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Marek Galewski, Joanna Smejda

SOME REMARKS ON NONLINEAR DISCRETE BOUNDARY VALUE PROBLEMS

Abstract. Using critical point theory and some monotonicity results we consider the existence and multiplicity of solutions to nonlinear discrete boundary value problems represented as a nonlinear system $Au = \lambda f(u)$ with parameter $\lambda > 0$ and with matrix A being not necessarily positive definite. We provide applications for discrete version of the Emden–Fowler equation.

1. Introduction

Discrete boundary value problems have attracted a lot of attention recently. The boundary value problems connected with discrete equations can be tackled with almost similar methods as their continuous counterparts. The variational techniques applied for discrete problems include, among others, the mountain pass methodology, the linking theorem, the Morse theory, the three critical point, compare with [2], [3], [10], [15], [16]. Moreover, the fixed point approach is in fact much more prolific in the case of discrete problem, see for example [1], [7].

In this submission we are going to employ variational techniques and monotonicity methods for a nonlinear system

$$Au = \lambda f(u),\ u \in \mathbb{R}^n \tag{1.1}$$

with a parameter $\lambda > 0$ and a continuous nonlinear term $f : \mathbb{R}^n \to \mathbb{R}^n$ in case when the necessarily symmetric $n \times n$ matrix A need not be positive definite. We recall that for a given $\lambda > 0$, a column of vector $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ is a solution corresponding to λ, if substitution of λ and u into (1.1) renders it an identity.

System (1.1) can be treated as a representation of some discrete boundary value problem which in turn arises as discretization of some continuous

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models. Let us consider this in a more detailed manner by means of some example. As it is well known difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance. One of such models is the Emden–Fowler equation

$$\frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right)+t^{\delta}u^{\gamma}=0$$

which originated in the gaseous dynamics in astrophysics and further was used in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemically reacting systems, see [14]. The discrete version of the generalized Emden–Fowler equation, namely of the following second order ODE

$$(p(t)y^{'})^{'}+q(t)y=f(t,y)$$

received some considerable interest lately mainly by the use of critical point theory, see for example [6], [8], [9] with a wide display of variational techniques.

The discretization of the generalized Emden–Fowler type boundary value problem can be put as follows

$$\Delta\left(p\left(k-1\right)\Delta x\left(k-1\right)\right)+q\left(k\right)x\left(k\right)+\lambda f\left(k,x\left(k\right)\right)=0 \tag{1.2}$$

with boundary conditions

$$x\left(0\right)=x\left(T\right),\ p\left(0\right)\Delta x\left(0\right)=p\left(T\right)\Delta x\left(T\right) \tag{1.3}$$

and where $f\in C\left(\left[1,T\right]\times\mathbb{R},\mathbb{R}\right)$, $p\in C\left(\left[0,T+1\right],\mathbb{R}\right)$, $q\in C\left(\left[1,T\right],\mathbb{R}\right)$, $p\left(T\right)\neq 0$. The realization of the form of (1.1) requires the following matrices, see [8]

$$M=\begin{bmatrix} p(0)+p(1) & -p(1) & 0 & \dots & 0 & -p(0)\\ -p(1) & p(1)+p(2) & -p(2) & \dots & 0 & 0\\ 0 & -p(2) & p(2)+p(3) & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & p(T-2)+p(T-1) & -p(T-1)\\ -p(0) & 0 & 0 & \dots & -p(T-1) & p(T-1)+p(0) \end{bmatrix}$$

and

$$Q=\begin{bmatrix} -q(1) & 0 & 0 & \dots & 0 & 0\\ 0 & -q(2) & 0 & \dots & 0 & 0\\ 0 & 0 & -q(3) & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -q(T-1) & 0\\ 0 & 0 & 0 & \dots & 0 & -q(T) \end{bmatrix}.$$

Setting $A = M + Q$ and using the assumption that $p(T) \neq 0$ we see that problem (1.2)–(1.3) has a form of a nonlinear system (1.1). Indeed, in this case there is a $1-1$ correspondence between solutions to (1.1) with $n = T$ and solutions to (1.2)–(1.3). There is a clear relation between the forward difference operator Δ and operator A since matrix M provided above multiplied by n-dimensional vector x provides in fact another n-dimensional vector with coordinates $\Delta(p(k-1)\Delta x(k-1))$ for $k = 1, 2, ..., n$.

