DEMONSTRATIO MATHEMATICA
Vol. XLV No 3 2012

10.1515/dema-2013-0404

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SPECTRUM AND FINE SPECTRUM OF GENERALIZED SECOND ORDER FORWARD DIFFERENCE OPERATOR Δ^2_{uvw} ON SEQUENCE SPACE l_1

Abstract. The purpose of this paper is to determine spectrum and fine spectrum of newly introduced operator Δ^2_{uvw} on sequence space l_1. The operator Δ^2_{uvw} on sequence space l_1 is defined by $\Delta^2_{uvw}x = (u_n x_n + v_{n-1}x_{n-1} + w_{n-2}x_{n-2})_{n=0}^{\infty}$ with $x_{-1}, x_{-2} = 0$, where $x = (x_n) \in l_1$, $u = (u_k)$ is either constant or strictly increasing sequence of positive real numbers with $U = \lim_{k\to\infty} u_k$, $v = (v_k)$ is a sequence of real numbers such that $v_k \neq 0$ for each $k \in \mathbb{N}_0$ with $V = \lim_{k\to\infty} v_k \neq 0$ and $w = (w_k)$ is a non-increasing sequence of positive real numbers such that $w_k \neq 0$ for each $k \in \mathbb{N}_0$ with $W = \lim_{k\to\infty} w_k \neq 0$. In this paper we have obtained the results on spectrum and point spectrum for the operator Δ^2_{uvw} over sequence space l_1. We have also obtained the results on continuous spectrum $\sigma_c(\Delta^2_{uvw}, l_1)$, residual spectrum $\sigma_r(\Delta^2_{uvw}, l_1)$ and fine spectrum of the operator Δ^2_{uvw} on sequence space l_1.

1. Introduction

The study of spectrum and fine spectrum for various operators are made by various authors. The fine spectra of the Cèsaro operator on the sequence space l_p has been studied by Gonzàlez [8], where $1 < p < \infty$. Also weighted mean operators on l_p have been investigated by Cartlidge [5]. The fine spectra of difference operator Δ over the sequence spaces l_p and bv_p is determined by Akhmedov and Basar [1, 2]. Also the fine spectra of difference operator Δ over the sequence spaces l_1 and bv is studied by Kayaduman and Furkan [9]; later the fine spectrum of the generalized difference operator $B(r, s)$ over sequence spaces l_1 and bv is established by Furkan et al. [6]. The fine spectrum of the same operator over the sequence space l_p and bv_p, $(1 < p < \infty)$ has been studied by Bilgic and Furkan [4]. The fine spectrum of the generalized

2000 *Mathematics Subject Classification*: 47A10, 47B39, 46A45.
Key words and phrases: spectrum of an operator, generalized second order forward difference operator, sequence space l_1.

difference operator $B(r,s,t)$ over sequence spaces l_1 and bv is established by [3], where r,s,t are taken as scalars.

The present work is a continuation of earlier papers which give the characterization of spectrum and fine spectrum of the generalized second order forward difference operator Δ^2_{uvw} for various real sequences $u=(u_k), v=(v_k)$ and $w=(w_k)$ under certain restrictions over the sequence space l_1. It is easy to verify that by choosing suitably u,v and w sequences, i.e., for suitable Δ^2_{uvw} one can get easily the operators such as $B(r,s,t)$. If $u=(r), v=(s)$ and $w=(t)$, then the operator Δ^2_{uvw} reduces to $B(r,s,t)$. Similarly, if $u=(1), v=(-2)$ and $w=(1)$ are constant sequences, then the operator Δ^2_{uvw} reduces to second order forward difference operator Δ^2. Thus, the results of this paper unify the corresponding results of many operator whose matrix representation is a triple-band matrix.

2. Preliminaries and notation

Let X and Y be the Banach spaces and $T: X \to Y$ be a bounded linear operator. We denote the range of T as $R(T)$, where $R(T)=\{y \in Y : y = Tx,\ x \in X\}$, and the set of all bounded linear operators on X into itself is denoted by $B(X)$. Further, the adjoint T^* of T is a bounded linear operator on the dual space X^* of X defined by

$$(T^*\phi)(x) = \phi(Tx) \text{ for all } \phi \in X^* \text{ and } x \in X.$$

Let $X \neq \{0\}$ be a complex normed space and $T: D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\alpha = (T - \alpha I)$, where α is a complex number and I is the identity operator on $D(T)$. The inverse of T_α (if exists) is denoted by T_α^{-1}, where $T_\alpha^{-1} = (T-\alpha I)^{-1}$ and is known as the resolvent operator of T. It is easy to verify that T_α^{-1} is linear, if T_α is linear. Since the spectral theory is concerned with many properties of T_α and T_α^{-1} which depend on α, so we are interested in the set of those α in the complex plane for which T_α^{-1} exists or T_α^{-1} is bounded or domain of T_α^{-1} is dense in X. For this, we need some definitions and known results given below which will be used in the sequel.

Definition 2.1. ([10], pp. 371) Let $X \neq \{0\}$ be a complex normed space and $T: D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value of T is a complex number α such that
(R1) T_α^{-1} exists,
(R2) T_α^{-1} is bounded,
(R3) T_α^{-1} is defined on a set which is dense in X.

Resolvent set $\rho(T,X)$ of T is the set of all regular values α of T. Its complement $\sigma(T,X) = \mathbb{C} \setminus \rho(T,X)$ in the complex plane $\mathbb{C}$ is called spectrum of T. The spectrum $\sigma(T,X)$ is further partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual spectrum as follows:

Point Spectrum $\sigma_p(T,X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} does not exists, i.e., condition (R1) fails. The element of $\sigma_p(T,X)$ is called eigenvalue of T.

Continuous spectrum $\sigma_c(T,X)$ is the set of all $\alpha \in \mathbb{C}$ such that conditions (R1) and (R3) hold but condition (R2) fails, i.e., T_α^{-1} exists, domain of T_α^{-1} is dense in X but T_α^{-1} is unbounded.

