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FUZZY ANTI-NORM AND FUZZY α-ANTI-CONVERGENCE

Abstract. In this paper the definition of fuzzy antinorm is modified. Some properties of finite dimensional fuzzy antinormed linear space are studied. Fuzzy α-anti-convergence and fuzzy α-anti-complete linear spaces are defined and some of their properties are studied.

1. Introduction

During the last few years there is a growing interest in the extension of fuzzy set theory which is a useful tool to describe the situation in which data are imprecise or vague or uncertain. Fuzzy set theory handle the situation by attributing a degree of membership to which a certain object belongs to a set. It has a wide range of application in the field of population dynamics [5], chaos control [9], computer programming [10], medicine [4] etc.

The concept of fuzzy set theory was first introduced by Zadeh [17] in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways, by several authors. The concept of fuzzy norm was introduced by Katsaras [12] in 1984. In 1992, Felbin [8] introduced the idea of fuzzy norm on a linear space. Cheng–Moderson [6] introduced another idea of fuzzy norm on a linear space whose associated metric is same as the associated metric of Kramosil–Michalek [14]. In 2003, Bag and Samanta [1] modified the definition of fuzzy norm of Cheng–Moderson [6] and established the concept of continuity and boundednes of a linear operator with respect to their fuzzy norm in [2].

Later on, Jebril and Samanta [11] introduced the concept of fuzzy anti-norm on a linear space depending on the idea of fuzzy anti norm, introduced by Bag and Samanta [3]. The motivation of introducing fuzzy anti-norm is

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to study fuzzy set theory with respect to the non-membership function. It is useful in the process of decision making.

In this paper we generalize the definition of fuzzy anti-norm on a linear space. Later on we prove Riesz lemma and some important properties of finite dimensional fuzzy anti-normed linear space. Also, we define fuzzy α-anti-convergence, fuzzy α-anti-Cauchy sequence, fuzzy α-anti-completeness and study the relations among them.

2. Preliminaries

This section contains some basic definition and preliminary results which will be needed in the sequel.

Definition 2.1. [16, 13] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$, $\forall a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0,1]$.

A few examples of continuous t-conorm are $a \diamond b = a + b - ab$, $a \diamond b = \max\{a, b\}$, $a \diamond b = \min\{a + b, 1\}$.

Definition 2.2. [3] Let X be a linear space over F (field of real/complex numbers). Let N^* be a fuzzy subset of $X \times \mathbb{R}$ such that for all $x, y \in X$ and $c \in F$

(N^*1) $\forall t \in \mathbb{R}$ with $t \leq 0$, $N^*(x,t) = 1$,
(N^*2) $\forall t \in \mathbb{R}$ with $t > 0$, $N^*(x,t) = 0$ if and only if $x = \theta$,
(N^*3) $\forall t \in \mathbb{R}$ with $t > 0$, $N^*(cx,t) = N^*(x, \frac{t}{|c|})$ if $c \neq 0$,
(N^*4) $\forall s, t \in \mathbb{R}$ with $N^*(x+y, s+t) \leq \max\{N^*(x,s), N^*(y,t)\}$,
(N^*5) $N^*(x, \cdot)$ is a non-increasing of $t \in \mathbb{R}$ and $\lim\limits_{t \to \infty} N^*(x,t) = 0$.

Then N^* is called a B-S-fuzzy antinorm on X.

We assume that

(N^*6) For all $t \in \mathbb{R}$ with $t > 0$, $N^*(x,t) < 1$ implies $x = \theta$.

Definition 2.3. [11] Let (U, N^*) be a B-S-fuzzy antinormed linear space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in U is said to *converge* to $x \in U$ if given $t > 0$, $r \in (0,1)$ there exists an integer $n_0 \in \mathbb{N}$ such that

$$N^*(x_n - x, t) < r, \ \forall n \geq n_0.$$

Definition 2.4. [11] Let (U, N^*) be a B-S-fuzzy antinormed linear space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in U is said to be *Cauchy sequence* if for given $t > 0$,

$r \in (0,1)$ there exists an integer $n_0 \in \mathbb{N}$ such that

$$N^*(x_{n+p} - x_n, t) < r \quad \forall n \geq n_0,\, p = 1, 2, 3, \ldots.$$

Definition 2.5. [11] A subset A of a B-S-fuzzy antinormed linear space (U, N^*) is said to be *bounded* if and only if there exist $t > 0$, $r \in (0,1)$ such that

$$N^*(x,t) < r,\ \forall x \in A.$$

Definition 2.6. [11] A subset A of a B-S-fuzzy antinormed linear space (U, N^*) is said to be *compact* if any sequence $\{x_n\}_{n\in\mathbb{N}}$ in A has a subsequence converging to an element of A.

Definition 2.7. [11] Let (U, N^*) be a B-S-fuzzy antinormed linear space. A subset B of U is said to be **closed** if any sequence $\{x_n\}_{n\in\mathbb{N}}$ in B converges to $x \in B$, that is

$$\lim_{n\to\infty} N^*(x_n - x, t) = 0, \quad \forall t > 0 \Rightarrow x \in B.$$

3. Fuzzy anti-normed linear space

In this section, the definition of B-S-fuzzy antinorm is modified, and after modification it will be termed as fuzzy antinorm with respect to a t-conorm $\diamond$. Thereafter some important results will be deduced.

