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I-UNIFORM CONTINUITY AND I-UNIFORM BOUNDEDNESS OF A FUNCTION

Abstract. The concepts of I-convergence and I-Cauchy condition are a generalization of statistical convergence and statistical Cauchy conditions and are dependent on the notion of the ideal I of subsets of the set $\mathbb{N}$ of positive integers. In this paper, we shall introduce two new notions of I-uniform continuity and I-uniform boundedness of a function with values in $\mathbb{R}$ or in a metric space and then study their basic properties.

1. Introduction background

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [5] and Steinhaus [15] (see also Schoenberg [14]) as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of the set K_n. The natural density of the subset K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$$

provided the limit exists.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, d_X) is said to be statistically convergent to $x \in X$ if for every $\epsilon > 0$, the set $A(\epsilon) = \{n \in \mathbb{N} : d_X(x_n, x) \geq \epsilon\}$ has natural density zero. A lot of investigations have been done on this convergence after the initial works by Fridy [6] and Šalát [13]. It should be also mentioned that the notion of statistical convergence has been considered, in other contexts, by R. A. Bernstein, Z. Frolik, and other authors.

Statistical convergence has several applications in various fields of mathematics: summability theory [1, 6], number theory, trigonometric series, probability theory, measure theory, optimization [12] and approximation theory.

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An interesting generalization of the notion of statistical convergence was proposed in [9] (see also [10]). Namely, it is easy to check that the family $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible ideal of $\mathbb{N}$ (recall that $I \subset 2^{\mathbb{N}}$ is called an ideal if $\phi \in I$, $A, B \in I$ implies $A \cup B \in I$, and $A \in I$, $B \subset A$ implies $B \in I$. I is called non-trivial if $I \neq \{\phi\}$ and $\mathbb{N} \notin I$. I is admissible if it contains all singletons. If I is a proper non-trivial ideal then the family of sets $F(I) = \{M \subseteq \mathbb{N} : (\exists\ A \in I)\ M = \mathbb{N} \setminus A\}$ is a filter in $\mathbb{N}$. It is called the filter associated with the ideal). Thus one may consider an arbitrary ideal I of $\mathbb{N}$ and define I-convergence of a sequence as follows:

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, d_X) is said to be I-convergent to $x \in X$ (in short $x = I-\lim\limits_{n\to\infty} x_n$) if $A(\epsilon) \in I$ for each $\epsilon > 0$, where $A(\epsilon) = \{n \in \mathbb{N} : d_X(x_n, x) \geq \epsilon\}$.

Fridy [6] formulated the statistical Cauchy condition for sequence of real numbers from the idea of classical Cauchy condition. The notion of statistical Cauchy condition was further extended to I-Cauchy condition in a metric space independently by Dems [4] and also by Gurdal [8]. (More results on this convergence can be found in [3,11]). In the present paper we shall introduce two new notions: of I-uniform continuity and I-uniform boundedness of a function with values in $\mathbb{R}$ or in a metric space, and then study their basic properties.

Throughout the paper we assume I to be a non-trivial, admissible ideal of $\mathbb{N}$.

Main definitions and results

In [4] the notion of an I-Cauchy sequence was introduced as follows:

Definition 1. Let (X, d_X) be a metric space and $I \subset 2^{\mathbb{N}}$ be an admissible ideal. Then a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called an I-Cauchy sequence in X if for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : d_X(x_n, x_k) \geq \epsilon\} \in I.$$

In [4] it was shown that $\{x_n\}_{n\in\mathbb{N}}$ is I-Cauchy if and only if for every $\epsilon > 0$, there exists a set $B \in I$ such that $m, n \notin B \;\Rightarrow\; d_X(x_m, x_n) < \epsilon$.

Now we introduce the following definitions:

Definition 2. Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be I-uniform continuous on X, if for every I-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, $\{f(x_n)\}_{n\in\mathbb{N}}$ is also an I-Cauchy sequence in Y.