Residual Spectrum $\sigma_r(T,X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} exists but do not satisfy conditions (R3), i.e., domain of T_α^{-1} is not dense in X. The condition (R2) may or may not holds good.

Goldberg's classification of operator T_α ([7], pp. 58): Let X be a Banach space and $T_\alpha \in B(X)$, where α is a complex number. Again let $R(T_\alpha)$ and T_α^{-1} denote the range and inverse of the operator T_α, respectively. Then the following possibilities may occur;

(A) $R(T_\alpha) = X$,
(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(C) $\overline{R(T_\alpha)} \neq X$,

and

(1) T_α is injective and T_α^{-1} is continuous,
(2) T_α is injective and T_α^{-1} is discontinuous,
(3) T_α is not injective.

Remark 2.2. Combining (A), (B), (C) and (1), (2), (3); we get nine different cases. These are labelled by A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2 and C_3. The notation $\alpha \in A_2\sigma(T,X)$ means the operator $T_\alpha \in A_2$, i.e., $R(T_\alpha) = X$ and T_α is injective but T_α^{-1} is discontinuous. Similarly others.

Remark 2.3. If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then α belongs to the resolvent set $\rho(T,X)$ of T on X. The other classification gives rise to the fine spectrum of T.

Definition 2.4. ([11], pp. 220–221) Let λ, μ be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n,k \in \mathbb{N}_0 = \{0,1,2,\dots\}$. For every $x = (x_k) \in \lambda$ and every integer n, we write

$$A_n(x) = \sum_k a_{nk}x_k,$$

where the sum without limits is always taken from $k=0$ to $k=\infty$. The sequence $Ax=(A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A\in(\lambda,\mu)$ if and only if $Ax\in\mu$ whenever $x\in\lambda$.

Lemma 2.5. ([12], pp. 126) *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.*

Note: The operator norm of T is the supremum of the l_1 norms of the columns.

Lemma 2.6. ([7], pp. 59) *T has a dense range if and only if T^* is one to one, where T^* denotes the adjoint operator of the operator T.*

Lemma 2.7. ([7], pp. 60) *The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

3. Spectrum and point spectrum of the operator Δ^2_{uvw} on sequence space l_1

In this section we introduce the new second order forward difference operator Δ^2_{uvw} and compute spectrum, point spectrum of the operator Δ^2_{uvw} over space l_1.

Let $u=(u_k)$ is either constant or strictly increasing sequence of positive real numbers with $U=\lim_{k\to\infty}u_k$, and $v=(v_k)$ be a sequence of real numbers such that $v_k\neq 0$ for each $k\in\mathbb{N}_0$ with $V=\lim_{k\to\infty}v_k\neq 0$ and $w=(w_k)$ is a non-increasing sequence of positive real numbers such that $w_k\neq 0$ for each $k\in\mathbb{N}_0$ with $W=\lim_{k\to\infty}w_k\neq 0$. We define the operator Δ^2_{uvw} on sequence space l_1 as

$$\Delta^2_{uvw}x=(u_nx_n+v_{n-1}x_{n-1}+w_{n-2}x_{n-2})_{n=0}^{\infty} \text{ with } x_{-1},x_{-2}=0, \quad \text{where } x=(x_n)\in l_1.$$

It is easy to verify that the operator Δ^2_{uvw} can be represented by the matrix

$$\Delta^2_{uvw}=\begin{pmatrix} u_0 & 0 & 0 & 0 & \dots \\ v_0 & u_1 & 0 & 0 & \dots \\ w_0 & v_1 & u_2 & 0 & \dots \\ 0 & w_1 & v_2 & u_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Through out this work, we take $\sqrt{z}$, if z is a complex number, as the square root of z with non-negative real part. If $\text{Re}(\sqrt{z})=0$ then $\sqrt{z}$ represents the square root of z with $\text{Im}(\sqrt{z})\geq 0$.

Theorem 3.1. *The generalized second order forward difference operator $\Delta^2_{uvw} : l_1 \to l_1$ is a bounded linear operator and $\|\Delta^2_{uvw}\|_{(l_1,l_1)} = \sup_k(|u_k| + |v_k| + |w_k|)$.*

Proof. Proof is simple. So we omit. ■

Theorem 3.2. *Define the set $\mathcal{S}$ by $\mathcal{S} = \left\{\alpha \in \mathbb{C} : \frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1\right\}$ and assume $\sqrt{V^2} = -V$. Then spectrum of the operator Δ^2_{uvw} on sequence space l_1 is given by $\sigma(\Delta^2_{uvw}, l_1) = \mathcal{S}$.*

Proof. The proof of the theorem is divided into two parts.
In the first part, we show that $\sigma(\Delta^2_{uvw}, l_1) \subseteq \mathcal{S}$, which we prove by contradiction. That is assuming $\alpha \in \mathbb{C}$ with $\left|\frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}}\right| > 1$, we will show that $\alpha \in \rho(\Delta^2_{uvw},\ l_1)$.

In second part, we establish the reverse inequality, i.e., $\mathcal{S} \subseteq \rho(\Delta^2_{uvw},\ l_1)$.

Part I: Let $\alpha \in \mathbb{C}$ with $\left|\frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}}\right| > 1$. Clearly, $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$ as it does not satisfy the condition. Further, $(\Delta^2_{uvw} - \alpha I)$ reduces to a triangle and hence has an inverse. Thus, $(\Delta^2_{uvw} - \alpha I)^{-1} = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{u_0-\alpha} & 0 & 0 & 0\dots \\ \frac{-v_0}{(u_0-\alpha)(u_1-\alpha)} & \frac{1}{u_1-\alpha} & 0 & 0\dots \\ \frac{v_0v_1}{(u_0-\alpha)(u_1-\alpha)(u_2-\alpha)} - \frac{w_0}{(u_0-\alpha)(u_2-\alpha)} & \frac{-v_1}{(u_1-\alpha)(u_2-\alpha)} & \frac{1}{u_2-\alpha} & 0\dots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix},$$

and $b_{n,k} = \frac{-v_{n-1}b_{n-1,k} - w_{n-2}b_{n-2,k}}{(u_n-\alpha)}$.