2. The assumptions

Before providing our main results, we give the assumptions.

M1 $f_k : \mathbb{R} \to \mathbb{R}$, for $k = 1, ..., n$, are continuous functions;

M2 $A = (a_{ij})_{(n\times n)}$ is an $n\times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} < 0 < \lambda_k \leq \cdots \leq \lambda_n$;

M3 $A = (a_{ij})_{(n\times n)}$ is an $n\times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1} = 0 < \lambda_k \leq \cdots \leq \lambda_n$;

M4 A is positive definite, with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ ordered as $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$;

M5 A is negative definite, with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 0$.

For eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ we consider the corresponding orthonormal eigenvectors $\xi_1, \xi_2, \dots, \xi_n$.

Concerning the nonlinear term, we assume that for any $k \in \{1, \dots, n\}$

F1 there exist $a_k > 0$ such that $\liminf_{|z|\to\infty} \frac{F_k(z)}{z^2} > a_k$, for $z \in \mathbb{R}$;

F2 there exist $b_k > 0$ such that $\limsup_{|z|\to\infty} \frac{F_k(z)}{z^2} < b_k$, for $z \in \mathbb{R}$;

F3 there exist $c_k > 0$ such that $\liminf_{|z|\to 0} \frac{f_k(z)}{z} > c_k$, for $z \in \mathbb{R}$;

F4 there exist $d_k > 0$ such that $F_k(z) \leq d_k z^2$, for $z \in \mathbb{R}$;

F5 there exist $\mu \in (\frac{1}{2}, 1)$ and $M > 0$ such that $F_k(z) \geq \mu z f_k(z)$, for $|z| \geq M$;

F6 there exist $\mu \in (0, \frac{1}{2})$ and $M > 0$ such that $F_k(z) \geq \mu z f_k(z)$, for $|z| \geq M$;

F7 there exist $\delta, A_k, B_k \in (0, +\infty)$ and an integer $i \geq 1$, which satisfy $A_k > B_k > \frac{\lambda_i}{\lambda_{i+1}} A_k > 0$, such that $B_k z^2 \leq F_k(z) \leq A_k z^2$, for all $|z| \leq \delta$;

F8 there exist $p_k > 0$ such that $\big(f_k(z_1) - f_k(z_2)\big)(z_1 - z_2) \geq p_k |z_1 - z_2|^2$ for $z_1, z_2 \in \mathbb{R}$;

F9 there exist $q_k > 0$ such that $f_k(z) z \geq q_k |z|^2$ for $z \in \mathbb{R}$.

Let us denote

$$a = \min_{1\leq k\leq n} \{a_k\}, \qquad b = \max_{1\leq k\leq n} \{b_k\},$$

$$c = \min_{1 \le k \le n} \{c_k\}, \qquad d = \max_{1 \le k \le n} \{d_k\},$$
$$p = \min_{1 \le k \le n} \{p_k\}, \qquad q = \min_{1 \le k \le n} \{q_k\},$$
$$A = \max_{1 \le k \le n} \{A_k\}, \qquad B = \min_{1 \le k \le n} \{B_k\}.$$

We will assume that inequalities $2b < c$, $2d < c$ and $a > B$ are satisfied without further recalling them.

Concerning for example the discrete Emden–Fowler equation, the assumptions M2–M5 have impact on the coefficients of matrix A and so on the sign of the values of function p. Let us consider for example $T = 4$. Then

$$M = \begin{bmatrix} p(0) + p(1) & -p(1) & 0 & -p(0) \\ -p(1) & p(1) + p(2) & -p(2) & 0 \\ 0 & -p(2) & p(2) + p(3) & -p(3) \\ -p(0) & 0 & -p(3) & p(3) + p(0) \end{bmatrix}.$$

Let $q(k) = 0$ for $k = 1, 2, 3, 4$. When $p(0) = -1$, $p(1) = 1$, $p(2) = -1$, $p(3) = -1$ we have condition M2 satisfied with eigenvalues $\sqrt{2} - 2, -\sqrt{2} - 2, \sqrt{2}, -\sqrt{2}$, while when $p(0) = p(3) = 0$, $p(1) = -1$, $p(2) = 2$ we have M3 with eigenvalues $\sqrt{7} + 1, 1 - \sqrt{7}, 0$.

