U)\times GL(V)$ acts as a graded Lie algebra automorphism group of $\mathfrak{m}^2(U,V)$.

Let $G_0 \subset Sp(n,\mathbb{R})$, $n = p+k$, be the Lie subgroup whose Lie algebra is equal to $\mathfrak{g}_0$ under the decomposition (6) of $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$. Under the identification of $\mathfrak{m}^2(U,V)$ with the tangent space at the base point of C^k_n, the group $Im(\rho)$ coincides with the image of the isotropy representation of $G_0 \subset P$ at the base point of $Sp(n,\mathbb{R})/P = C^k_n$.

Define a subgroup $G^\sharp_0$ of $GL(\mathfrak{m}^2(U,V))$ by

$$G^\sharp_0 = \left\{ \begin{pmatrix} \sigma(B) & 0 \\ C & A\otimes B \end{pmatrix} \,\middle|\, A\in Sp(U), B\in GL(V), C\in End(\mathfrak{g}_{-2},\mathfrak{g}_{-1}) \right\}.$$

A $2p\times q$ matrix F and $B\in GL(V)$ defines a linear map $\mu(B,F) : \mathfrak{g}_{-2} \to \mathfrak{g}_{-1}$ by the matrix multiplication: $\mathfrak{g}_{-2} \ni C \mapsto FCB^{-1} \in \mathfrak{g}_{-1}$, where we identify $\mathfrak{g}_{-2} = S$ and $\mathfrak{g}_{-1} = M(2p,k)$ as in §6. We define a closed subgroup $\widetilde{G}$ of $G^\sharp_0$ by

$$\widetilde{G} = \left\{ \begin{pmatrix} \sigma(B) & 0 \\ \mu(B,F) & A\otimes B \end{pmatrix} \,\middle|\, A\in Sp(U), B\in GL(V), F\in M(2p,q) \right\}.$$

By calculating the adjoint representation of P on $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$, we see that the group $\widetilde{G}$ is the linear isotropy representation of P at the base point of $Sp(n,\mathbb{R})/P = C^k_n$. If $q\geq 2$, then $\widetilde{G} \subsetneq G^\sharp_0$.

Lie Tensor symplectic structure

A $\left(\dfrac{q(q+1)}{2} + 2pq\right)$-dimensional manifold M with a $G_0^\sharp$-structure is called a (p,q) **Lie tensor symplectic manifold** if the canonical 1-form θ satisfies

$$(C) \qquad d\theta_{-2} + \tfrac{1}{2}\,[\,\theta_{-1}, \theta_{-1}] \equiv 0 \ (\operatorname{mod} \theta_{-2}).$$

Here the bracket $[,]$ is that of $\mathfrak{m}^2(U,V)$. The Lie tensor symplectic manifold M carries a distribution $D \subset TM$, which is defined by $\theta_{-2} = 0$, where θ_{-2} is the $\mathfrak{g}_{-2}$-component of θ so that $\operatorname{rank} D = \dim \mathfrak{g}_{-1}$. Then the condition (C) is equivalent to the condition that (M,D) is a regular differential system of type $\mathfrak{m}^2(U,V)$, i.e., the symbol algebra $\mathfrak{m}(x)$ of (M,D) is isomorphic to $\mathfrak{m}^2(U,V)$ at each $x \in M$ [Ta].

We have

Proposition 7.1. *For $1 \le k < n,\ 2 \le n$, the generalized flag manifold C_n^k has a structure of $(n-k,k)$ Lie tensor symplectic manifold.*

The generalized flag manifold C_n^1 is a contact manifold and the automorphism group is of infinite dimension. If we give it a **projective contact structure**, the automorphism group of the structure is isomorphic to $Sp(n,\mathbb{R})$ and is of finite dimension.

Furthermore, when $2 \le k < n$, from Theorem 5.3 [Ya1], it follows that $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$ is the prolongation of $\mathfrak{m}^2(n-k,k)$. This implies that $\rho(S(U) \times GL(V))$ coincides with the full group of graded Lie algebra automorphisms of $\mathfrak{m}^2(U,V)$. Hence, a regular differential system (M,D) of type $\mathfrak{m}^2(U,V)$ admits the $(n-k,k)$ Lie tensor symplectic structure. Thus the parabolic geometry modeled after C_n^k, when $2 \le k < n$, is the geometry of regular differential systems of type $\mathfrak{m}^2(n-k,k)$.

By Tanaka theory [Ta] and Proposition 5.5 [Ya1], we have the following;

Theorem 7.2. *For $p \ge 1,\ q \ge 2$, let M be a (p,q) Lie tensor symplectic manifold, i.e., M carries a regular differential system D of type $\mathfrak{m}^2(p,q)$. Write $C_{p+q}^q = Sp(p+q,\mathbb{R})/P$ together with the invariant differential system $D_{\mathfrak{g}}$. Then there exists a principal P-bundle Q over M and unique normal Cartan connection θ on Q. If $p \ge 2$ and $q \ge 3$, the curvature always vanishes. When $q = 2$, the curvature K has two components if $p = 1$ and has one component if $p \ge 2$. When $p = 1$ and $q \ge 3$, the curvature K has one component. The vanishing of the curvature is the condition that (M,D) is equivalent to the flat model $(C_{p+q}^q, D_{\mathfrak{g}})$.*

8. Examples

We show that the unit tangent bundle of semi-Riemannian manifolds are example of Lie tensor metric structure.