Definition 3.1. Let V be linear space over the field $F(= \mathbb{R} or \mathbb{C})$. A fuzzy subset ν of $V \times \mathbb{R}$ is called a *fuzzy antinorm* on V with respect to a t-conorm $\diamond$ if and only if for all $x, y \in V$

(i) $\forall t \in \mathbb{R}$ with $t \leq 0$, $\nu(x,t) = 1$;
(ii) $\forall t \in \mathbb{R}$ with $t > 0$, $\nu(x,t) = 0$ if and only if $x = \theta$;
(iii) $\forall t \in \mathbb{R}$ with $t > 0$, $\nu(cx,t) = \nu(x, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$;
(iv) $\forall s, t \in \mathbb{R}$ with $\nu(x+y, s+t) \leq \nu(x,s) \diamond \nu(y,t)$;
(v) $\lim_{t\to\infty} \nu(x,t) = 0$.

The Definition 3.1 is more general than the Definition 2.2; since, in (N^*4) instead of maximum function we have used more generalized function, conorm function and in the condition (N^*5) it is used that $N^*(x,\cdot)$ is non-increasing function of $t(\in \mathbb{R})$, which is redundant and later on it will be deduced.

Remark 3.2. *Let ν be a fuzzy anti-norm on V with respect to a t-conorm $\diamond$, then $\nu(x,t)$ is non-increasing with respect to t for each $x \in V$.*

Proof. Let $t < s$. Then $k = s - t > 0$ and we have

$$\nu(x,t) = \nu(x,t) \diamond 0 = \nu(x,t) \diamond \nu(0,k) \geq \nu(x,s).$$

Hence the proof. ■

Definition 3.3. If $A^* = \{((x,t),\nu(x,t)) : (x,t) \in V \times \mathbb{R}\}$ is a fuzzy antinorm on a linear space V with respect to a t-conorm $\diamond$ over a field F, then (V, A^*) is called a fuzzy antinormed linear space with respect to the t-conorm $\diamond$ over the field F.

We further assume that for any fuzzy anti-normed linear space (V, A^*) with respect to a t-conorm $\diamond$,

(vi) $\nu(x,t) < 1$, $\forall t > 0 \Rightarrow x = \theta$.
(vii) $\nu(x,\cdot)$ is a continuous function of $\mathbb{R}$ and strictly decreasing on the subset $\{t : 0 < \nu(x,t) < 1\}$ of $\mathbb{R}$.
(viii) $a \diamond a = a$, $\forall a \in [0,1]$.

Example 3.4. Let $(V, \|\cdot\|)$ be a normed linear space and consider $a \diamond b = a + b - ab$. Define $\nu : V \times \mathbb{R} \to [0,1]$ by

$$\nu(x,t) = \begin{cases} 0, & \text{if } t > \|x\|, \\ 1, & \text{if } t \le \|x\|. \end{cases}$$

Then ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V,ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Solution. (i) $\forall x \in V$ and $\forall t \in \mathbb{R}$, $t \le 0$ we have $\nu(x,t) = 1$.

(ii) $\forall t \in \mathbb{R}$, $t > 0$ we have $\nu(\theta,t) = 0$. Again

$$\nu(x,t) = 0, \quad \forall t > 0 \Leftrightarrow \|x\| < t, \quad \forall t(>0) \in \mathbb{R} \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$$

(iii) $\nu(cx,t) = 0 \Leftrightarrow t > \|cx\| \Leftrightarrow t > |c|\|x\| \Leftrightarrow \dfrac{t}{|c|} > \|x\| \Leftrightarrow \nu\left(x, \dfrac{t}{|c|}\right) = 0.$

$$\nu(cx,t) = 1 \Leftrightarrow t \le \|cx\| \Leftrightarrow t \le |c|\|x\| \Leftrightarrow \frac{t}{|c|} \le \|x\| \Leftrightarrow \nu\left(x, \frac{t}{|c|}\right) = 1.$$

(iv) $\nu(x,s) \diamond \nu(y,t) = \nu(x,s) + \nu(y,t) - \nu(x,s)\nu(y,t)$.

If $s > \|x\|$ and $t > \|y\|$ then $\nu(x+y, s+t) = 0$, since $s + t > \|x\| + \|y\|$ and $\nu(x,s) \diamond \nu(y,t) = 0 + 0 - 0 = 0$. So, $\nu(x+y, s+t) = \nu(x,s) \diamond \nu(y,t)$.
If $s > \|x\|$ and $t \le \|y\|$ then $\nu(x,s) \diamond \nu(y,t) = 0 + 1 - 0 = 1$.
If $s \le \|x\|$ and $t > \|y\|$ then $\nu(x,s) \diamond \nu(y,t) = 1 + 0 - 0 = 1$.
If $s \le \|x\|$ and $t \le \|y\|$ then $\nu(x,s) \diamond \nu(y,t) = 1 + 1 - 1 = 1$.

Therefore in any of the above three cases

$$\nu(x,s) \diamond \nu(y,t) = 1 \ge \nu(x+y, s+t).$$

Thus

$$\nu(x+y, s+t) \le \nu(x,s) \diamond \nu(y,t).$$

(v) From the definition it is clear that $\lim_{t\to\infty} \nu(x,t) = 0$. Thus ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V,ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Note 3.5. The above example satisfes the condition (vi) but does not satisfy the condition (vii).

Example 3.6. Let $(V, \|\cdot\|)$ be a normed linear space and consider $a \diamond b = \min\{a+b, 1\}$. Define $\nu : V \times \mathbb{R} \to [0,1]$ by

$$\nu(x,t) = \begin{cases} 0, & \text{if } t > \|x\|,\ t > 0, \\ \frac{\|x\|}{t+\|x\|}, & \text{if } t \le \|x\|,\ t > 0, \\ 1, & \text{if } t \le 0. \end{cases}$$

Then ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V, ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Solution. (i) From the definition we have $\nu(x,t) = 1$ if $t \le 0$, $\forall t \in \mathbb{R}$.

