

Mieczysław Cichoń, Ireneusz Kubiaczyk, Aneta Sikorska-Nowak
and Ahmet Yantir

EXISTENCE OF SOLUTIONS OF THE DYNAMIC CAUCHY PROBLEM IN BANACH SPACES

Abstract. In this paper we obtain the existence of solutions and Carathéodory type solutions of the dynamic Cauchy problem in Banach spaces for functions defined on time scales

$$\begin{aligned}x^\Delta(t) &= f(t, x(t)), \\ x(0) &= x_0, \quad t \in I_a,\end{aligned}$$

where f is continuous or f satisfies Carathéodory conditions and some conditions expressed in terms of measures of noncompactness. The Mönch fixed point theorem is used to prove the main result, which extends these obtained for real valued functions.

1. Introduction

Time scale (or a measure chain) was introduced by Hilger in his Ph. D. thesis in 1988 in order to unify discrete and continuous analysis [27]. Since Hilger formed the definitions of derivative and integral on time scale several authors have extended on various aspects of the theory [1, 14, 15]. Time scale has been shown that it is applicable to any field which can be described with various kinds of discrete or continuous models. This notion is also useful when we consider q -difference calculus (quantum calculus cf. [29]). In recent years there have been many research activities on dynamic equations, in order to unify the results from difference equations and differential equations [1, 2, 3, 7, 14, 15, 24]. However, the dynamic equations in Banach spaces are quite new research area. A first result on this topic is due to Hilger [28, Theorem 5.7], but this idea is not sufficiently developed. Nevertheless, from the results of Aulbach and Hilger, it is obvious, that the constructed calculus on time scales was designed for Banach-space valued dynamic equations right

2000 *Mathematics Subject Classification*: 34D09; 34D99.

Key words and phrases: Cauchy dynamic problem, Banach space, measure of non-compactness, Carathéodory solutions, time scales.

from the beginning. We follow this idea. Let us note, that the results of this paper are particularly useful for the discretization of continuous problems in Banach spaces. It is possible via difference equations with uniform or non-uniform steps which lead to different time scales: \mathbb{Z} or very useful for the discretization time scales of the form $\mathbb{T} = \{s\} \cup \{t_k : k \in \mathbb{Z}\}$, where $t_k \in \mathbb{R}$, $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$, where (t_k) is a convergent sequence (to some s). Due to a growing number of papers dealing with q -difference (quantum) models cf. [9, 12, 29], it seems to be very interesting to cover such a kind of discretization in Banach spaces too. This is also the reason to take into account the compactness assumption in the right-hand side of our problem (f is a countable α -contraction), which is not necessary for the case $\mathbb{T} = \mathbb{Z}$ (cf. Remark 3.4), but useful in the case of convergence for discretized models.

The Cauchy problems for differential equation $x'(t) = f(t, x(t))$ and difference equation $\Delta x(t) = f(t, x(t))$ in Banach spaces have been widely studied by many authors [4, 5, 17, 18, 21, 22, 23, 26, 34] in the literature. Under various kind of conditions, the existence and the properties of the solutions are presented, but there is no new results which can be treated as a unification and extension for different kind of time scales (q -difference equations in Banach spaces, for instance).

In this paper we focus on the existence of solution and Carathéodory solution of the dynamic Cauchy problem (in a Banach space):

$$(1.1) \quad \begin{aligned} x^\Delta(t) &= f(t, x(t)), \\ x(0) &= x_0, \quad t \in I_a. \end{aligned}$$

The existence results for differential problem are well-known (see [23] for a survey). But we were motivated, among others, by some interesting papers on difference equations in Banach spaces [4, 21, 26]. Their authors present results which guarantee the existence of one or more solutions to particular cases of (1.1). The theorems of this paper extends also those results. Let us stress, that for some papers the authors considered simultaneously continuous and discrete problems ([32] or [6], for instance). We skip such an approach, by considering dynamic equations, which include, as particular cases, the results mentioned above.