Definition 3. Let (X, d_X) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be I-bounded, if there exists an element $x \in X$ and a positive real number r such that $\{n \in \mathbb{N} : d_X(x_n, x) > r\} \in I$.

Proposition 1. *In a metric space (X, d_X), every I-Cauchy sequence is also I-bounded.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence in X. Then for each $\epsilon > 0$ there exists a positive integer k such that $\{n \in \mathbb{N} : d_X(x_n, x_k) \geq \epsilon\} \in I$. Choose $x \in X$ and put $M = \epsilon + d_X(x, x_k)$. Then $\{n \in \mathbb{N} : d_X(x_n, x_k) \geq \epsilon\} = \{n \in \mathbb{N} : d_X(x, x_k) + d_X(x_n, x_k) \geq M\} \supseteq \{n \in \mathbb{N} : d_X(x, x_n) \geq M\} \supset \{n \in \mathbb{N} : d_X(x, x_n) > 2M\}$. Hence the result. ∎

It is quite clear that every function $f : X \to Y$ uniformly continuous on X is also I-uniformly continuous on X. But the converse is not true.

Example 1. Let us consider the function $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x) = x^2$. The function is I-uniformly continuous on $\mathbb{Q}$ but not uniformly continuous on $\mathbb{Q}$.

It is sufficient to prove that for every I-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$, $\{x_n^2\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence in $\mathbb{Q}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence, $\{x_n\}_{n\in\mathbb{N}}$ is I-bounded, hence there exists a positive real number r such that $M = \{n \in \mathbb{N} : |x_n| \leq r\} \in F(I)$. Since $\{x_n\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence, for each $\epsilon > 0$ there exists $A \in I$ such that $m, n \notin A$ implies $|x_m - x_n| < \frac{\epsilon}{2r}$. Let $m, n \in (\mathbb{N} \setminus A) \cap M \in F(I)$ (m, n exist since $\phi \notin F(I)$) then $|x_m^2 - x_n^2| \leq |x_m + x_n||x_m - x_n| \leq (|x_m| + |x_n|)|x_m - x_n| < 2r\frac{\epsilon}{2r} = \epsilon$.

Let us define sets

$$C(X, Y) = \{f : (X, d_X) \to (Y, d_Y) : f \text{ is continuous on } X\}$$

and

$$UC(X, Y, I) = \{f : (X, d_X) \to (Y, d_Y) : f \text{ is } I\text{-uniform continuous on } X\}.$$

The following examples show that $C(X, Y)\Delta UC(X, Y, I) \neq \phi$ (where Δ is the symmetric difference).

First we construct a function $f : (0, 1] \to \mathbb{R}$ such that f is continuous on $(0, 1]$ but not I-uniformly continuous on $(0, 1]$, i.e. there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(0, 1]$ which is I-Cauchy, while $\{f(x_n)\}_{n\in\mathbb{N}}$ is not.

It is easy to verify that $f(x) = \frac{1}{x}$, $x \in (0, 1]$ is such a function (I can be any admissible ideal).

Next we construct a function $f : (0, 1] \to \mathbb{R}$ such that f is I-uniformly continuous, but not continuous on $(0, 1]$, i.e., for every I-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(0, 1]$, $\{f(x_n)\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence in $\mathbb{R}$, but f is not continuous on $(0, 1]$.

Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ and } m, n \text{ are relatively prime.} \end{cases}$$

Also let $I(\subset 2^{\mathbb{N}})$ be any admissible ideal. Then the function f is discontinuous at every rational point in $(0,1]$. We skip the straightforward verification.