(ii) If $t > 0$ and $t > \|x\|$ then

$$\nu(x,t) = 0 \Leftrightarrow \|x\| < t, \quad \forall t(>0) \in \mathbb{R} \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$$

If $t > 0$ and $t \le \|x\|$ then

$$\nu(x,t) = 0 \Leftrightarrow \frac{\|x\|}{t + \|x\|} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$$

(iii) $\nu(cx,t)=0 \Leftrightarrow t > \|cx\| \Leftrightarrow t > |c|\|x\| \Leftrightarrow \frac{t}{|c|} > \|x\| \Leftrightarrow \nu\left(x, \frac{t}{|c|}\right)=0.$

$$\nu(cx,t) = \frac{\|cx\|}{t+\|cx\|} \Leftrightarrow t \le \|cx\| \Leftrightarrow \frac{t}{|c|} \le \|x\|$$
$$\Leftrightarrow \nu(x, \frac{t}{|c|}) = \frac{\|x\|}{\frac{t}{|c|} + \|x\|} = \frac{\|cx\|}{t+\|cx\|}.$$

(iv) $\nu(x,s) \diamond \nu(y,t) = \min\{\nu(x,s) + \nu(y,t), 1\}$. If $\|x\| \ge s$ and $\|y\| \ge t$ then

$$\nu(x,s) + \nu(y,t) = \frac{\|x\|}{s+\|x\|} + \frac{\|y\|}{t+\|y\|}$$
$$= \frac{(t\|x\| + \|x\|\|y\| + s\|y\|) + \|x\|\|y\|}{(t\|x\| + \|x\|\|y\| + s\|y\|) + st} \ge 1 \text{ since } \|x\|\|y\| \ge st.$$

In this case $\nu(x,s) \diamond \nu(y,t) = 1 \ge \nu(x+y, s+t)$.
If $\|x\| \ge s$ and $\|y\| < t$ then either $\|x+y\| \ge s+t$ or $\|x+y\| < s+t$. Now,

$$\nu(x,s) + \nu(y,t) = \frac{\|x\|}{s+\|x\|} + 0 < 1.$$

Hence

$$\nu(x,s) \diamond \nu(y,t) = \frac{\|x\|}{s+\|x\|}.$$

If $\|x+y\| \geq s+t$ then

$$\begin{aligned}\nu(x+y,s+t)-\nu(x,s)\diamond\nu(y,t) &= \frac{\|x+y\|}{s+t+\|x+y\|}-\frac{\|x\|}{s+\|x\|} \\ &\leq \frac{\|x\|+\|y\|}{s+t+\|x\|+\|y\|}-\frac{\|x\|}{s+\|x\|}=\frac{s\|y\|-t\|x\|}{(s+t+\|x\|+\|y\|)(s+\|x\|)} \\ &< \frac{st-t\|x\|}{(s+t+\|x\|+\|y\|)(s+\|x\|)}, \quad \text{since } \|y\|<t, \\ &\leq 0, \quad \text{since } s\leq\|x\| \Rightarrow st<t\|x\|.\end{aligned}$$

Therefore, $\nu(x+y,s+t)<\nu(x,s)\diamond\nu(y,t)$.
If $\|x+y\|<s+t$ then

$$\nu(x+y,s+t)=0\leq\frac{\|x\|}{s+\|x\|}=\nu(x,s)\diamond\nu(y,t).$$

If $\|x\|<s$ and $\|y\|\geq t$ then in the similar manner (as in the case when $\|x\|\geq s$ and $\|y\|<t$) we can show that $\nu(x+y,s+t)\leq\nu(x,s)\diamond\nu(y,t)$. If $\|x\|<s$ and $\|y\|<t$ then $\nu(x,s)+\nu(y,t)=0+0<1$. Therefore, $\nu(x,s)\diamond\nu(y,t)=0$. Also $\|x+y\|\leq\|x\|+\|y\|<s+t$ and hence $\nu(x+y,s+t)$ $=0$. Hence $\nu(x+y,s+t)=\nu(x,s)\diamond\nu(y,t)$. Thus, in any case

$$\nu(x+y,s+t)\leq\nu(x,s)\diamond\nu(y,t).$$

(v) If $t>\|x\|$ then from the definition it is clear that $\lim_{t\to\infty}\nu(x,t)=0$. If $x\neq\theta$ and $t\leq\|x\|$ then

$$\lim_{t\to\infty}\nu(x,t)=\lim_{t\to\infty}\frac{\|x\|}{t+\|x\|}=0.$$

If $x=\theta$ and $t\leq\|x\|$ then

$$\lim_{t\to\infty}\nu(x,t)=\lim_{t\to\infty}\nu(\theta,t)=\lim_{t\to\infty}\frac{0}{t}=0.$$

Hence

$$\lim_{t\to\infty}\nu(x,t)=0 \quad \forall x\in V.$$

Thus ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V,ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Note 3.7. The above example does not satisfy the conditions (vi) and (vii).

Example 3.8. Let $(V,\|\cdot\|)$ be a normed linear space and consider $a\diamond b=\max\{a,b\}$. Define $\nu:V\times\mathbb{R}\to[0,1]$ by

$$\nu(x,t)=\begin{cases}\frac{\|x\|}{t+\|x\|}, & \text{if } t>0,\\ 1, & \text{if } t\leq 0.\end{cases}$$

Then by Example 3.2 of [11] it follows that ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V, ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Note 3.9. The above example does not satisfy the condition (vi) and satisfies the condition (vii).