A Mönch fixed point theorem [33] and the techniques of the theory of the measure of noncompactness [13] are used to prove the existence of solution of the problem (1.1) (advantages of using of this theorem are clarified in Remark 3.5). By imposing some conditions expressed in terms of the measure of noncompactness on f , we define an operator over the Banach space (the space of rd-continuous functions from a time scale interval to a Banach space), whose fixed points are solutions of (1.1).

Finally, we need to remark, that the paper form a basis for future work with different kind of problems in Banach spaces (cf. [20], for instance), including differential inclusions and the problems for equations of the type $x^\Delta(t) = f(t, x(t), x(\sigma(t)))$. Although the presence of both x and $x(\sigma(t))$ in this equation may appear to be somewhat strange, their appearance can naturally occur, as the example from economics presented in [36] clearly illustrates. Let us note, that the results for ∇ derivatives are quite the same, so we omit this case.

2. Preliminaries

To understand so-called dynamic equation and follow this paper easily we present some preliminary definitions and notations of time scale which are very common in the literature (see [1, 14, 15, 27, 28, 30] and references therein).

A time scale \mathbb{T} is a nonempty closed subset of real numbers \mathbb{R} , with the subspace topology inherited from the standard topology of \mathbb{R} . Thus \mathbb{R} , \mathbb{Z} , \mathbb{N} , $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$, where $q > 1$ and the Cantor set are the examples of time scales while \mathbb{Q} , \mathbb{C} and $(0, 1)$ are not time scales.

Throughout this paper, by an interval $[a, b]$ we mean a time scale interval i.e. $\{t \in \mathbb{T} : a \leq t \leq b\}$. In particular, $I_a = \{t \in \mathbb{T} : 0 \leq t \leq a\}$. By a subinterval I_b of I_a we mean the time scale subinterval.

Let $(E, \|\cdot\|)$ be a Banach space and let $B_r = \{x \in E : \|x\| \leq r\}$, $r > 0$ be a ball in E .

DEFINITION 2.1. The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively. We put $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if \mathbb{T} has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if \mathbb{T} has a minimum m).

The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

We next define the so-called delta derivative and delta integral for Banach valued functions similar as Δ -derivative and Δ -integral on time scales [14, 15].

DEFINITION 2.2. Fix $t \in \mathbb{T}$. Let $u : \mathbb{T} \rightarrow E$. Then we define $u^\Delta(t)$ by

$$u^\Delta(t) = \lim_{s \rightarrow t} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}.$$

The Δ -derivative turns out that

- (i) $u^\Delta = u'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,
- (ii) $u^\Delta = \Delta u$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$ and
- (iii) $u^\Delta = D_q u$ is the q -differential operator i.e. $D_q u(x) = \frac{u(qx) - u(x)}{(q-1)x}$ if $\mathbb{T} = q^{\mathbb{Z}} = \{q^t : t \in \mathbb{Z}\}$.

Hence time scale allows us the unification of differential, difference and q -difference equations as particular cases (but our results hold also for more exotic time scales which appear in mathematical biology or economics cf. [14, 15, 36], for instance).

DEFINITION 2.3. We say that $u : \mathbb{T} \rightarrow E$ is right dense continuous (rd-continuous) if u is continuous at every right dense point $t \in I_a$ and $\lim_{s \rightarrow t^-} u(s)$ exists and is finite at every left dense point $t \in I_a$.

DEFINITION 2.4. If $U^\Delta(t) = u(t)$ then we define the integral by

$$\int_a^t u(\tau) \Delta \tau = U(t) - U(a).$$

REMARK 2.5. [14] (existence of antiderivatives) Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$ then U defined by

$$U(t) := \int_{t_0}^t u(\tau) \Delta \tau, \quad t \in \mathbb{T}$$

is an antiderivative of u .

