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AN EXISTENCE THEOREM FOR SOLUTIONS OF AN INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

Abstract. The paper contains an existence theorem for local solutions of an initial value problem for a nonlinear integro-differential equation in Banach spaces. The assumptions and proofs are expressed in terms of measures of noncompactness.

Consider the following Cauchy problem

$$x^{(m)}(t) = f\Big(t, x(t), \int_0^t g(t,s,x(s))ds\Big), \tag{1}$$

$$x(0) = 0, x'(0) = \eta_1, \ldots, x^{(m-1)}(0) = \eta_{m-1} \tag{2}$$

in a Banach space E, where $m \geq 1$ is a natural number. Throughout this paper we shall assume that $D = [0, a]$ is a compact interval in $\mathbb{R}$, $B = \{x \in E : \| x \| \leq b\}$, $f\colon D \times B \times E \to E$ is a continuous function, and $g\colon D^2 \times B \to E$ is a bounded continuous function. Moreover, we suppose that $\|f(t, x, z)\| \leq M$ for $t \in D$, $x \in B$, $z \in W$, where

$$W = \bigcup_{0 \leq \lambda \leq a} \lambda \overline{\text{conv}} g(D^2 \times B).$$

Denote by α the Kuratowski measure of noncompactness in E (cf. [1]).

1. Main result

In this section we shall prove an existence theorem for local solutions of the above initial value problem for the nonlinear integro-differential equation in Banach spaces.

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Theorem. *Let $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous nondecreasing function such that $w(0) = 0$, $w(r) > 0$ for $r > 0$ and*

$$\int_{0+} \frac{dr}{\sqrt[m]{r^{m-1} w(r)}} = \infty.$$

If

$$\alpha\left(f(t, X \times Y)\right) \leq w\left(\alpha(X)\right) + \alpha(Y) \tag{3}$$

for $t \in D$, $X \subset B$ and bounded $Y \subset E$, and the set $g(D^2 \times B)$ is relatively compact in E, then there exists an interval $J = [0, d]$ such that the problem $(1) - (2)$ has at least one solution defined on J.

Our results extend the Aronszajn type theorem for the equation $x^{(m)} = f(t, x)$ in Banach space obtained in [6, Th. 2.1] (see also [5]).

Proof. We choose a positive number d such that $d \leq a$ and

$$\sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M \frac{d^m}{m!} < b. \tag{4}$$

Put $J = [0, d]$. Denote by $C = C(J, E)$ the Banach space of continuous functions $y\colon J \to E$ with the usual norm $\|y\|_C = \max_{t \in J} \|y(t)\|$.

Let $\widetilde{B} \subset C$ be the subset of those functions with values in B. For $t \in J$ and $x \in \widetilde{B}$ put

$$\widetilde{g}(t, x) = \int_0^t g(t, s, x(s)) ds.$$

Fix $\tau \in J$ and $x \in \widetilde{B}$. As the set $J \times x(J)$ is compact, from the continuity of g it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g(t, s, x(s)) - g(\tau, s, x(s))\| < \varepsilon \ \text{ for } \ t, s \in J \ \text{ with } \ |t - \tau| < \delta.$$

In view of the inequality

$$\|\widetilde{g}(t, x) - \widetilde{g}(\tau, x)\| \leq K|t - \tau| + \int_0^\tau \|g(t, s, x(s)) - g(\tau, s, x(s))\| \, ds,$$

where $K = \sup\{\|g(t, s, x)\| : \ t, s \in D, \ x \in B\}$, this implies the continuity of the function $t \to \widetilde{g}(t, x)$. On the other hand, the Lebesgue dominated convergence theorem proves that for each fixed $t \in J$ the function $x \to \widetilde{g}(t, x)$ is continuous on $\widetilde{B}$. Moreover

$$\|\widetilde{g}(t, x)\| \leq Kt \ \text{ for } \ t \in J \ \text{ and } \ x \in \widetilde{B}.$$

By the Mazur lemma the set $\overline{\text{conv}} g(D^2 \times B)$ is relatively compact. Therefore from the following properties of the Kuratowki measure of noncompactness $\alpha(\bigcup_{0 \leq \lambda \leq a} \lambda A) = a\alpha(A)$ it follows that $W = \bigcup_{0 \leq \lambda \leq a} \lambda \overline{\text{conv}} g(D^2 \times B)$ is relatively compact.