Example 3.10. Let $(V, \|\cdot\|)$ be a normed linear space and consider $a \diamond b = \min\{a + b, 1\}$. Define $\nu : V \times \mathbb{R} \to [0, 1]$ by

$$\nu(x,t) = \begin{cases} \frac{\|x\|}{2t-\|x\|}, & \text{if } t > \|x\|, \\ 1, & \text{if } t \leq \|x\|. \end{cases}$$

Then ν satisfies all conditions of Definition 3.1. Therefore, ν is a fuzzy antinorm on V with respect to the t-conorm $\diamond$ and (V, ν) is a fuzzy anti-normed linear space with respect to the t-conorm $\diamond$.

Note 3.11. The above example satisfies both conditions (vi) and (vii).

Theorem 3.12. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ satisfying* (vi) *and* (viii). *Then for any $\alpha \in (0, 1)$ the function $\|x\|_\alpha^* : X \to [0, \infty)$ defined as*

$$\text{(ix)} \qquad \|x\|_\alpha^* = \bigwedge\{t > 0 : \nu(x,t) \leq 1 - \alpha\}, \quad \alpha \in (0,1)$$

is a norm on V.

Proof. (i) For $x \in V$, $\nu(x, t) = 1$ for $t \leq 0 \Rightarrow \bigwedge\{t > 0 : \nu(x, t) \leq 1-\alpha\} \geq 0$, $\alpha \in (0, 1) \Rightarrow \|x\|_\alpha^* \geq 0, \alpha \in (0, 1)$.

(ii) $\|x\|_\alpha^* = 0 \Rightarrow \nu(x, t) \leq 1 - \alpha < 1$, $\forall t \in \mathbb{R}$, $t > 0 \Rightarrow x = \theta$, [by (vi)]. Conversely, $x = \theta \Rightarrow \nu(x, t) = 0$, $\forall t > 0 \Rightarrow \bigwedge\{t > 0 : \nu(x, t) \leq 1 - \alpha\} = 0$, $\forall \alpha \in (0, 1) \Rightarrow \|x\|_\alpha^* = 0$.

(iii) If $c \neq 0$ then

$$\begin{aligned} \|cx\|_\alpha^* &= \bigwedge\{s > 0 : \nu(cx, s) \leq 1 - \alpha\} \\ &= \bigwedge\left\{s > 0 : \nu\left(x, \frac{s}{|c|}\right) \leq 1 - \alpha\right\} \\ &= \bigwedge\{|c|t > 0 : \nu(x,t) \leq 1 - \alpha\} \\ &= \bigwedge |c|\{t > 0 : \nu(x,t) \leq 1 - \alpha\} = |c|\|x\|_\alpha^*. \end{aligned}$$

If $c = 0$ then $\|cx\|_\alpha^* = \|\theta\|_\alpha^* = 0 = 0. \|x\|_\alpha^* = |c|\ \|x\|_\alpha^*$.

(iv) $\|x\|_\alpha^* + \|y\|_\alpha^*$

$$= \bigwedge\{t > 0 : \nu(x,t) \leq 1 - \alpha\} + \bigwedge\{s > 0 : \nu(y,s) \leq 1 - \alpha\},\ \forall \alpha \in (0,1)$$

$$\geq \bigwedge\{t+s>0 : \nu(x,t) \leq 1-\alpha, \nu(y,s) \leq 1-\alpha\}$$
$$\geq \bigwedge\{t+s>0 : \nu(x+y,t+s) \leq 1-\alpha \quad \text{[by (viii)]}$$
$$= \|x+y\|_\alpha^* .$$

Hence, $\{\|\cdot\|_\alpha^*\}$ is a norm on V. ■

Remark 3.13. The norm defined above is more general than the norm defined in Theorem 3.2 of [3]; since instead of $\nu(x,t) < \alpha$ we write $\nu(x,t) \leq 1-\alpha$.

Theorem 3.14. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$. If $\alpha_1 \leq \alpha_2$, then $\|x\|_{\alpha_1}^* \leq \|x\|_{\alpha_2}^*$ i.e., $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ is an increasing family of norms on V.*

Proof. $\alpha_1 \leq \alpha_2$ we have

$$\{t>0 : \nu(x,t) \leq 1-\alpha_2\} \subset \{t>0 : \nu(x,t) \leq 1-\alpha_1\}$$
$$\Rightarrow \bigwedge\{t>0 : \nu(x,t) \leq 1-\alpha_2\} \geq \bigwedge\{t>0 : \nu(x,t) \leq 1-\alpha_1\}$$
$$\Rightarrow \|x\|_{\alpha_2}^* \geq \|x\|_{\alpha_1}^* . \ ■$$

In the following theorem we describe another one equivalent expression for ν, which will be useful to describe Riesz theorem in fuzzy environment.

Theorem 3.15. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ satisfying* (vi), (vii), (viii) *and let $\nu' : V \times R \to [0,1]$ be defined by*

$$\text{(x)} \qquad \nu'(x,t) = \begin{cases} \bigwedge\{1-\alpha : \|x\|_\alpha^* \leq t\}, & \text{if } (x,t) \neq (\theta,0), \\ 1, & \text{if } (x,t) = (\theta,0). \end{cases}$$

Then $\nu' = \nu$, where $\|\cdot\|_\alpha^*$ is a increasing family of norms given by (ix).

To prove this theorem we use the following lemma.