3. Existence by critical point theory

We will use the following functional $J : \mathbb{R}^n \to \mathbb{R}$

$$J(u) = \lambda \sum_{k=1}^{n} \int_0^{u_k} f_k(s)ds - \frac{1}{2}(u, Au) \tag{3.1}$$

whose critical points are in fact solution to (1.1) and in turn solutions to (1.1) are precisely critical points to (3.1). The solutions which we investigate are the strong ones. This is in contrast to the infinite dimensional case, when the critical point theory allows usually for obtaining weak solutions. We shall start with the results describing some properties of the action functional.

Lemma 3.1. *We assume* (M1) *and also one of* (M2), (M3), (M4), (M5). *Let f satisfy* (F1). *Then for all $\lambda \in (\max\{0, \frac{\lambda_n}{2a}\}, \infty)$ functional J is coercive on $\mathbb{R}^n$ and it therefore satisfies the* (PS) *condition.*

Proof. We see that in either case $(u, Au) \le \lambda_n \|u\|^2$, where λ_n is the largest eigenvalue of A. Hence, the arguments used are the same as in [5], Lemma 3.1. ∎

Lemma 3.2. *We assume* (M1). *Let f satisfy either* (F5), (M4) *or* (F6) *and one of* (M2), (M3), (M5). *For all $\lambda > 0$, J satisfies the* (PS) *condition.*

Proof. Case (F5), (M4) follows by [5]. Concerning the other one we reason as in [5], proof of Lemma 3.2, for some fixed constant $C^* > 0$ that

$$-\mu\|J^{'}(u^m)\|\|u^m\| - \lambda nc_1 + \left(\mu - \frac{1}{2}\right)\lambda_1\|u^m\|^2 \leq C^*.$$

Fix $\varepsilon > 0$. Since $J^{'}(u^m) \to 0$, there exists $N_0 \in \mathbb{N}$ such that $\|J^{'}(u^m)\| \leq \varepsilon$ for all $m \geq N_0$. Therefore

$$\left(\mu - \frac{1}{2}\right)\lambda_1\|u^m\|^2 - \mu\varepsilon\|u^m\| - \lambda nc_1 - C^* \leq 0$$

which indicates that the sequence $\{u^m\} \subset \mathbb{R}^n$ is bounded and thus it has a convergent subsequence. ∎

Theorem 3.1. *We assume* (M1) *and also one of* (M2), (M3), (M4), (M5). *If f satisfies* (F1), *then for* $\lambda \in (\max\{0, \frac{\lambda_n}{2a}\}, \infty)$, (1.1) *has at least one solution in* $\mathbb{R}^n$.

Proof. By Lemma 3.1 J is coercive and obviously it is continuous. Since also J is Gâteaux differentiable, it has a critical point on $\mathbb{R}^n$ which solves (1.1). ∎

Theorem 3.2. *We assume either* (M1), (M4), (F2), (F3), (F5) *or* (M1), (F2), (F3), (F6) *with one of assumptions* (M2), (M3). *Then for* $\lambda \in (\frac{\lambda_n}{c}, \frac{\lambda_n}{2b})$, *problem* (1.1) *has at least one nonzero solution in* $\mathbb{R}^n$.

Proof. The lines of the proof follow [5] with the use of Mountain pass lemma—see proof of Theorem 3.5 therein. ∎

Next result requires the version of the Linking Theorem, which we provide below.