We consider a Cauchy problem on a time scale \mathbb{T} . Without losing the generality we will assume, that $0 \in \mathbb{T}$

$$\begin{aligned} x^\Delta(t) &= f(t, x(t)), \\ x(0) &= x_0, \quad t \in I_a, \end{aligned}$$

where f is a function with values in a Banach space E .

Let us denote by $C_{rd}(I_a, E)$ the space of all rd-continuous functions mapping a time scale interval I_a into a Banach space E . Note that this space is a Banach space with the supremum norm.

Our fundamental tool is the (Kuratowski) measure of noncompactness [13]. For any bounded subset A of E we denote by $\alpha(A)$ the Kuratowski measure of noncompactness of A , i.e. the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter smaller than ε . For the convenience we present the properties of the measure of noncompactness α .

- (i) If $A \subset B$ then $\alpha(A) \leq \alpha(B)$,
- (ii) $\alpha(A) = \alpha(\bar{A})$, where \bar{A} denotes the closure of A ,

- (iii) $\alpha(A) = 0$ if and only if A is relatively compact,
- (iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,
- (v) $\alpha(\lambda A) = |\lambda|\alpha(A)$ ($\lambda \in \mathbb{R}$),
- (vi) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$,
- (vii) $\alpha(\text{conv} A) = \alpha(A)$, where $\text{conv}(A)$ denotes the convex extension of A .

For the proof of our main result we need the following lemmas.

LEMMA 2.6. [31] *If $\|E_1\| = \sup\{\|x\| : x \in E_1\} < 1$ then*

$$\alpha(E_1 + E_2) \leq \alpha(E_2) + \|E_1\|\alpha(K(E_2, 1))$$

where $K(E_2, 1) = \{x \in E_2 : d(x, 1) \leq 1\}$.

Next we prove Ambrosetti's Lemma for the functions defined on a time scale (cf. [8]).

LEMMA 2.7. *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_a$, $H(t) = \{h(t) \in E : h \in H\}$. Then*

$$\alpha_C(H) = \sup_{t \in I_a} \alpha(H(t)) = \alpha(H(I_a)),$$

where $\alpha_C(H)$ denotes the measure of noncompactness in $C(I_a, E)$ and the function $t \mapsto \alpha(H(t))$ is continuous.

Proof. Since $H(t) \subset H(I_a)$ by the first property of measure of noncompactness $\alpha(H(t)) \leq \alpha(H(I_a))$ and consequently

$$(2.1) \quad \sup_{t \in I_a} \alpha(H(t)) \leq \alpha(H(I_a)).$$

Let $u \in H$ be arbitrary. In order to prove the converse inequality suppose that for $\epsilon > 0$, $0 < t_0 < t_1 < \dots < t_n = a$ is a partition of the real interval $[0, a]$ such that $\|u(t) - u(s)\| \leq \epsilon$ for all $t, s \in [t_i, t_{i+1}] \cap \mathbb{T}$, $i = 0, 1, \dots, n-1$, $u \in H$. As

$$u(t) = u_i + u(t) - u_i \in u_i + \epsilon K(0, 1),$$

where

$$u_i = \begin{cases} u(t_i), & t_i \in \mathbb{T}; \\ u(\tilde{\sigma}(t_i)), & t_i \notin \mathbb{T}, \end{cases}$$

we have

$$u(t) \in \bigcup_{i=1}^n H(t_i) + \epsilon K(0, 1) \quad \text{and} \quad H(I_a) \subset \bigcup_{i=1}^n H(t_i) + \epsilon K(0, 1).$$

Here $\tilde{\sigma}$ is the deformation of forward jump operator σ and defined for $t \in \mathbb{R}$ by $\tilde{\sigma}(t) = \inf\{s \in \mathbb{R} : s > t\}$. By the properties of measure of noncompactness and Lemma 2.6, we obtain

$$\begin{aligned}
\alpha(H(I_a)) &\leq \alpha\left(\bigcup_{i=1}^n H(t_i)\right) + \|\epsilon K(0, 1)\| \cdot \alpha\left(K\left(\bigcup_{i=1}^n H(t_i), 1\right)\right) \\
&\leq \sup_{t_i \in I_a} \alpha(H(t_i)) + \epsilon \alpha(K(H(I_a), 1)) \\
&\leq \sup_{t \in I_a} \alpha(H(t)) + \epsilon \alpha(K(H(I_a), 1)).
\end{aligned}$$