According to the above and $\left\{\widetilde{g}(s,x):\ x\in\widetilde{B}\right\}\subset W$, we have

$$\alpha\left(\left\{\widetilde{g}(s,x):\ x\in\widetilde{B}\right\}\right)\le\alpha(W)=0. \tag{5}$$

Let us remark that the problem (1)–(2) is equivalent to the integral equation

$$x(t)=p(t)+\frac{1}{(m-1)!}\int_0^t(t-s)^{m-1}f(s,x(s),\widetilde{g}(s,x))ds\quad(t\in J),$$

where $p(t)=\sum_{j=1}^{m-1}\eta_j\frac{t^j}{j!}$. We define a mapping F by

$$F(x)(t)=p(t)+\frac{1}{(m-1)!}\int_0^t(t-s)^{m-1}f(s,x(s),\widetilde{g}(s,x))ds\quad(t\in J,x\in\widetilde{B}).$$

Owing to (4), it is known (cf. [3]) that F is a continuous mapping $\widetilde{B}\mapsto\widetilde{B}$ and the set $F(\widetilde{B})$ is equicontinuous.

For any positive integer n put

$$v_n(t)=\begin{cases}p(t) & \text{if } 0\le t\le\frac{d}{n}\\ p(t)+\frac{1}{(m-1)!}\int_0^{t-\frac{d}{n}}(t-s)^{m-1}f(s,v_n(s),\widetilde{g}(s,v_n))ds & \text{if } \frac{d}{n}\le t\le d.\end{cases}$$

Then, by (4), $v_n\in\widetilde{B}$ and

$$\lim_{n\to\infty}\|v_n-F(v_n)\|_C=0. \tag{6}$$

Put $V=\{v_n:n\in N\}$ and $V(t)=\{v_n(t):n\in N\}$ for $t\in J$.

As $V\subset\{v_n-F(v_n):n\in N\}+F(V)$ and $V\subset\widetilde{B}$, from (6) it follows that the set V is equicontinuous. Thus the function $t\mapsto v(t)=\alpha(V(t))$ is continuous on J. Since

$$V(t)\subset\{v_n(t)-F(v_n)(t):n\in N\}+F(V)(t)$$

and $\alpha(\{v_n(t)-F(v_n)(t):\ n\in N\})=0$, we have

$$\alpha(V(t))\le\alpha\left(F(V)(t)\right). \tag{7}$$

By (3), (5) and Heinz's lemma [2] we obtain

$$\alpha(F(V)(t)) = \alpha\Big(\Big\{\frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s, v_n(s), \widetilde{g}(s, v_n))ds : n \in N\Big\}\Big)$$
$$\leq \frac{2}{(m-1)!}\int_0^t \alpha\left(\left\{(t-s)^{m-1} f(s, v_n(s), \widetilde{g}(s, v_n)) : n \in N\right\}\right) ds$$
$$\leq \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1}\alpha(f(s, V(s), \widetilde{g}(s, V))ds$$
$$\leq \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1}\left(w(\alpha(V(s))) + \alpha(\{\widetilde{g}(s,x) : \ x \in \widetilde{B}\})\right) ds$$
$$= \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(\alpha(V(s)))ds.$$

Applying (7) we have

$$v(t) \leq \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(v(s))ds \ \text{ for } \ t \in J.$$

Putting $h(t) = \frac{2}{(m-1)!}\int_0^t (t-s)^{m-1} w(v(s))ds$, we see that $h \in C^m$, $v(t) \leq h(t)$, $h^{(j)}(t) \geq 0$ for $j = 0, 1, \ldots, m$, $h^{(j)}(0) = 0$ for $j = 0, 1, \ldots, m-1$ and $h^{(m)}(t) = 2w(v(t)) \leq 2w(h(t))$ for $t \in J$. By Th. 1 of [4] from this we deduce that $h(t) = 0$ for $t \in J$. Thus $\alpha(V(t)) = 0$ for $t \in J$. Therefore, for each $t \in J$ the set $V(t)$ is relatively compact in E. Hence by Ascoli's theorem, V is relatively compact subset of C. Hence, we can find a subsequence (v_{n_k}) of (v_n) which converges in C to a limit u. As F is continuous, from (6) we conclude that $u = F(u)$, so that u is a solution of (1)–(2). ∎

2. The set of solutions

Put

$$\overline{f}(t, x, z) = f\big(t, r(x), z\big),$$
$$\overline{g}(t, s, x) = g\big(t, s, r(x)\big),$$

where

$$r(x) = \begin{cases} x & \text{for } x \in B \\ \dfrac{bx}{\|x\|} & \text{for } x \in E \setminus B \end{cases}$$

and define a mapping $\overline{F}$ by

$$\overline{F}(x)(t) = p(t) + \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1}\overline{f}\Big(s, x(s), \int_0^s \overline{g}\big(s,\tau,x(\tau)\big)\,d\tau\Big)ds.$$

It can be shown that $\overline{F}$ satisfies the assumptions of Theorem 1.3 from [6] (see also Vidossich [7]). By this theorem we conclude that under the assumptions of the Theorem, the set of all solutions of $(1)-(2)$ defined on J is a compact R_δ in $C(J,E)$, i.e. it is homeomorphic to the intersection od a decreasing sequence of compact absolute retracts.

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