Theorem 3.3. (Linking theorem) [11] *Let $E = V \oplus X$ be a real Banach space with* $\dim V < \infty$. *Let $\rho > r > 0$ and let $z \in X$ be such that $\|z\| = r$. Define*

$$M = \{u = y + \lambda z : \|u\| \leq \rho,\ \lambda \geq 0,\ y \in V\},$$

$$M_0 = \{u = y + \lambda z : y \in V,\ \|u\| = \rho,\ \lambda \geq 0,\ \textit{or}\ \|u\| \leq \rho,\ \lambda = 0\},$$

$$N = \{u \in X : \|u\| = r\}.$$

Let $J \in C^1(E, \mathbb{R})$ be such that

$$b = \inf_{u \in N} J(u) > a = \max_{u \in M_0} J(u).$$

If J satisfies the $(\mathrm{PS})_\mathrm{c}$ *condition with*

$$c = \inf_{\gamma \in \Gamma} \max_{u \in M} J(\gamma(u)), \quad \Gamma = \{\gamma \in C(\mathrm{M}, \mathrm{E}) : \gamma|_{M_0} = id\},$$

then c is a critical value of J.

Theorem 3.4. *We assume* (M1) *and* (M2). *Assume that conditions* (F3), (F4) *and* (F6) *hold. Let* $\lambda_i < 0$, $\lambda_{i+1} > 0$. *Then for* $\lambda \in (0, \frac{\lambda_{i+1}}{2d}]$, *equation* (1.1) *has at least one nonzero solution in* $\mathbb{R}^n$.

Proof. We use Theorem 3.3. We will verify that functional J, defined by (3.1), satisfies the conditions of Theorem 3.3. By Lemma 3.2, it follows that J satisfies $(\mathrm{PS})_c$ condition. We fix $\lambda \in (0, \frac{\lambda_{i+1}}{2d}]$. There exists $\delta > 0$ such that

$$F_k(z) = \int_0^z f_k(s)ds \geq \frac{1}{2}c_k z^2,\ |z| \leq \delta,\ k \in \{1, \ldots, n\}.$$

Let $V_1 = \operatorname{span}\{\xi_1, \xi_2, \ldots, \xi_i\}$ $(i < n)$, $V_2 = V_1^{\perp}$. Hence on V_1 we see that $\frac{1}{2}(u, Au) < 0$ and so

$$\lambda \sum_{k=1}^{n} \int_0^{u_k} f_k(s)ds \geq \lambda \sum_{k=1}^{n} \frac{1}{2} c_k u_k^2.$$

Thus $\inf_{u \in V_1 \cap \partial B_\delta} J(u) > 0$.

By condition (F4), we get on V_2

$$J(u) \leq \lambda \sum_{k=1}^{n} d_k u_k^2 - \frac{1}{2}\lambda_{i+1}\|u\|^2 \leq (\lambda d - \frac{1}{2}\lambda_{i+1})\|u\|^2 \leq 0.$$

Defining $\beta = \delta\xi_i$ we obtain that

$$J(u) \to -\infty \text{ as } u \in V_2 \oplus \mathbb{R}^1 \text{ and } \|u\| \to \infty,$$

and that there exists $r > \delta$ such that

$$b = \inf_{u \in V_1 \cap \partial B_\delta} J(u) \geq \alpha > 0 = \max_{u \in M_0} J(u),$$

where

$$M_0 = \{u = y + \gamma\beta : y \in V_2, \|u\| = r \text{ and } \gamma \geq 0,\ \text{or } \|u\| \leq r,\ \text{and } \gamma = 0\}.$$

Thus, according to Theorem 3.3, J has a critical value $c^* > 0$, i.e. there exists $v^* \in \mathbb{R}^n$ such that $J(v^*) = c^*$ and $J'(v^*) = \lambda f(v^*) - Av^* = 0$. It is obvious that $v^* \neq 0$ since $J(0) = 0$. ∎

Theorem 3.5. *We assume* (M1) *and* (M2). *Let* f *satisfies* (F1) *and* (F7). *Let* $\lambda_i < 0$, $\lambda_{i+1} > 0$ *then for* $\lambda \in (0, \frac{\lambda_{i+1}}{2A}] \subset (\frac{\lambda_n}{2a}, \infty)$ (1.1) *has at least two nontrivial solutions in* $\mathbb{R}^n$.

Remark 3.1. We also get Theorem 3.6 and Theorem 3.7 from [5] when we replace assumption (M2) imposed in [5] with less demanding assumption (M5).