By Lemma 2.5, the operator $(\Delta^2_{uvw} - \alpha I)^{-1} \in (l_1, l_1)$ if the supremum of l_1 norms of the columns of (b_{nk}) is bounded, i.e., $\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty$.

In order to show that $\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty$, first we prove that the series $\sum_{n=0}^{\infty} |b_{nk}|$ is convergent for each $k \in \mathbb{N}_0$.

For this, consider $S_k = \sum_{n=0}^{\infty} |b_{nk}| = |b_{k,k}| + |b_{k+1,k}| + |b_{k+2,k}| + \dots$. Clearly, series for k

$$\begin{aligned}
S_k = &\left|\frac{1}{(u_k-\alpha)}\right| + \left|\frac{(-1)v_k}{(u_k-\alpha)(u_{k+1}-\alpha)}\right| \\
&+ \left|\frac{(-1)^2 v_k v_{k+1}}{(u_k-\alpha)(u_{k+1}-\alpha)(u_{k+2}-\alpha)} + \frac{(-1)w_k}{(u_k-\alpha)(u_{k+2}-\alpha)}\right| \\
&+ \left|\frac{(-1)^3 v_k v_{k+1} v_{k+2}}{(u_k-\alpha)(u_{k+1}-\alpha)(u_{k+2}-\alpha)(u_{k+3}-\alpha)}\right. \\
&+ \frac{(-1)^2 w_k v_{k+2}}{(u_k-\alpha)(u_{k+2}-\alpha)(u_{k+3}-\alpha)} \\
&+ \left.\frac{(-1)^2 v_k w_{k+1}}{(u_k-\alpha)(u_{k+2}-\alpha)(u_{k+3}-\alpha)}\right| + \dots .
\end{aligned}$$

Now by letting

$$r_1 = \frac{-V + \sqrt{V^2 - 4W(U-\alpha)}}{2(U-\alpha)} \quad \text{and} \quad r_2 = \frac{-V - \sqrt{V^2 - 4W(U-\alpha)}}{2(U-\alpha)},$$

we can observe,

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{(u_k-\alpha)} &= \frac{1}{U-\alpha} = a_1 = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}[(r_1)-(r_2)] \\
\lim_{k\to\infty} \frac{-v_k}{(u_k-\alpha)(u_{k+1}-\alpha)} &= \frac{-V}{(U-\alpha)^2} = a_2 \\
&= \frac{1}{\sqrt{V^2-4W(U-\alpha)}}[(r_1)^2-(r_2)^2]
\end{aligned}$$

$$\begin{aligned}
&\lim_{k\to\infty} \frac{v_k v_{k+1}}{(u_k-\alpha)(u_{k+1}-\alpha)(u_{k+2}-\alpha)} - \frac{w_k}{(u_k-\alpha)(u_{k+2}-\alpha)} \\
&\quad = \frac{V^2}{(U-\alpha)^3} - \frac{W}{(U-\alpha)^2} = a_3 = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}[(r_1)^3-(r_2)^3].
\end{aligned}$$

Clearly, $a_n = \frac{1}{\sqrt{V^2-4W(U-\alpha)}}[(r_1)^n - (r_2)^n]$ for $n = 1, 2, \dots$.
Since α is not in $\mathcal{S}$, we have $|r_1| < 1$. Now we show that $|r_2| < 1$. Since $|r_1| < 1$, we have

$$\left|1 + \sqrt{1 - 4W(U-\alpha)/V^2}\right| < \left|\frac{2(U-\alpha)}{-V}\right|.$$

Since $|1-\sqrt{z}| \le |1+\sqrt{z}|$ for any $z \in \mathbb{C}$, we must have

$$\left|1 - \sqrt{1 - 4W(U-\alpha)/V^2}\right| < \left|\frac{2(U-\alpha)}{-V}\right|$$

which leads us to the fact that $|r_2| < 1$.

First consider the case $V^2 \neq 4W(U-\alpha)$, then $|r_2| < |r_1|$. Clearly, the series

$$S_k = \sum_{n=0}^{\infty} |b_{nk}| = |b_{k,k}| + |b_{k+1,k}| + |b_{k+2,k}| + \dots.$$

is convergent because,

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{b_{2k+1,k}}{b_{2k,k}}\right| = \lim_{k\to\infty}\frac{|a_{k+2}|}{|a_{k+1}|} &= \lim_{k\to\infty}\frac{|(r_1)^{k+2}-(r_2)^{k+2}|}{|(r_1)^{k+1}-(r_2)^{k+1}|}\\
&\leq \lim_{k\to\infty}\frac{|r_1|^{k+2}+|r_2|^{k+2}}{|r_1|^{k+1}-|r_2|^{k+1}}\\
&= \lim_{k\to\infty}\frac{|r_1|^{k+2}\left(1+\frac{|r_2|}{|r_1|}^{k+2}\right)}{|r_1|^{k+1}\left(1-\frac{|r_2|}{|r_1|}^{k+1}\right)}\\
&= |r_1| < 1.
\end{aligned}$$

So, S_k is convergent for each $k \in \mathbb{N}_0$. Now to show that $\sup_k S_k < \infty$.

Taking limit both sides of S_k and since $|r_1| < 1$ and $|r_2| < 1$, we get

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty} |a_n| \leq \frac{1}{|\sqrt{V^2-4W(U-\alpha)}|}\left(\sum_{n=1}^{\infty}|r_1|^n + \sum_{n=1}^{\infty}|r_2|^n\right) < \infty.$$

Since (S_k) is a sequence of positive real numbers and $\lim_{k\to\infty} S_k < \infty$, so $\sup_k S_k < \infty$.