Proposition 2. *Let (X,d_X) and (Y,d_Y) be two metric spaces and the functions $f,g : X \to Y$ be I-uniform continuous on X. Then the function $d_Y(f,g) : X \to \mathbb{R}$ is I-uniform continuous on X, where $d_Y(f,g)(x) = d_Y(f(x),g(x)), \forall\, x \in X$.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence in X. Since f and g are both I-uniformly continuous on X, then $\{f(x_n)\}_{n\in\mathbb{N}}$ and $\{g(x_n)\}_{n\in\mathbb{N}}$ are I-Cauchy sequences in Y, so for each $\epsilon > 0$ there exist $M_1, M_2 \in F(I)$ such that

$$\begin{aligned} m,n \in M_1 \ &\Rightarrow\ d_Y(f(x_m),f(x_n)) < \frac{\epsilon}{2} \\ \text{and}\ \ m,n \in M_2 \ &\Rightarrow\ d_Y(g(x_m),g(x_n)) < \frac{\epsilon}{2}. \end{aligned}$$

Since

$$\begin{aligned} d_Y(f(x_n),g(x_n)) \le\ & d_Y(f(x_n),f(x_m)) + d_Y(f(x_m),g(x_m)) \\ &+ d_Y(g(x_m),g(x_n)), \end{aligned}$$

we have $m,n \in M_1 \cap M_2 \in F(I)$ (m,n exist since $\phi \notin F(I)$), which shows that

$$|d_Y(f(x_n),g(x_n)) - d_Y(f(x_m),g(x_m))| < \epsilon. \ \blacksquare$$

Proposition 3. *Let X and Y be two norm linear spaces and functions $f,g : X \to Y$ be I-uniformly continuous on X. Then $f+g$ and $f\cdot g$ are both I-uniformly continuous on X.*

Proof. We only prove the result for multiplication. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence in X. Since an I-Cauchy sequence is I-bounded and f and g both are I-uniformly continuous, there exist two positive real number B_1 and B_2 such that

$$\begin{aligned} M_1 &= \{n \in \mathbb{N} : ||f(x_n)|| \le B_1\} \in F(I) \\ \text{and } M_2 &= \{n \in \mathbb{N} : ||g(x_n)|| \le B_2\} \in F(I). \end{aligned}$$

Since $\{f(x_n)\}_{n\in\mathbb{N}}$ and $\{g(x_n)\}_{n\in\mathbb{N}}$ are I-Cauchy sequences, for each $\epsilon > 0$ there exist $M_3, M_4 \in F(I)$ such that

$$\begin{aligned} m,n \in M_3 \ &\Rightarrow\ ||f(x_m) - f(x_n)|| < \frac{\epsilon}{2B_2} \\ \text{and}\ \ m,n \in M_4 \ &\Rightarrow\ ||g(x_m) - g(x_n)|| < \frac{\epsilon}{2B_1}. \end{aligned}$$

Then $m,n \in M_1 \cap M_2 \cap M_3 \cap M_4 \in F(I)$ (m,n exist because $\phi \notin F(I)$) implies $||(f\cdot g)(x_n) - (f\cdot g)(x_m)|| = ||f(x_n)\cdot g(x_n) - f(x_m)\cdot g(x_m)|| \le$

$||f(x_n)|| \cdot ||g(x_n) - g(x_m)|| + ||g(x_m)|| \cdot ||f(x_n) - f(x_m)|| < B_1 \frac{\epsilon}{2B_1} + B_2 \frac{\epsilon}{2B_2} = \epsilon$. Hence $f \cdot g$ is I-uniformly continuous on X. ∎

Proposition 4. *Let X, Y and Z be three norm linear spaces, and $f : X \to Y$ and $g : f(X) \to Z$ be I-uniformly continuous on X and $f(X)$, respectively. Then $g \circ f$ is also I-uniformly continuous on X.*

Proof is straightforward.