4. Existence by monotonicity theory

Theorem 4.6. *We assume* (M1) *and also one of* (M2)*,* (M3)*,* (M4)*,* (M5)*. Let f satisfies* (F8)*. Then for $\lambda \in (\max\{0, \frac{\lambda_n}{p}\}, \infty)$,* (1.1) *has a unique solution in $\mathbb{R}^n$.*

Proof. Recalling that $(u, Au) \leq \lambda_n \|u\|^2$, where λ_n is the largest eigenvalue of A, we define an operator $K : \mathbb{R}^n \to \mathbb{R}^n$ by $Ku = \lambda f(u) - Au$ and we obtain that

$$(Ku - Kv, u - v) \geq (\lambda p - \lambda_n)\|u - v\|^2.$$

Now the Strongly Monotone Principle, see [4], applies and $Ku = 0$ has a unique solution $u^* \in \mathbb{R}^n$ which in turn is a solution to (1.1). ∎

The following principle can be applied in investigating the existence of discrete equations.

Theorem 4.7. [4] *Let X be a finite dimensional real Banach space and let $T : X \to X^*$ be a continuous operator. Suppose that there exists a function $r : (0, \infty) \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} r\left(t\right) = +\infty \tag{4.2}$$

and that inequality

$$\left(T\left(x\right), x\right) \geq r\left(\|x\|_X\right) \|x\|_X$$

holds for all $x \in X$. Then for each $f \in X^*$ equation $T\left(x\right) = f$ has at least one solution.

Theorem 4.8. *We assume* (M1) *and also one of* (M2)*,* (M3)*,* (M4)*,* (M5)*. Suppose further that* (F9) *holds, then for all $\lambda \in (\max\{0, \frac{\lambda_n}{q}\}, \infty)$ equation* (1.1) *has at least one solution.*

Proof. We define K as above and we take $r\left(t\right) = \left(\lambda q - \lambda_n\right) t$. Relation (4.2) holds when $\lambda > \max\{0, \frac{\lambda_n}{q}\}$. ∎

5. Conclusions

We note that our results constitute somehow dual approach towards problem (1.1) when compared with [12]. That duality is based on the following observation in the context of variational methods applied to discrete problems: together with a functional

$$J(u) = \frac{1}{2}(u, Au) - \lambda \sum_{k=1}^{n} \int_0^{u_k} f_k(s)ds \tag{5.3}$$

which was investigated in [12], we can investigate the somehow dual functional (3.1). Since in [13] the authors already investigated the case when the eigenvalues are not necessarily positive with the use of functional (5.3), we complete these results in the sense that we can consider different kinds of nonlinear terms. Moreover, we use monotonicity methods - not applied in [13]. The reason that monotonicity methods were not applied in [13] was as follows. When monotonicity results are applied we are looking for fixed points of certain operators. In the context of discrete problems we can consider both, the operator $K(u) = Au - \lambda f(u)$, and operator $K_1(u) = \lambda f(u) - Au$ which involve different monotonicity and definiteness assumptions on f and A. While K requires that A must be positive definite, K_1 does not. Thus our observation concerning both functionals has also impact over the application of the monotonicity theory. Note that matrices arising in discrete problems are usually symmetric, while they need not be positive definite, compare with [8] where the already mentioned discretization of the Emden–Fowler boundary value problem is considered. Therefore, it seems that our results are more flexible.

Remark 5.2. Concerning the discrete Emden–Fowler equation we may provide existence results putting $A = M + Q$, and using the following action functional

$$J(x) = \lambda \sum_{k=1}^{T} F(k, x(k), u(k)) - \frac{1}{2} \langle (M+Q)x, x \rangle .$$

In fact Theorems 4.6, 4.8, 3.1, 3.2, 3.4 and 3.5 can be repeated for matrix A given as above.

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Marek Galewski
INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
Wólczańska 215
90-924 ŁÓDŹ, POLAND
E-mail: marek.galewski@p.lodz.pl

Joanna Smejda
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ŁÓDŹ UNIVERSITY
Banacha 22
90–238 ŁÓDŹ, POLAND
E-mail: jsmejda@math.uni.lodz.pl

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