Since the above inequality holds for any $\epsilon > 0$ we have

$$(2.2) \quad \alpha(H(I_a)) \leq \sup_{t \in I_a} \alpha(H(t)).$$

Hence from (2.1) and (2.2) we can conclude that $\alpha(H(I_a)) = \sup_{t \in I_a} \alpha(H(t))$. ■

In the proof of the main theorem we will apply the following results.

THEOREM 2.8. [33] [Mönch fixed point theorem] *Let D be a closed convex subset of E , and let F be a continuous map from D into itself. If for some $x \in D$ the implication*

$$\bar{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively compact}$$

holds for every countable subset V of D , then F has a fixed point.

THEOREM 2.9. [Mean Value Theorem] *If the function $f : I_a \rightarrow E$ is Δ -integrable then*

$$\int_{I_b} f(t) \Delta t \in \mu_{\Delta}(I_b) \cdot \overline{\text{conv}} f(I_b),$$

where I_b is an arbitrary subinterval of I_a and $\mu_{\Delta}(I_b)$ is the Lebesgue Δ -measure of I_b .

Proof. Let I_E be the set of left scattered points of I . By the properties of Δ -integral on Banach spaces (see Theorem 5.2 of [16]), we obtain

$$\int_I f(t) \Delta t = \int_{I \setminus I_E} f(t) dt + \sum_{t_i \in I_E} f(t_i) \mu(t_i).$$

By making use of Theorem 5.2 and Lemma 3.1 from [16] we get

$$\begin{aligned}
\int_I f(t) \Delta t &\in \text{mes}(I \setminus I_E) \cdot \overline{\text{conv}} f(I) + \sum_{t_i \in I_E} f(t_i) \mu(t_i) \\
&\subset \text{mes}(I) \cdot \overline{\text{conv}} f(I) + f(I) \cdot \sum_{t_i \in I_E} \mu(t_i) \\
&\subset (\text{mes}(I) + \sum_{t_i \in I_E} \mu(t_i)) \cdot \overline{\text{conv}} f(I) \\
&= \mu_{\Delta}(I) \cdot \overline{\text{conv}} f(I).
\end{aligned}$$

Here $\text{mes}(I)$ denotes the measure of the interval I . ■

Using the measure-theoretical approach of Hilger [28], the definition of the Lebesgue integral is obvious. For the properties of Lebesgue Δ -measure and Lebesgue Δ -integral we refer the readers to the papers of Aulbach and Neidhart [11] and Guseinov [25].

3. Main results

When we investigate the existence of solutions of (1.1) with non-continuous right-hand side it is natural to consider the so-called Carathéodory-type solutions. We recall that a function $f : I_a \times E \rightarrow E$ is a Carathéodory function if for each $x \in E$, $f(t, x)$ is measurable in $t \in I_a$ and for almost all $t \in I_a$, $f(t, x)$ is continuous with respect to x . Moreover, assume that f is bounded.

By a Carathéodory-type solution of (1.1) we understand a function $x \in C(I_a, E)$ such that $x(0) = x_0$ and $x(\cdot)$ satisfies (1.1) μ_Δ a.e. in I_a . For such solutions, problem (1.1) is equivalent to the integral problem

$$(3.1) \quad x(t) = x_0 + \int_0^t f(s, x(s)) \Delta s, \quad \mu_\Delta \text{ a.e. on } I_a.$$