"Let D be a bounded interval in $\mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous on D, then f is bounded on D." The uniform continuity of the function f on D is a sufficient condition for boundedness of the function f on D. There also exists a weaker condition such that the above statement holds:

Theorem 1. *Let I be any admissible ideal and D be a bounded interval in $\mathbb{R}$. If a function $f : D \to \mathbb{R}$ is I-uniformly continuous on D, then f is bounded on D.*

Proof. Let us assume that f is not bounded on D. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in D such that $|f(x_n)| > n$ for $n = 1, 2, 3, \ldots$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, it has a convergent subsequence, say $\{x_{r_n}\}_{n\in\mathbb{N}}$ (by Bolzano-Weierstrass theorem). Since $\{x_{r_n}\}_{n\in\mathbb{N}}$ is convergent in D, it is an I-Cauchy sequence in D. Since f is I-uniformly continuous on D, $\{f(x_{r_n})\}_{n\in\mathbb{N}}$ must be an I-Cauchy sequence in $\mathbb{R}$. But $|f(x_{r_n})| > r_n > n$ for $n = 1, 2, 3, \ldots$ so $\{f(x_{r_n})\}_{n\in\mathbb{N}}$ can not be I-bounded, hence $\{f(x_{r_n})\}_{n\in\mathbb{N}}$ is not an I-Cauchy sequence and we arrive at a contradiction. This proves that f is bounded on D. ∎

Next, we introduce the following definition:

Definition 4. Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be I-uniformly bounded on X, if for any I-bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in X $\{f(x_n)\}_{n\in\mathbb{N}}$ is also I-bounded in Y.

It is quite clear that if a function $f : X \to Y$ is bounded on X, then $f : X \to Y$ is I-uniformly bounded on X.

But the converse is not true. For example $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ and I be any admissible ideal in $\mathbb{N}$.

Next we define

$$UB(X, Y, I) = \{f : (X, d_X) \to (Y, d_Y) : f \text{ is } I\text{-uniform bounded on } X\}.$$

The following example shows that f is I_f-uniformly bounded but not I_f-uniformly continuous for the admissible ideal I_f of $\mathbb{N}$, where $I_f = \{A \subset \mathbb{N} : A \text{ is finite}\}$.

Setting, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0, \infty) \\ -1, & \text{if } x \in (-\infty, 0]. \end{cases}$$

Proof is straightforward.

Note 1. For the case when f is I-uniformly continuous but not I-uniformly bounded see [7].

Theorem 2. *Let $D \subset \mathbb{R}$ be an interval. If a function $f : D \to \mathbb{R}$ be such that f' exists and is I_f-uniformly bounded on D, then f is I-uniformly continuous on D.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence. Then $\{x_n\}_{n\in\mathbb{N}}$ is an I-bounded sequence so there exists a positive real number r such that $M = \{n \in \mathbb{N} : |x_n| \le r\} \in F(I)$.

Using MVT, we define a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that

$$\frac{f(x_n) - f(x_m)}{x_n - x_m} = f'(y_k) \text{ if } n, m \in M \text{ and } x_n \neq x_m,$$
$$\text{where } \min\{x_n, x_m\} < y_k < \max\{x_n, x_m\},$$

$y_k = 0$ otherwise (since there exists a bijection between $\mathbb{N}$ and $\mathbb{N}^2$).

Then $|y_k| \le r$ for all $k \in \mathbb{N}$. Since f' is I_f-uniformly bounded on D, there exists a positive real number B such that $|f'(y_k)| \le B \; \forall \; k \in \mathbb{N}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence, for each $\epsilon > 0$ there exists $A \in I$ such that $m, n \notin A$ implies $|x_m - x_n| < \frac{\epsilon}{B}$. Let $m, n \in (\mathbb{N} \setminus A) \cap M \in F(I)$ (m, n exist since $\phi \notin F(I)$), then $|f(x_m) - f(x_n)| = |f'(y_k)||x_m - x_n| < B\frac{\epsilon}{B} = \epsilon$. Hence the result. ■

In [2] the notion of I-uniform convergence of sequences of functions was introduced as follows:

Definition 5. Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$, $f_n : X \to Y$ be functions on X, $n \in \mathbb{N}$. Then the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be I-uniformly convergent on X to a function f if

$$(\forall\ \epsilon > 0)\ (\exists\ M \in I)\ (\forall\ n \notin M)\ (\forall\ x \in X)\ \Rightarrow\ d_Y(f_n(x), f(x)) < \epsilon.$$

Theorem 3. *Let (X, d_X) and (Y, d_Y) be two metric spaces then the set $UB(X, Y, I)$ is closed with respect to I-uniform convergence for any ideal I.*

Proof. Let the sequence $\{f_n\}_{n\in\mathbb{N}}$ be I-uniformly convergent to f, where $f_n \in UB(X, Y, I)$ for each $n \in \mathbb{N}$, then for $\epsilon = 1$ there exists $A \in I$ such that for all $m \notin A$, for all $x \in X \Rightarrow\ d_Y(f_m(x), f(x)) < 1$. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-bounded sequence in X. Since each f_m is I-uniformly bounded, then $\{f_m(x_n)\}_{n\in\mathbb{N}}$ is also I-bounded in Y, then there exist $y \in Y$ and $r \in \mathbb{R}^+$

such that $M = \{n \in \mathbb{N} : d_Y(y, f_m(x_n)) \le r\} \in F(I)$. Let $n \in M$ then $d_Y(y, f(x_n)) \le d_Y(y, f_m(x_n)) + d_Y(f_m(x_n), f(x_n)) \le r + 1$. Hence the result. ∎

Note 2. If each f_n be I-uniformly bounded on D, the I-uniform convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ on D is a sufficient but not a necessary condition for I-uniform boundedness of the limit function f on D.

For example, let $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in [0,1]$. Then the limit function f is defined by $f(x) = 0$, $x \in [0,1]$. Let I be an admissible ideal. Each f_n is I-uniformly bounded on $[0,1]$. Also the limit function f is I-uniformly bounded on $[0,1]$. Let $\epsilon < \frac{1}{2}$ then there does not exist any $A \in I$ such that for all $n \notin A$ for all $x \in [0,1]$, $|f_n(x) - f(x)| < \epsilon$, (since $f_n(\frac{1}{n}) = \frac{1}{2}$, $\forall\, n \in \mathbb{N}$).

Theorem 4. *Let (X, d_X) and (Y, d_Y) be two metric spaces then the set $UC(X,Y,I)$ is closed with respect to I-uniform convergence for any ideal I.*

Proof. Let the sequence $\{f_n\}_{n\in\mathbb{N}}$ is I-uniformly convergent to f, where $f_n \in UC(X,Y,I)$ for each $n \in \mathbb{N}$, then for $\epsilon > 0$ there exists $A \in I$, for all $n \notin A$, for all $x \in X \Rightarrow\ d_Y(f_n(x), f(x)) < \frac{\epsilon}{3}$. For fixed $m \notin A$ $\Rightarrow\ d_Y(f_m(x), f(x)) < \frac{\epsilon}{3}\ \forall\, x \in X$. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence in X. Since each f_m is I-uniformly continuous on X, there exists $k \in \mathbb{N}$ such that $M = \{n \in \mathbb{N} : d_Y(f_m(x_n), f_m(x_k)) < \frac{\epsilon}{3}\} \in F(I)$. Hence

$$\begin{aligned} d_Y(f(x_n), f(x_k)) \le d_Y(f(x_n), f_m(x_n)) &+ d_Y(f_m(x_n), f_m(x_k)) \\ &+ d_Y(f_m(x_k), f(x_k)). \end{aligned}$$

This implies $(\mathbb{N} \setminus A) \cap M \subset \{n \in \mathbb{N} : d_Y(f(x_n), f(x_k)) < \epsilon\} \in F(I)$. ∎

Note 3. If each $f_n : D(\subset \mathbb{R}) \to \mathbb{R}$ be I-uniformly continuous on D, the I-uniform convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ on D is a sufficient but not a necessary condition for I-uniform continuous of the limit function f on D.

Same as example in Note 2.

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