Suppose $V^2 = 4W(U-\alpha)$ then

$$a_n = \left(\frac{2n}{-V}\right)\left[\frac{-V}{2(U-\alpha)}\right]^n,$$

so, the series S_k is convergent because,

$$\lim_{k\to\infty}\left|\frac{b_{2k+1,k}}{b_{2k,k}}\right| = \lim_{k\to\infty}\frac{|a_{k+2}|}{|a_{k+1}|} = \left|\frac{-V}{2(U-\alpha)}\right| < 1,$$

since $\alpha \notin \mathcal{S}$, implies $\left|\frac{-V}{2(U-\alpha)}\right| < 1$. So, S_k is convergent for each $k \in \mathbb{N}_0$. Now to show that $\sup_k S_k < \infty$. Then

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{2n}{-V}\right|\left|\frac{-V}{2(U-\alpha)}\right|^n < \infty,$$

by using ratio test and since $\left|\frac{-V}{2(U-\alpha)}\right| < 1$. Therefore $\alpha \notin \mathcal{S}$ implies

$\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty$. Thus,

$$(3.1)\quad (\Delta^2_{uvw} - \alpha I)^{-1} \in B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| > 1.$$

Next, we will show that domain of the operator $(\Delta^2_{uvw} - \alpha I)^{-1}$ is dense in l_1. This statement holds if and only if range of the operator $(\Delta^2_{uvw} - \alpha I)$ is dense in l_1. Since $(\Delta^2_{uvw} - \alpha I)^{-1} \in (l_1, l_1)$, which implies that range of the operator $(\Delta^2_{uvw} - \alpha I)$ is dense in l_1. Hence we have

$$(3.2)\quad \sigma(\Delta^2_{uvw}, l_1) \subseteq \{\alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| \leq 1\}.$$

Part (II): We now prove the reverse inequality, i.e.,

$$(3.3)\quad \{\alpha \in \mathbb{C} : \left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| \leq 1\} \subseteq \sigma(\Delta^2_{uvw}, l_1).$$

First we prove the inclusion 3.3 under the assumption that $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$, i.e., we want to show that one of the conditions of Definition 2.1 fails. Let $\alpha \in \mathcal{S}$. Clearly, $(\Delta^2_{uvw} - \alpha I)$ is a triangle and hence $(\Delta^2_{uvw} - \alpha I)^{-1}$ exists. So, condition (R1) is satisfied but condition (R2) fails as can be seen below:

Let us first consider $V^2 \neq 4W(U-\alpha)$ implies $|r_1| > |r_2|$.
Suppose $\alpha \in \mathbb{C}$ with $\left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| < 1$. Then $|r_1| > 1$, consequently,

$$\lim_{k\to\infty} \left| \frac{b_{2k+1,k}}{b_{2k,k}} \right| = \lim_{k\to\infty} \frac{|a_{k+2}|}{|a_{k+1}|} = \lim_{k\to\infty} \frac{|(r_1)^{k+2} - (r_2)^{k+2}|}{|(r_1)^{k+1} - (r_2)^{k+1}|}$$
$$= \lim_{k\to\infty} \frac{|r_1|^{k+2} |1 - (\frac{r_2}{r_1})^{k+2}|}{|r_1|^{k+1} |1 - (\frac{r_2}{r_1})^{k+1}|} = |r_1| > 1,$$

which gives S_k is divergent for each $k \in \mathbb{N}_0$. Hence

$$(3.4)\quad (\Delta^2_{uvw} - \alpha I)^{-1} \notin B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| < 1.$$

Next, we consider $\alpha \in \mathbb{C}$ with $\left| \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \right| = 1$ implies $|r_1| = 1$

and $1 > |r_2|$. So,

$$\lim_{k\to\infty}\left|\frac{b_{2k+1,k}}{b_{2k,k}}\right| = \lim_{k\to\infty}\frac{|a_{k+2}|}{|a_{k+1}|} = |r_1| = 1.$$

Thus, ratio test fails. Now we apply Raabe's test.

We have $\left(\left|\frac{b_{2k,k}}{b_{2k+1,k}}\right| - 1\right) = 0$ for all $k \geq 1$. Therefore,

$$\lim_{k\to\infty} k\left(\left|\frac{b_{2k,k}}{b_{2k+1,k}}\right| - 1\right) = 0 < 1,$$

so, the series S_k diverges for $k \in \mathbb{N}_0$. Hence

$$(3.5)\qquad (\Delta^2_{uvw} - \alpha I)^{-1} \notin B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left|\frac{2(U-\ \alpha)}{-V + \sqrt{V^2 - 4W(U-\alpha)}}\right| = 1.$$

Now consider the case $V^2 = 4W(U-\alpha)$, then $a_n = \left(\frac{2n}{-V}\right)\left[\frac{-V}{2(U-\alpha)}\right]^n$. Then,

$$\lim_{k\to\infty}\left|\frac{b_{2k+1,k}}{b_{2k,k}}\right| = \lim_{k\to\infty}\frac{|a_{k+2}|}{|a_{k+1}|} = \left|\frac{-V}{2(U-\alpha)}\right|,$$

when $\left|\frac{-V}{2(U-\alpha)}\right| > 1$, the series S_k divergent for each $k \in \mathbb{N}_0$. But when $\left|\frac{-V}{2(U-\alpha)}\right| = 1$, then ratio test fails. Then we apply Raabe's test.

We have $\left(\left|\frac{b_{2k,k}}{b_{2k+1,k}}\right| - 1\right) = 0$ for all $k \geq 1$. Therefore,

$$\lim_{k\to\infty} k\left(\left|\frac{b_{2k,k}}{b_{2k+1,k}}\right| - 1\right) = 0 < 1$$

so, the series S_k diverges for $k \in \mathbb{N}_0$. Hence (R2) fails.