To verify the equivalence, let a continuous function $x : I_a \rightarrow E$ be a solution of the problem (1.1). Since $\int_A f(s, x(s)) \Delta s = 0$ (see [25]) where $A = \{t \in I_a : x^\Delta \neq f(t, x(t))\}$ ($\mu_\Delta(A) = 0$), by definition 2.4 we have

$$\begin{aligned} \int_0^t f(s, x(s)) \Delta s &= \int_A f(s, x(s)) \Delta s + \int_{I_t \setminus A} x^\Delta(s) \Delta s \\ &= \int_0^t x^\Delta(s) \Delta s = x(t) - x(0) = x(t) - x_0, \quad \mu_\Delta \text{ a.e. on } I_a, \end{aligned}$$

which means that a function x is the Carathéodory solution of the problem (3.1). Here the integral is taken in the sense of Lebesgue.

Now, let the function x be the solution of the problem (3.1). Then by definition 2.2 we obtain that if F is a function such that $F^\Delta(t, x(t)) = f(t, x(t))$, μ_Δ a.e. then

$$\begin{aligned} x^\Delta(t) &= \left(x_0 + \int_0^t f(s, x(s)) \Delta s \right)^\Delta = \left(\int_0^t f(s, x(s)) \Delta s \right)^\Delta \\ &= (F(t, x(t)) - F(0, x(0)))^\Delta = F^\Delta(t, x(t)) = f(t, x(t)), \quad \mu_\Delta \text{ a.e.} \end{aligned}$$

Conclude, the function x is the Carathéodory solution of the problem (1.1).

For $x \in C_{rd}(I_a, E)$, we define the norm of x by: $\|x\| = \sup\{\|x(t)\|, t \in I_a\}$.

Let $f : I_a \times B_r \rightarrow E$ satisfy the sublinearity condition

$$(3.2) \quad \|f(t, x(t))\| \leq m_1(t) + m_2(t) \|x(t)\|,$$

for each $(t, x) \in I_a \times B_r$, where $m_1(t)$ and $m_2(t)$ are bounded functions taken from $C_{rd}(I_a, E)$. Also let $M(t)$ be the solution of the initial value problem

$$\begin{aligned} M^\Delta(t) &= m_1(t) + m_2(t)M(t), \\ M(0) &= x_0. \end{aligned}$$

By Theorem 5.3 in [2],

$$(3.3) \quad M(t) = x_0 \cdot e_{m_2}(t, 0) + \int_0^t e_{m_2}(t, \sigma(s)) m_1(s) \Delta s, \quad \text{for } t \in J.$$

Assume that m_1 and m_2 are such functions that,

$$M_0 = \sup_{t \in I_a} M(t) < \infty.$$

Define the ball \tilde{B}_{M_0} as follows:

$$\begin{aligned} \tilde{B}_{M_0} &= \{x \in C_{rd}(I_a, E) : \|x(t)\| \leq M_0, \\ &\quad \|x(t) - x(s)\| \leq (\|m_1\| + \|m_2\| \cdot M_0) \cdot |t - s|, (t, x) \in I_a \times B_r\}. \end{aligned}$$

Note that \tilde{B}_{M_0} is nonempty, closed, bounded, convex and equicontinuous. For convenience, we denote by $M_1 = \|m_1\| + \|m_2\| \cdot M_0$ and $\tilde{B}_{M_0} = \tilde{B}$.

Let the operator $G : C(I_a, E) \rightarrow C(I_a, E)$ be defined by

$$(3.4) \quad G(x)(t) = x_0 + \int_0^t f(s, x(s)) \Delta s.$$

The fixed point of G is the solution of (1.1). As claimed in the Introduction, for simplicity of the proof, we will assume in the main theorem, that f is a countable α -contraction. This condition is more general than the condition of α -contractness (cf. [35]), but can be also generalized to another (countable) conditions (see Remark 3.5).