Lemma 3.16. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ satisfying* (vi), (vii), (viii) *and $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ be increasing family of norms of V, defined by* (ix). *Then for $x_0 (\neq \theta) \in V$, $\alpha \in (0,1)$ and $s (> 0) \in \mathbb{R}$,*

$$\|x_0\|_\alpha^* = s \Leftrightarrow \nu(x_0, s) = 1-\alpha.$$

Proof. Let $\|x_0\|_\alpha^* = s$, then $s > 0$. Then there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$, $s_n > 0$ such that $\nu(x_0, s_n) \leq 1-\alpha$, for all $n \in \mathbb{N}$ and $s_n \to s$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \nu(x_0, s_n) \leq 1-\alpha \Rightarrow \nu(x_0, \lim_{n\to\infty} s_n) \leq 1-\alpha \quad \text{by (vii)}$$
$$\Rightarrow \nu(x_0, \|x_0\|_\alpha^*) \leq 1-\alpha, \quad \forall \alpha \in (0,1).$$

Let $\alpha \in (0,1)$, $x_0(\neq \theta) \in V$ and $s = \|x_0\|_\alpha^* = \bigwedge\{t : \nu(x_0,t) \leq 1-\alpha\}$. Since $\nu(x,\cdot)$ is continuous (by (vii)) we have

$$\nu(x_0,s) \leq 1-\alpha. \tag{1}$$

If possible, let $\nu(x_0,s) < 1-\alpha$, then by (vii) there exists $s' > s$ such that $\nu(x_0,s') < \nu(x_0,s) < 1-\alpha$, which is impossible since $s = \bigwedge\{t : \nu(x_0,t) \leq 1-\alpha\}$. Thus

$$\nu(x_0,s) \geq 1-\alpha. \tag{2}$$

From (1) and (2) it follows that $\nu(x_0,s) = 1-\alpha$. Thus

$$\|x_0\|_\alpha^* = s \Rightarrow \nu(x_0,s) = 1-\alpha. \tag{3}$$

Next, if $\nu(x_0,s) = 1-\alpha, \alpha \in (0,1)$ then by (vii)

$$\|x_0\|_\alpha^* = \bigwedge\{t : \nu(x_0,t) \leq 1-\alpha\} = s. \tag{4}$$

Hence, from (3) and (4), we have for $\alpha \in (0,1)$, $x(\neq \theta) \in V$ and for $s > 0$, $\|x_0\|_\alpha^* = s \Leftrightarrow \nu(x_0,s) = 1-\alpha$. ∎

Proof of the main theorem. Let $(x_0,t_0) \in V \times \mathbb{R}$. To prove this theorem, we consider the following cases:

Case 1: For any $x_0 \in V$ and $t \leq 0$, $\nu(x_0,t_0) = \nu'(x_0,t_0) = 1$.

Case 2: If $x_0 = \theta$, $t_0 > 0$. Then $\nu(x_0,t_0) = \nu'(x_0,t_0) = 0$.

Case 3: $x_0 \neq \theta$, $t_0(>0) \in \mathbb{R}$ such that $\nu(x_0,t_0) = 1$. By Lemma 3.16 we have, $\nu(x_0,\|x\|_\alpha^*) = 1-\alpha$ for all $\alpha \in (0,1)$. Since $\nu(x_0,t_0) = 1 > 1-\alpha$ it follows that $\nu(x_0,\|x\|_\alpha^*) \leq 1-\alpha < \nu(x_0,t_0)$ and since $\nu(x_0,\cdot)$ is strictly non increasing $t_0 < \|x_0\|_\alpha^*$, $\forall \alpha \in (0,1)$. So, $\nu'(x_0,t_0) = \bigwedge\{1-\alpha : \|x_0\|_\alpha^* \leq t_0\} = 1$. Thus, $\nu(x_0,t_0) = \nu'(x_0,t_0) = 1$.

Case 4: $x_0 \neq \theta$, $t_0(>0) \in \mathbb{R}$ such that $\nu(x_0,t_0) = 0$. From (ix) it follows that $\|x_0\|_\alpha^* < t_0$, $\forall \alpha \in (0,1)$. Therefore, $\|x_0\|_\alpha^* < t_0 \Rightarrow \nu'(x_0,t_0) = 0$, by (x). Thus, $\nu(x_0,t_0) = \nu'(x_0,t_0) = 0$.

Case 5: $x_0 \neq \theta$, $t_0(>0) \in \mathbb{R}$ such that $0 < \nu(x_0,t_0) < 1$. Let $\nu(x_0,t_0) = 1-\beta$, then from (ix) we have

$$\|x\|_\beta^* \leq t_0. \tag{5}$$

Using (5) from (x) we get, $\nu'(x_0,t_0) \leq 1-\beta$. Therefore,

$$\nu(x_0,t_0) \geq \nu'(x_0,t_0). \tag{6}$$

Now, from Lemma 3.16 we have $\nu(x_0,t_0) = 1-\beta \Leftrightarrow \|x\|_\beta^* = t_0$. Now, for $\beta < \alpha < 1$, let $\|x\|_\alpha^* = t'$. Then again by Lemma 3.16, we have $\nu(x_0,t') = 1-\alpha$. So, $\nu(x_0,t') = 1-\alpha < 1-\beta = \nu(x_0,t_0)$. Since $\nu(x_0,\cdot)$ is strictly monotonically decreasing and $\nu(x_0,t') < \nu(x_0,t_0)$ therefore $t' > t_0$. Then for