Thus,

$$(3.6)\qquad (\Delta^2_{uvw} - \alpha I)^{-1} \notin B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left|\frac{2(U-\ \alpha)}{-V + \sqrt{V^2 - 4W(U-\alpha)}}\right| \leq 1.$$

Finally, we prove the inclusion 3.3 under the assumption that $\alpha = U$ and

$\alpha = u_k$ for each $k \in \mathbb{N}_0$. We have

$$(\Delta^2_{uvw} - \alpha I)x = \begin{pmatrix} (u_0-\alpha)x_0 \\ v_0x_0 + (u_1-\alpha)x_1 \\ w_0x_0 + v_1x_1 + (u_2-\alpha)x_2 \\ w_1x_1 + v_2x_2 + (u_3-\alpha)x_3 \\ \vdots \end{pmatrix}.$$

Case (i): If (u_k) is a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$, then

$$(\Delta^2_{uvw} - UI)x = 0 \quad \Rightarrow x_0 = 0, x_1 = 0, x_2 = 0, \ldots$$

This shows that the operator $(\Delta^2_{uvw} - UI)$ is one to one, but $R(\Delta^2_{uvw} - UI)$ is not dense in l_1. So, condition (R3) fails. Hence $U \in \sigma(\Delta^2_{uvw}, l_1)$.

Case (ii): If (u_k) is a strictly increasing sequence of positive real numbers, then for fixed k,

$$(\Delta^2_{uvw} - u_kI)x = 0$$

$\Rightarrow x_0 = 0, x_1 = 0, \ldots, x_{k-1} = 0,\ x_{k+1} = \left(\frac{-v_k}{u_{k+1}-u_k}\right)x_k.$

We take $x_k \neq 0$ gives non zero solution of $(\Delta^2_{uvw} - u_kI)$. This shows that $(\Delta^2_{uvw} - u_kI)$ is not injective. So, condition (R1) fails. Hence $u_k \in \sigma(\Delta^2_{uvw}, l_1)$ for all $k \in \mathbb{N}_0$. Hence we have

$$\{\alpha \in \mathbb{C} : \left|\frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}}\right| \leq 1\} \subseteq \sigma(\Delta^2_{uvw}, l_1). \tag{3.7}$$

From inclusions 3.2 and 3.7, we get

$$\sigma(\Delta^2_{uvw}, l_1) = \left\{\alpha \in \mathbb{C} : \left|\frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}}\right| \leq 1\right\}.$$

This completes the proof. ∎

Remark 3.3. If $\sqrt{V^2} = V$, then spectrum of the operator Δ^2_{uvw} on sequence space l_1 is given by $\sigma(\Delta^2_{uvw}, l_1) = \left\{\alpha \in \mathbb{C} : \frac{2|(U-\alpha)|}{|-V-\sqrt{V^2-4W(U-\alpha)}|} \leq 1\right\}$.

Theorem 3.4. *Point spectrum of the operator Δ^2_{uvw} on sequence space l_1 is*

$$\sigma_p(\Delta^2_{uvw}, l_1) = \begin{cases} \emptyset, & \textit{if } (u_k) \textit{ is a constant sequence,} \\ \{u_0, u_1, \ldots\}, & \textit{if } (u_k) \textit{ is a strictly increasing sequence.} \end{cases}$$

Proof. The proof of this theorem divided into two cases.

Case (i): Suppose (u_k) is a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$. Consider $\Delta^2_{uvw}x = \alpha x$ for $x \neq \mathbf{0} = (0, 0, \ldots,) \in l_1$, which gives

$$\left.\begin{aligned} u_0x_0 &= \alpha x_0 \\ v_0x_0 + u_1x_1 &= \alpha x_1 \\ w_0x_0 + v_1x_1 + u_2x_2 &= \alpha x_2 \\ w_1x_1 + v_2x_2 + u_3x_3 &= \alpha x_3 \\ &\vdots \\ w_{k-2}x_{k-2} + v_{k-1}x_{k-1} + u_kx_k &= \alpha x_k \\ &\vdots \end{aligned}\right\} \tag{3.8}$$

Let (x_t) is the first non-zero entry of the sequence $x = (x_n)$. So equation $w_{t-2}x_{t-2} + v_{t-1}x_{t-1} + Ux_t = \alpha x_t$, implies $\alpha = U$, and from the equation $w_{t-1}x_{t-1} + v_tx_t + Ux_{t+1} = \alpha x_{t+1}$, we get $x_t = 0$, which is a contradiction to our assumption, therefore

$$\sigma_p(\Delta^2_{uvw}, l_1) = \emptyset.$$

Case (ii): Suppose (u_k) is a strictly increasing sequence. Consider $\Delta^2_{uvw}x = \alpha x$ for $x \neq \mathbf{0} = (0, 0, \ldots,) \in l_1$, which gives system of equation (3.8).

If $\alpha = u_k$ for all $k \geq 1$, then $(\Delta^2_{uvw} - u_kI)x = 0$ gives $\Rightarrow x_0 = 0, x_1 = 0, \ldots, x_{k-1} = 0$ and

$$x_{k+1} = \left(\frac{-v_k}{u_{k+1} - u_k}\right) x_k.$$

If we take $x_k \neq 0$, then we get the non-zero solution of $(\Delta^2_{uvw} - \alpha I)x = 0$. Hence,

$$\sigma_p(\Delta^2_{uvw}, l_1) = \{u_0, u_1, u_2, \ldots\}.$$

This completes the proof. ∎

4. Point spectrum of the adjoint operator Δ^{2*}_{uvw} of Δ^2_{uvw} on dual sequence space l_1

Let $T : X \to X$ be a bounded linear operator having matrix representation A and the dual space of X denoted by X^*. Again, let T^* be its adjoint operator on X^*. Then the matrix representation of T^* is the transpose of the matrix A.