Let (M, g) be a semi-Riemannian manifold of index (r, s) such that $r+s = n = \dim M$. A non-zero vector $z \in T_pM$ is called **spacelike, null, timelike** if $g(z, z) \gtreqless 0$ respectively and **non null** if $g(z, z) \neq 0$.

Put $S_p^+(M) = \{z \in T_pM \mid g(z, z) = 1\} \subset T_pM$. Then $S^+M = \bigcup_{p \in M} S_p^+ M$ is called the unit spacelike bundle. Put $S_p^-(M) = \{z \in T_pM \mid g(z, z) = -1\} \subset T_pM$. Then $S^-M = \bigcup_{p \in M} S_p^- M$ is called the unit timelike bundle. Then we have (cf. [SY1])

Theorem 8.1. *The unit spacelike bundle S^+M of semi-Riemannian manifold of signature (r, s), $r+s = n = \dim M$ has the structure of $(n-1, 2)$ Lie tensor metric structure with signature $(r, s-1)$. The unit timelike bundle S^-M of semi-Riemannian manifold of signature (r, s), $r + s = n = \dim M$ has the structure of $(n-1, 2)$ Lie tensor metric structure with signature $(r-1, s)$.*

For B_n^k, if $k = n$, we have

$$\dim \mathfrak{g}_{-2} = \frac{n(n-1)}{2}, \quad \dim \mathfrak{g}_{-1} = n.$$

The manifold B_n^n is diffeomorphic to $SO(n+1)$, which is diffeomorphic to the oriented orthonormal frame bundle of S^n. We have $\dim D_{\mathfrak{g}} = n$. Since $\mathcal{D} + [\mathcal{D}, \mathcal{D}] = \Gamma(TM)$, $D_{\mathfrak{g}}$ gives a maximally nondegenerate structure on B_n^n.

If $k = n-1$, $\dim \mathfrak{g}_{-2} = \dfrac{(n-1)(n-2)}{2}$, $\dim \mathfrak{g}_{-1} = 3(n-1)$. The manifold B_n^{n-1} is diffeomorphic to $(SO(n+1) \times SO(n)) \,/\, (SO(2) \times \Delta SO(n-1))$.

For a manifold M with a distribution $D \subset TM$, a curve γ is called a D-curve if tangent vectors of γ are contained in D. A D-curve γ in M is called singular (= irregular) if it is a critical point of the endpoint map $end : \Omega(x_0, D) \to M$ (see [Mo, p.83]).

Proposition 8.2. *The manifold B_n^{n-1} is a set of singular D-curves of B_n^n.*

Proof. Let $B_n^{n-1,n}$ denote the generalized flag manifold defined by choosing the subset Δ_1 of the set of simple roots Δ of $G = O(n+1, n)$ as $\Delta_1 = \{\alpha_{n-1}, \alpha_n\}$. Then $B_n^{n-1,n}$ is diffeomorphic to $(SO(n+1) \times SO(n)) \,/\, \Delta SO(n-1)$, which in turn is naturally diffeomorphic to the set of oriented D-lines of the manifold (B_n^n, D). The tangent space $\mathfrak{m} = T_0 B_n^{n-1,n}$ has the decomposition

$$\mathfrak{m} = \mathfrak{g}_{-4} + \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$$

whose dimensions are given

$$\frac{n^2+3n-2}{2} = \frac{n^2-3n+2}{2} + (n-1) + (n-1) + n.$$

The n-dimensional space $\mathfrak{g}_{-1}$ is equal to the restriction to $B_n^{n-1,n}$ of the tautological distribution of the Grassmann bundle consisting of oriented lines of B_n^n. The set of vectors $\mathfrak{g}_{-1}^1$ tangent to the fiber of the fibration $p^n : B_n^{n-1,n} \to B_n^n$ is $(n-1)$-dimensional subspace of $\mathfrak{g}_{-1}$. The set of vectors $\mathfrak{g}_{-1}^2$ tangent to the fiber of the fibration $p^{n-1} : B_n^{n-1,n} \to B_n^{n-1}$ is 1-dimensional subspace of $\mathfrak{g}_{-1}$ transversal to $\mathfrak{g}_{-1}^1$ such that $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^1 \oplus \mathfrak{g}_{-1}^2$. In the Lie algebra (1), put

$$v_{p,q} = E_{n+p,q} - E_{2n+2-q,n+2-p},$$

where $E_{i,j}$ denotes the matrix whose (i,j) component is equal to 1 and others 0. Let $e \in B_n^n$ be the base point and let $x \in D_e \subset T_e B_n^n$. By the action of G, we may assume that $x = \{v_{1,n}\} \in T_e B_n^n$. Denote by $\langle x \rangle_+$ the oriented line generate by x. Then $T_{\langle x \rangle_+} B_n^{n-1,n}$ is equal to a subspace of (1) so that

$$\mathfrak{g}_{-1}^1 = \langle\, v_{0,j} \mid 1 \le j \le n-1 \,\rangle, \quad \mathfrak{g}_{-1}^2 = \langle\, v_{1,n} \,\rangle.$$