$\beta < \alpha < 1$, we have $\|x\|_\alpha^* = t' > t_0$. So,

$$\nu'(x_0, t_0) \geq 1 - \beta = \nu(x_0, t_0). \tag{7}$$

Thus, from (6) and (7) we have $\nu(x_0, t_0) = \nu'(x_0, t_0)$. Since $(x_0, t_0) \in V \times \mathbb{R}$ is arbitrary, $\nu'(x, t) = \nu(x, t_0)$ for all $(x, t) \in V \times \mathbb{R}$. ∎

Lemma 3.17. *In a fuzzy antinormed linear space (V, A^*) with respect to a t-conorm $\diamond$ satisfying* (vi), (vii) *and* (viii), *every sequence is convergent if and only if it is convergent with respect to its corresponding α-norms, $\alpha \in (0, 1)$.*

Proof. $\Rightarrow$ Part: Let (V, A^*) be a fuzzy antinormed linear space satisfying (vi) and (vii) and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in V such that $x_n \to x$

$$\lim_{n\to\infty} \nu(x_n - x, t) = 0, \quad \forall t > 0.$$

Choose $0 < \alpha < 1$. So, $\lim_{n\to\infty} \nu(x_n - x, t) = 0 < 1 - \alpha \Rightarrow$ there exists $n_o(t)$ such that

$$\nu(x_n - x, t) < 1 - \alpha, \quad \forall n \geq n_0(t, \alpha). \tag{8}$$

Now,

$$\|x_n - x\|_\alpha^* = \bigwedge\{t > 0 : \nu(x_n - x, t) \leq 1 - \alpha\}$$
$$\Rightarrow \|x_n - x\|_\alpha^* \leq t, \quad \forall n \geq n_0(t, \alpha).$$

Since $t > 0$ is arbitrary,

$$\|x_n - x\|_\alpha^* \to 0 \quad \text{as } n \to \infty, \forall \alpha \in (0, 1).$$

$\Leftarrow$ Part: Next we suppose that, $\|x_n - x\|_\alpha^* \to 0$ as $n \to \infty$, $\forall \alpha \in (0, 1)$.

Then for $\alpha \in (0, 1), \epsilon > 0$ there exists $n_0(\alpha, \epsilon)$ such that

$$\|x_n - x\|_\alpha^* < \epsilon, \quad \forall n \geq n_0(\alpha, \epsilon),\ \alpha \in (0, 1). \tag{9}$$

Now,

$$\nu(x_n - x, \epsilon) = \bigwedge\{1 - \alpha : \|x_n - x\|_\alpha^* \leq \epsilon\}$$
$$\Rightarrow \nu(x_n - x, \epsilon) \leq 1 - \alpha, \quad \forall n \geq n_0(\alpha, \epsilon), \alpha \in (0, 1)$$
$$\Rightarrow \lim_{n\to\infty} \nu(x_n - x, \epsilon) = 0.$$

Thus x_n converges to x. ∎

Corollary 3.18. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ satisfying* (vi), (vii) *and* (viii). *$W(\subseteq V)$ is closed in (V, A^*) if and only if it is closed with respect to its corresponding α-norms, $\alpha \in (0, 1)$.*

In the following lemma, a finite dimensional space is characterized by compact set in fuzzy environment and this will lead us to one of the fundamental differences between finite dimensional and infinite dimensional normed spaces with respect to fuzzy antinorms.

Lemma 3.19. (Riesz) *Let W be a closed and proper subspace of a fuzzy antinormed linear space (V, ν) with respect to a t-conorm $\diamond$, satisfying* (vi), (vii) *and* (viii). *Then for each $\epsilon > 0$ there exists $y \in V - W$ such that $\nu(y, 1) \leq 1 - \alpha$ and $\nu(y - w, 1 - \epsilon) \leq 1 - \alpha$ for all $\alpha \in (0, 1)$ and $w \in W$.*

Proof. Recall that, $\|x\|_\alpha^* = \bigwedge\{t : \nu(x, t) \leq 1 - \alpha\}$, $\alpha \in (0, 1)$ and $\{\|\cdot\|_\alpha^* : \alpha \in (0, 1)\}$ is an increasing family of α-norms on a linear space V. Now, by applying Riesz lemma for normed linear space, it follows that for any $\epsilon > 0$ there exists $y \in V - W$ such that

$$\|y\|_\alpha^* = 1, \tag{10}$$

$$\|y - w\|_\alpha^* > 1 - \epsilon, \quad \forall w \in W. \tag{11}$$

Now, from Theorem 3.15 for all $\alpha \in (0, 1)$ we have

$$\begin{aligned} \nu(y, t) &= \bigwedge\{1 - \alpha : \|y\|_\alpha^* \leq t\} \\ &\Rightarrow \nu(y, 1) = \bigwedge\{1 - \alpha : \|y\|_\alpha^* \leq 1\} \\ &\Rightarrow \nu(y, 1) \leq 1 - \alpha. \end{aligned}$$

Again,

$$\begin{aligned} \nu(y - w, t) &= \bigwedge\{1 - \alpha : \|y - w\|_\alpha^* \leq t\} \\ &\Rightarrow \nu(y - w, \epsilon) = \bigwedge\{1 - \alpha : \|y - w\|_\alpha^* \leq \epsilon\} \\ &\Rightarrow \nu(y - w, \epsilon) \leq 1 - \alpha. \end{aligned}$$

Hence the proof. ∎

Theorem 3.20. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying* (vi), (vii) *and* (viii). *If the set $\{x : \nu(x, 1) \leq 1 - \alpha\}$, $\alpha \in (0, 1)$ is compact, then V is a space of finite dimension.*

Proof. It can be easily verified that $\{x : \nu(x, 1) \leq 1 - \alpha\} = \{x : \|x\|_\alpha^* \leq 1\}$, $\alpha \in (0, 1)$. By applying Lemma 3.19, it can be proved that if for some $\alpha \in (0, 1)$ the set $\{x : \|x\|_\alpha^* \leq 1\}$ is compact, then V is of finite dimensional. Using Lemma 3.17, it follows that, for some $\alpha \in (0, 1)$, $\{x : \nu(x, 1) \leq 1 - \alpha\}$ is compact, then V is a space of finite dimension. ∎

4. Fuzzy α-anti-convergence

In this section, the relations of fuzzy α-anti-convergence, fuzzy α-anti-Cauchyness, fuzzy α-anti-compactness with respect to their corresponding increasing family norms are studied.