Theorem 4.1. *Point spectrum of the adjoint operator Δ^{2*}_{uvw} over l_1^* is $\sigma_p(\Delta^{2*}_{uvw}, l_1^*) = \mathcal{S}$.*

Proof. The set $\mathcal{S}$ is already proved as $\sigma(\Delta^2_{uvw}, l_1)$. To prove this theorem, we first need to show that $\sigma(\Delta^2_{uvw}, l_1) \subset \sigma_p(\Delta^{2*}_{uvw}, l_1^*)$.

Let $\alpha \in \mathcal{S} = \sigma(\Delta^2_{uvw}, l_1)$ then $\left|\frac{1}{r_1}\right| \leq 1$. Suppose $\Delta^{2*}_{uvw} f = \alpha f$ for $0 \neq f \in l_1^* \cong l_\infty$, where

$$\Delta^{2*}_{uvw} = \begin{pmatrix} u_0 & v_0 & w_0 & 0 & 0 & \dots \\ 0 & u_1 & v_1 & w_1 & 0 & \dots \\ 0 & 0 & u_2 & v_2 & w_2 & \dots \\ 0 & 0 & 0 & u_3 & v_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{pmatrix}.$$

Then we get the system of linear equations

$$\left.\begin{aligned} u_0 f_0 + v_0 f_1 + w_0 f_2 &= \alpha f_0 \\ u_1 f_1 + v_1 f_2 + w_1 f_3 &= \alpha f_1 \\ u_2 f_2 + v_2 f_3 + w_2 f_4 &= \alpha f_2 \\ &\vdots \end{aligned}\right\} \tag{4.9}$$

Solving the system of linear equations (4.9) in terms of f_0 and f_1, we obtain

$$f_k = (b_{k-1,0} f_1 - b_{k-1,1} f_0) \frac{(u_0 - \alpha)(u_1 - \alpha) \dots (u_{k-1} - \alpha)}{w_0 w_1 \dots w_{k-2}},$$

where $b_{k-1,0}$ and $b_{k-1,1}$ are defined as in last section.
For $\alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$ and $u = (u_k)$ is strictly increasing sequence and $w = (w_k)$ non-increasing positive real sequence, we get for $n = 0, 1, \dots, k-2$

$$\left|\frac{u_n - \alpha}{w_n}\right| = \frac{1}{|w_n|}((u_n - \alpha_1)^2 + (\alpha_2)^2)^{\frac{1}{2}} \leq \frac{1}{|W|}((U - \alpha_1)^2 + (\alpha_2)^2)^{\frac{1}{2}} = \left|\frac{U - \alpha}{W}\right|.$$

Then

$$|f_k| \leq |b_{k-1,0} f_1 - b_{k-1,1} f_0||U - \alpha|\left|\frac{U - \alpha}{W}\right|^{k-1}.$$

Taking limit on both sides and choosing $f_0 = 1$ and $f_1 = \frac{1}{r_1}$, we obtain

$$\begin{aligned} \lim_{k\to\infty} |f_k| &\leq \lim_{k\to\infty} \left\{ |a_k f_1 - a_{k-1} f_0| \left|\frac{U - \alpha}{W}\right|^{k-1} |U - \alpha| \right\} \\ &= \lim_{k\to\infty} \left\{ \frac{|(r_1^k - r_2^k) f_1 - (r_1^{k-1} - r_2^{k-1}) f_0|}{|\sqrt{V^2 - 4W(U - \alpha)}|} \left|\frac{U - \alpha}{W}\right|^{k-1} |U - \alpha| \right\} \end{aligned} \tag{4.10}$$

$$= \lim_{k\to\infty} \frac{|r_2|^{k-1}|r_1 - r_2|}{|r_1||\sqrt{V^2 - 4W(U-\alpha)}|} \left|\frac{U-\alpha}{W}\right|^{k-1} |U-\alpha|$$

$$= \lim_{k\to\infty} \frac{|r_2|^{k-1}}{|r_1|} \left|\frac{U-\alpha}{W}\right|^{k-1}.$$

We have the relation

$$\frac{U-\alpha}{W} = \frac{2(U-\alpha)}{-V+\sqrt{V^2-4W(U-\alpha)}} \times \frac{2(U-\alpha)}{-V-\sqrt{V^2-4W(U-\alpha)}} = \frac{1}{r_1 r_2}. \tag{4.11}$$

Then using (4.11) in (4.10), we obtain

$$\lim_{k\to\infty} |f_k| \le \lim_{k\to\infty} \frac{|r_2|^{k-1}}{|r_1|} \frac{1}{|r_1 r_2|^{k-1}} = \lim_{k\to\infty} \left|\frac{1}{r_1}\right|^k < \infty,$$

since $\alpha \in \mathcal{S}$. Hence,

$$\sigma(\Delta^2_{uvw}, l_1) \subset \sigma_p(\Delta^{2*}_{uvw}, l_1^*).$$

In the second part, we have to show that $\sigma_p(\Delta^{2*}_{uvw}, l_1^*) \subset \sigma(\Delta^2_{uvw}, l_1)$. It is clear that $\sigma_p(\Delta^{2*}_{uvw}, l_1^*) \subset \sigma(\Delta^{2*}_{uvw}, l_1^*)$ and from Corollary II.5.3 (i) [7], $\sigma(\Delta^2_{uvw}, l_1) = \sigma(\Delta^{2*}_{uvw}, l_1^*)$. So, combining we get

$$\sigma_p(\Delta^{2*}_{uvw}, l_1^*) \subset \sigma(\Delta^2_{uvw}, l_1).$$

This completes the proof. ■

5. Residual and continuous spectrum of the operator Δ^2_{uvw} on sequence space l_1

Theorem 5.1. *Residual spectrum $\sigma_r(\Delta^2_{uvw}, l_1)$ of the operator Δ^2_{uvw} over l_1 is*

$$\sigma_r(\Delta^2_{uvw}, l_1) = \begin{cases} \mathcal{S}, & \text{if } (u_k) \text{ is a constant sequence,} \\ \mathcal{S} \setminus \{u_0, u_1, \dots\}, & \text{if } (u_k) \text{ is a strictly increasing sequence.} \end{cases}$$

Proof. The proof of the theorem is divided into two cases.