Put $v = v_{1,n}$. Then v is mapped by p_*^n to x and is tangent to the fiber of the projection p^{n-1}. We will show that x is the projection of a characteristic direction at some $\lambda \in D^\perp$. Denote by $\lambda^{p,q}$ the dual basis of $v_{p,q}$ in (1). Then $D = \langle\, v_{1,q} \mid 1 \le q \le n \,\rangle$ and $D^\perp$ is equal to $\langle\, \lambda^{p,q} \mid 2 \le p,\ 1 \le q \le n \,\rangle$. Let $\lambda = \lambda^{p,q} \in D^\perp$ such that $3 \le p$. By an easy calculation, we have $\lambda([x,y]) = 0$ for any $y \in D$. Let $w : D^\perp \to \wedge^2 D^*$ be the dual curvature [Mo, 4.2]. Then $w(\lambda)(x,y) = \lambda(-[x,y])$ for $y \in D$. Thus we conclude that x is the projection of a characteristic direction. ∎

References

[AG] M. A. Akivis, V. V. Goldberg, *Conformal Differential Geometry and its Generalizations*, John Wiley and Sons, Inc., New York, 1996.

[BE] T. N. Bailey, M. G. Eastwood, *Complex paraconformal manifolds – their differential geometry and twistor theory*, Forum Math. 3 (1991), 61–103.

[Bo] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitre 4, 5 et 6, Hermann, Paris, 1968.

[Br] R. Bryant, *Conformal geometry and 3-plane fields on 6-manifolds*, in Developments of Cartan Geometry and Related Mathematical Problems, RIMS Symposium Proceedings, vol. 1502 (July, 2006), pp. 1–15.

[BH] R. Bryant, L. Hsu, *Rigidity of integral curves of rank 2 distributions*, Invent. Math. 114 (1993), 435–461.

[Ca] E. Cartan, *Sur la structure des groupes infinis de transformations*, Ann. Ecole Norm. Supp. 21 (1904), 153–206, (1905), 219–308.

[Go] A. B. Goncharov, *Generalized conformal structures on manifolds*, Selecta Math. Sov. 6 (1987), 307–340.

[GS] V. Guillemin, S. Sternberg, *Sur les systèmes de formes différentielles*, Ann. Inst. Fourier, Grenoble 13(2) (1963), 61–74.

[Ha] T. Hangan, *Géométrie différentielle Grassmannienne*, Rev. Roumaine Math. Pures Appl. 11 (1966), 519–531.

[Hs] L. Hsu, *Calculus of variations via the Griffiths formalism*, J. Differential Geom. 36(3) (1992), 551–589.

[Ma] Y. I. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[Mc] B. McKay, *Rational curves and ordinary differential equations*, Math. DG/0507087.

[Mi] Y. I. Mikhailov, *On the structure of almost Grassmannian manifolds*, Soviet Math. 22 (1978), 54–63.

[MS] Y. Machida, H. Sato, *Twistor theory of manifolds with Grassmannian structures*, Nagoya Math. J. 160 (2000), 17–102

[Mo] R. Montgomery, *A Tour of Subriemannian Geometry, their Geodesics and Applications*, Mathematical Surveys and Monographs, vol. 91, 2002.

[SY1] H. Sato, K. Yamaguchi, *Lie Contact Manifolds, in Geometry of Manifolds*, K. Shiohama ed., 191–238, Boston Academic Press, 1989.

[SY2] H. Sato, K. Yamaguchi, *Lie contact manifolds II*, Math. Ann. 297(1) (1993), 33–57.

[Ta] N. Tanaka, *On the equivalence problem associated with simple graded Lie algebras*, Hokkaido Math. J. 8 (1979), 23–84.

[Ya1] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Adv. Studies in Pure Math. 22 (1993), 413–494.

[Ya2] K. Yamaguchi, *G_2-geometry of overdetermined systems of second order*, Trends in Math. (Analysis and Geometry in Several Complex Variables) (1999), Birkhäuser, Boston, 289–314.

H. Sato
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, 464-8602 JAPAN
E-mail: hsato@math.nagoya-u.ac.jp

K. Yamaguchi
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPORO, 060-0810 JAPAN
E-mail: yamaguch@math.sci.hokudai.ac.jp

Received April 1, 2011; revised version July 1, 2011.