Theorem 4.1. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying* (vi), (vii), (viii) *and $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ be increasing family of norms of V, defined by* (ix). *Then, for any increasing (or, decreasing) sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ in $(0,1), \alpha_n \to \alpha$ in $(0,1)$ implies $\|x\|_{\alpha_n}^* \to \|x\|_\alpha^*$, $\forall x \in V$.*

Proof. For $x = \theta$, it is clear that α_n converges to $\alpha \Rightarrow \|x\|_{\alpha_n}^* \to \|x\|_\alpha^*$.

Suppose $x \neq \theta$. Then, from Lemma 3.16, for $x \neq \theta, \alpha \in (0,1)$ and $t' > 0$, we have

$$\|x\|_\alpha^* = t' \Leftrightarrow \nu(x_0, t') = 1 - \alpha.$$

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $(0,1)$, such that α_n converges to α in $(0,1)$. Let $\|x\|_{\alpha_n}^* = s_n$ and $\|x\|_\alpha^* = s$. Then,

$$\nu(x, s_n) = 1 - \alpha_n \quad \text{and} \quad \nu(x, s) = 1 - \alpha. \tag{12}$$

Since $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ is an increasing family of norms, $\{s_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers. Since $\{s_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers and is bounded above by s, $\{s_n\}_{n\in\mathbb{N}}$ is convergent. Thus,

$$\lim_{n\to\infty} \nu(x, s_n) = 1 - \lim_{n\to\infty} \alpha_n \Rightarrow \nu(x, \lim_{n\to\infty} s_n) = 1 - \alpha. \tag{13}$$

From (12) and (13) we have $\nu(x, \lim_{n\to\infty} s_n) = \nu(x, s)$. This implies $\lim_{n\to\infty} s_n = s$, by (vii). Therefore,

$$\lim_{n\to\infty} \|x\|_{\alpha_n}^* = \|x\|_\alpha^*.$$

Similarly, if $\{\alpha_n\}_{n\in\mathbb{N}}$ is a decreasing sequence in $(0,1)$ and α_n converges to α in $(0,1)$ then, it can be easily shown that $\|x\|_{\alpha_n}^* \to \|x\|_\alpha^*$, $\forall x \in V$. ∎

Definition 4.2. Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ and $\alpha \in (0,1)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in V is said to be fuzzy α-anti-convergent in (V, A^*), if there exists $x \in V$ such that for all $t > 0$

$$\lim_{n\to\infty} \nu(x_n - x, t) < 1 - \alpha.$$

Then x is called fuzzy α-antilimit of x_n.

Theorem 4.3. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ satisfying* (vi) *and* (viii). *Then fuzzy α-antilimit of a fuzzy α-anti-convergent sequence is unique.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a fuzzy α-anti-convergent sequence and suppose it converges to x and y in V. Then for all $t > 0$

$$\lim_{n\to\infty} \nu(x_n - x, t) < 1 - \alpha \quad \text{and} \quad \lim_{n\to\infty} \nu(x_n - y, t) < 1 - \alpha.$$

Now,

$$\begin{aligned}\nu(x-y,t) &= \nu(x-x_n+x_n-y,t), \quad \forall n\\ &= \nu(x_n-x,t)\diamond\nu(x_n-y,t), \quad \forall n.\end{aligned}$$

Taking limit we have

$$\begin{aligned}\nu(x-y,t) &= \lim_{n\to\infty}\nu(x_n-x,t)\diamond\lim_{n\to\infty}\nu(x_n-y,t)\\ &< (1-\alpha)\diamond(1-\alpha)=(1-\alpha), \qquad \text{(by (viii))}.\end{aligned}$$

That is, $\nu(x-y,t)<1$, $\forall t>0$. Therefore, $x-y=\theta$ by (vi) $\Rightarrow x=y$. ∎

Theorem 4.4. *Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying* (vi) *and* (viii). *If $\{x_n\}_{n\in\mathbb{N}}$ is a fuzzy α-anti-convergent sequence in (V,A^*) such that x_n converges to x, then $\|x_n-x\|_\alpha^*\to 0$ as $n\to\infty$.*

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ be a fuzzy α-anti-convergent sequence, suppose it converges to x, then for all $t>0$,

$$\begin{aligned}&\lim_{n\to\infty}\nu(x_n-x,t)<1-\alpha\\ &\qquad\Rightarrow \exists n_0(t)>0 \text{ such that } \nu(x_n-x,t)<1-\alpha,\ \forall n\geq n_0(t)\\ &\qquad\Rightarrow \exists n_0(t)>0 \text{ such that } \|x_n-x\|_\alpha^*\leq t, \quad \forall n\geq n_0(t).\end{aligned}$$