Case (i): Let (u_k) be a constant sequence, say $u_k = U$ for each $k \in \mathbb{N}_0$. For $\alpha \in \mathbb{C}$ with $2|U - \alpha| \le |-V + \sqrt{V^2 - 4W(U-\alpha)}|$, the operator $(\Delta^2_{uvw} - \alpha I)$ is a triangle except $\alpha = U$ and consequently $(\Delta^2_{uv} - \alpha I)$ has an inverse. Further by Theorem 3.4, the operator $(\Delta^2_{uvw} - \alpha I)$ is one to one for $\alpha = U$ and hence has an inverse.

But by Theorem 4.1, the operator $(\Delta^2_{uvw} - \alpha I)^* = \Delta^{2*}_{uvw} - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ with $\frac{2|U-\alpha|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \le 1$. Hence by Lemma 2.6, range of the operator $(\Delta^2_{uvw} - \alpha I)$ is not dense in l_1. Thus,

$$\sigma_r(\Delta^2_{uvw}, c_0) = \left\{\alpha \in \mathbb{C} : \frac{2|U-\alpha|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1\right\}.$$

Case (ii): Let (u_k) be a strictly increasing sequence. For $\alpha \in \mathbb{C}$ such that $\frac{2|U-\alpha|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1$, the operator $(\Delta^2_{uvw} - \alpha I)$ is a triangle except $\alpha = u_k$ for all $k \in \mathbb{N}_0$ and consequently the operator $(\Delta^2_{uvw} - \alpha I)$ has an inverse. Further by Theorem 3.4, the operator $(\Delta^2_{uv} - \alpha I)$ is not one to one for $\alpha = u_k$ for all $k \in \mathbb{N}_0$. So, $(\Delta^2_{uvw} - \alpha I)^{-1}$ does not exist.

On the basis of argument as given in Case (i), it is easy to verify that the range of the operator $(\Delta^2_{uvw} - \alpha I)$ is not dense in l_1. Thus,

$$\sigma_r(\Delta^2_{uvw}, l_1) = \left\{\alpha \in \mathbb{C} : \frac{2|U-\alpha|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1\right\} \setminus \{u_0, u_1, u_2, \dots\}.$$

■

Theorem 5.2. *Continuous spectrum $\sigma_c(\Delta^2_{uvw}, l_1)$ of the operator Δ^2_{uvw} over l_1 is*

$$\sigma_c(\Delta^2_{uvw}, l_1) = \emptyset.$$

Proof. It is known that $\sigma_p(\Delta^2_{uvw}, l_1)$, $\sigma_r(\Delta^2_{uvw}, l_1)$, and $\sigma_c(\Delta^2_{uvw}, l_1)$ are pairwise disjoint and union of these is $\sigma(\Delta^2_{uvw}, l_1)$. But by Theorems 3.2, 3.4, and 5.1, we get

$$\sigma(\Delta^2_{uvw}, l_1) = \sigma_p(\Delta^2_{uvw}, l_1) \cup \sigma_r(\Delta^2_{uvw}, l_1),$$

for both constant and non-constant sequence. Therefore, $\sigma_c(\Delta^2_{uvw}, l_1) = \emptyset$. ■

6. Fine spectrum of the operator Δ^2_{uvw} on sequence space l_1

Theorem 6.1. *If α satisfies $\frac{2|U-\alpha|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} > 1$, then $(\Delta^2_{uvw} - \alpha I) \in A_1$.*

Proof. It is required to show that the operator $(\Delta^2_{uvw} - \alpha I)$ is bijective and has an inverse for $\alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} > 1$. Since $\alpha \neq U$, therefore the operator $(\Delta^2_{uvw} - \alpha I)$ is a triangle. Hence it has an inverse. The operator $(\Delta^2_{uvw} - \alpha I)^{-1}$ is continuous for $\alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} > 1$ by statement 3.1. Also the equation

$(\Delta^2_{uvw} - \alpha I)x = y$ gives $x = (\Delta^2_{uvw} - \alpha I)^{-1} y$,

i.e., $x_n = ((\Delta^2_{uvw} - \alpha I)^{-1} y_n), n \in \mathbb{N}_0$.

Thus, for every $y \in l_1$, we can find $x \in l_1$ such that

$$(\Delta^2_{uvw} - \alpha I)x = y, \text{ since } (\Delta^2_{uvw} - \alpha I)^{-1} \in (l_1, l_1).$$

This shows that the operator $(\Delta^2_{uvw} - \alpha I)$ is onto and hence $(\Delta^2_{uvw} - \alpha I) \in A_1$. This completes the proof. ■

Theorem 6.2. *Let (u_k) be a constant sequence, say $u_k = U$ and $\alpha = U$. Then $\alpha \in C_1\sigma(\Delta^2_{uvw}, l_1)$.*

Proof. We have

$$\sigma_p(\Delta^{2*}_{uvw}, l_1^*) = \left\{\alpha \in \mathbb{C} : \frac{2|(U - \alpha)|}{|-V + \sqrt{V^2 - 4W(U-\alpha)}|} \leq 1\right\}.$$

For $\alpha = U$, the operator $(\Delta^2_{uvw} - \alpha I)^*$ is not one to one. By Lemma 2.6, $R(\Delta^2_{uvw} - \alpha I)$ is not dense in l_1. Again by Theorem 3.4, since $\alpha = U$ does not belong to the set $\sigma_p(\Delta^2_{uvw}, l_1)$, therefore the operator $(\Delta^2_{uvw} - \alpha I)$ has an inverse.