Since $t>0$ is arbitrary, $\|x_n-x\|_\alpha^*\to 0$ as $n\to\infty$. ∎

Definition 4.5. Let (V,A^*) be a fuzzy anti-normed linear space with respect to a t-conorm $\diamond$ and $\alpha\in(0,1)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in V is said to be fuzzy α-anti-Cauchy sequence if

$$\lim_{n\to\infty}\nu(x_n-x_{n+p},t)\leq 1-\alpha, \qquad \forall t>0,\ p=1,2,3,\ldots.$$

Theorem 4.6. *Let (V,A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying* (viii) *and $\alpha\in(0,1)$. Then every fuzzy α-anti-convergent sequence in (V,A^*) is a fuzzy α-anti-Cauchy sequence in (V,A^*).*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a fuzzy α-anti-convergent sequence and it converging to x. Then

$$\lim_{n\to\infty}\nu(x_n-x,t)<1-\alpha.$$

Now,

$$\begin{aligned}\nu(x_n-x_{n+p},t) &= \nu(x_n-x+x-x_{n+p},t), \quad \text{for } p=1,2,3,\ldots\\ &= \nu\left(x_n-x,\frac{t}{2}\right)\diamond\nu\left(x_{n+p}-x,\frac{t}{2}\right), \quad \text{for } p=1,2,3,\ldots\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \nu(x_n - x_{n+p}, t) \le \lim_{n\to\infty} \nu\left(x_n - x, \frac{t}{2}\right) \diamond \lim_{n\to\infty} \nu\left(x_{n+p} - x, \frac{t}{2}\right)$$
$$< (1-\alpha) \diamond (1-\alpha) = (1-\alpha), \quad \text{(by (viii))}.$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is a fuzzy α-anti-Cauchy sequence in (V, A^*). ■

Note 4.7. *Every constant sequence in a fuzzy antinormed linear space (V, A^*), with respect to a t-conorm $\diamond$, is a fuzzy α-anti-Cauchy sequence in (V, A^*), $\alpha \in (0,1)$.*

Proof. Obvious. ■

Theorem 4.8. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying* (vi) *and* (viii)*. Then every Cauchy sequence in $(V, \|\cdot\|_\alpha^*)$ is a fuzzy α-anti-Cauchy sequence in (V, A^*), where $\|\cdot\|_\alpha^*$ denotes the increasing family of norms on V defined by* (ix)*, $\alpha \in (0,1)$.*

Proof. Choose $\alpha_0 \in (0,1)$ arbitrary but fixed. Let $\{y_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in V with respect to $\|\cdot\|_{\alpha_0}^*$. Then

$$\lim_{n\to\infty} \|y_n - y_{n+p}\|_{\alpha_0}^* = 0.$$

Then for any given ϵ (>0) there exists a positive integer $n_0(\epsilon)$ such that $\|y_n - y_{n+p}\|_{\alpha_0}^* < \epsilon$, $\forall n \ge n_0(\epsilon)$ and $p = 1, 2, 3, \ldots$.

$\Rightarrow \bigwedge\{t > 0 : \nu(y_n - y_{n+p}, t) \le 1 - \alpha_0\} < \epsilon$,

$\Rightarrow$ there exists $t(n, p, \epsilon) < \epsilon$ such that

$$\nu(y_n - y_{n+p}, t(n,p,\epsilon)) \le 1 - \alpha_0, \quad \forall n \ge n_0(\epsilon) \quad \text{and} \quad p = 1, 2, 3, \ldots.$$

$\Rightarrow \nu(y_n - y_{n+p}, \epsilon) \le 1 - \alpha_0$.

Since ϵ (>0) is arbitrary, $\lim_{n\to\infty} \nu(y_n - y_{n+p}, t) \le 1 - \alpha_0$, $\forall t > 0 \Rightarrow \{y_n\}_{n\in\mathbb{N}}$ is fuzzy α_0-anti-Cauchy sequence in (V, A^*).

Since $\alpha_0 \in (0,1)$ is arbitrary, every Cauchy sequence in $(V, \|\cdot\|_\alpha^*)$ is fuzzy α-anti-Cauchy sequence in (V, A^*) for each $\alpha \in (0,1)$. ■

Definition 4.9. Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$ and $\alpha \in (0,1)$. It is said to be fuzzy α-anti-complete if every fuzzy α-anti-Cauchy sequence in V fuzzy α-anti-converges to a point of V.

Theorem 4.10. *Let (V, A^*) be a fuzzy antinormed linear space with respect to a t-conorm $\diamond$, satisfying* (vi) *and* (viii)*. If (V, A^*) is fuzzy α-anti-complete then V is complete with respect to $\|\cdot\|_\alpha^*$, $\alpha \in (0,1)$.*

Proof. Choose $\alpha_0 \in (0,1)$ arbitrary but fixed. Let $\{y_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in V with respect to $\|\cdot\|^*_{\alpha_0}$, then $\{y_n\}_{n\in\mathbb{N}}$ is fuzzy α_0-anti-Cauchy sequence in (V, A^*).

Since (V, A^*) is fuzzy α_0-anti-complete, there exists $y \in V$ such that $\lim_{n\to\infty} \nu(y_n - y, t) < 1 - \alpha_0$, $\forall t > 0 \Rightarrow \lim_{n\to\infty} \|y_n - y\|^*_{\alpha_0} = 0$, by Theorem 4.4.

$\Rightarrow y_n \to y$ with respect to $\|\cdot\|^*_{\alpha_0}$.

$\Rightarrow (V, \|\cdot\|^*_{\alpha_0})$ is complete.

Since α_0 is arbitrary, $(V, \|\cdot\|^*_{\alpha})$ is complete. ■

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