Next, we show that the operator $(\Delta^2_{uvw} - \alpha I)^{-1}$ is continuous. By Lemma 2.7, it is enough to show that $(\Delta^2_{uvw} - \alpha I)^*$ is onto, i.e., for given $y = (y_n) \in l_1^*$, we have to find $x = (x_n) \in l_1^*$ such that $(\Delta^2_{uvw} - \alpha I)^* x = y$. Now, $(\Delta^2_{uvw} - UI)^* x = y$, i.e.,

$$\begin{aligned}
v_0x_1 + w_0x_2 &= y_0\\
v_1x_2 + w_1x_3 &= y_1\\
&\vdots\\
v_{i-1}x_i + w_{i-1}x_{i+1} &= y_{i-1}\\
&\vdots
\end{aligned}$$

Thus, $v_{n-1}x_n + w_{n-1}x_{n+1} = y_{n-1}$ for all $n \geq 1$, which implies $\sup_n |x_n| < \infty$, since $y \in l_\infty$. This shows that the operator $(\Delta^2_{uvw} - \alpha I)^*$ is onto and hence $\alpha \in C_1\sigma(\Delta^2_{uvw}, l_1)$. ■

Theorem 6.3. *Let (u_k) be a constant sequence, say $u_k = U$ and $\alpha \neq U$, $\alpha \in \sigma_r(\Delta^2_{uvw}, l_1)$. Then $\alpha \in C_2\sigma(\Delta^2_{uvw}, l_1)$.*

Proof. Since $\alpha \neq U$, therefore the operator $(\Delta^2_{uvw} - \alpha I)$ is a triangle. Hence it has an inverse. For $U \neq \alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1$, the operator $(\Delta^2_{uvw} - \alpha I)^{-1}$ is discontinuous by statement (3.4) and (3.5). Thus, $(\Delta^2_{uvw} - \alpha I)$ is injective and $(\Delta^2_{uvw} - \alpha I)^{-1}$ is discontinuous.

Again by Theorem 4.1, the operator $(\Delta^2_{uvw} - \alpha I)^*$ is not one to one for $\alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1$. But Lemma 2.6 yields the fact that range of the $(\Delta^2_{uvw} - \alpha I)$ is not dense in l_1 and $\alpha \in C_2\sigma(\Delta^2_{uvw}, l_1)$. ■

Theorem 6.4. *Let (u_k) be constant sequence and if $|w_k| < |v_k|$ for each k, then $U \in C_1\sigma(\Delta^2_{uvw}, l_1)$. If $|w_k| \geq |v_k|$ for each k, then $U \in C_2\sigma(\Delta^2_{uvw}, l_1)$.*

Proof. If $\alpha = U$, then by Theorem 5.1 $(\Delta^2_{uvw} - \alpha I)$ is in state C_1 or C_2. A left inverse of Δ^2_{uvw} is

$$B = (\Delta^2_{uvw} - UI)^{-1} = \begin{pmatrix} 0 & \left(\frac{1}{v_0}\right) & 0 & 0 & \dots \\ 0 & \left(\frac{-w_0}{v_0 v_1}\right) & \left(\frac{1}{v_1}\right) & 0 & \dots \\ 0 & \left(\frac{w_0 w_1}{v_0 v_1 v_2}\right) & \left(\frac{-w_1}{v_1 v_2}\right) & \left(\frac{1}{v_2}\right) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix B is in $B(l_1)$ for $|w_k| < |v_k|$ and is not in $B(l_1)$ for $|w_k| \geq |v_k|$, $k \in \mathbb{N}_0$. That is $(\Delta^2_{uvw} - UI)$ has a continuous inverse for $|w_k| < |v_k|$, $k \in \mathbb{N}_0$ but it does not have a continuous inverse for $|w_k| \geq |v_k|$, $k \in \mathbb{N}_0$. Therefore, $U \in C_1\sigma(\Delta^2_{uvw}, l_1)$ for $|w_k| < |v_k|$, $k \in \mathbb{N}_0$, and $U \in C_2\sigma(\Delta^2_{uvw}, l_1)$ for $|w_k| \geq |v_k|$, $k \in \mathbb{N}_0$. This completes the proof. ∎

Theorem 6.5. *Let (u_k) be non-constant sequence and $\alpha \in \sigma_r(\Delta^2_{uvw}, l_1)$. Then $\alpha \in C_2\sigma(\Delta^2_{uvw}, l_1)$.*

Proof. We have,

$$\sigma_r(\Delta^2_{uvw}, l_1) = \{\alpha \in \mathbb{C} : \frac{2|(U - \alpha)|}{|-V + \sqrt{V^2 - 4W(U-\alpha)}|} \leq 1\} \backslash \{u_0, u_1, u_2, \dots\}.$$

Since $\alpha \neq u_k$ for all k, therefore the operator $(\Delta^2_{uvw} - \alpha I)$ is a triangle. Hence it has an inverse. For $u_k \neq \alpha \in \mathbb{C}$ with $\frac{2|(U-\alpha)|}{|-V+\sqrt{V^2-4W(U-\alpha)}|} \leq 1$, the inverse of the operator $(\Delta^2_{uvw} - \alpha I)$ is discontinuous by statement (3.4) and (3.5). Thus $(\Delta^2_{uvw} - \alpha I)$ injective and $(\Delta^2_{uvw} - \alpha I)^{-1}$ is discontinuous.

On the basis of argument as given in Theorem 6.3, it is easy to verify that the range of the operator $(\Delta^2_{uvw} - \alpha I)$ is not dense in l_1 and hence $\alpha \in C_2\sigma(\Delta^2_{uvw}, l_1)$. ∎

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Received June 26, 2010.