THEOREM 3.1. *Suppose that a function $f : I_a \times B_r \rightarrow E$ is a Carathéodory function and there exists a constant $c > 0$ satisfying*

$$(3.5) \quad \alpha(f(I_b, X)) \leq c \cdot \alpha(X), 0 \leq ca < 1,$$

for each $X \subset B_r$ and for each subinterval I_b of I_a . Assume that there exist bounded functions $m_1, m_2 \in C_{rd}(I_a, E)$ such that

$$M_0 = \sup_{t \in I_a} \left\{ x_0 \cdot e_{m_2}(t, 0) + \int_0^t e_{m_2}(t, \sigma(s)) m_1(s) \Delta s \right\} < \infty$$

and $\|f(t, x(t))\| \leq m_1(t) + m_2(t) \|x(t)\|$ for $(t, x) \in I_a \times B_r$. Then there exists at least one Carathéodory solution of the problem (1.1) on I_a .

Proof. We verify that the conditions of Mönch fixed point theorem (Theorem 2.8) are fulfilled. First we show that the operator G maps \tilde{B} into \tilde{B} . By

the equivalence of (1.1) and (3.1), the sublinearity condition (3.2) and the operator (3.4) we conclude that

$$\|G(x)(t)\| \leq x_0 \cdot e_{m_2}(t, 0) + \int_0^t e_{m_2}(t, \sigma(s)) m_1(s) \Delta s = M(t) \leq M_0.$$

(See the comparison theorem (Theorem 5.4 from [2]).)

Consequently we show that the set $G(\tilde{B})$ is equicontinuous.

$$\begin{aligned} \|G(x)(t) - G(x)(\tau)\| &= \left\| \int_{\tau}^t f(s, x(s)) \Delta s \right\| \leq \int_{\tau}^t \|f(s, x(s))\| \Delta s \\ &\leq \int_{\tau}^t (m_1(s) + m_2(s) \cdot \|x\|) \Delta s \leq M_1 |t - \tau|, \end{aligned}$$

for every $x \in \tilde{B}$. Hence $G(x) \in \tilde{B}$.

Now we show continuity of G . Let $x_n \rightarrow x$ in \tilde{B} . Then

$$\begin{aligned} \|G(x_n)(t) - G(x)(t)\| &= \left\| \int_0^t [f(s, x_n(s)) - f(s, x(s))] \Delta s \right\| \\ &\leq \int_0^t \|f(s, x_n(s)) - f(s, x(s))\| \Delta s, \end{aligned}$$

(see [14, 15, 25] for the above inequality). Since f is a Carathéodory function by using Lebesgue's dominated convergence theorem for the delta integral (see [15]), we deduce that $\|G(x_n) - G(x)\| \rightarrow 0$.

Hence the operator G is well defined, continuous and maps \tilde{B} into \tilde{B} .

Observe that the fixed point of the operator G is the solution of the problem (1.1). Now we prove that a fixed point of the operator G exists using fixed point Theorem 2.8.

Let V be a countable subset of \tilde{B} and satisfy the condition

$$\bar{V} = \overline{\text{conv}}(\{x\} \cup G(V)) \quad \text{for some } x \in \tilde{B}.$$

Let, for $t \in I_a$, $V(t) = \{v(t) \in E, v \in V\}$. Since V is an equicontinuous, by Lemma 2.7 a map $t \rightarrow v(t) = \alpha(V(t))$ is continuous on I_a . Let

$$G(V)(t) = \left\{ x_0 + \int_0^t f(s, x(s)) \Delta s, \quad x \in V, t \in I_a \right\} = x_0 + \int_0^t f(s, V(s)) \Delta s.$$

By the definition of the operator G , properties of the Kuratowski measure of noncompactness, Lebesgue Δ -measure, Theorem 2.9 and the assumption

(3.5) we have

$$\begin{aligned}\alpha(G(V)(t)) &= \alpha\left(x_0 + \int_0^t f(s, V(s))\Delta s\right) \leq \alpha\left(\int_0^t f(s, V(s))\Delta s\right) \\ &\leq \alpha(\mu_\Delta(I_t) \cdot \overline{\text{conv}}f([0, t] \times V([0, t]))) \\ &\leq \alpha(t \cdot \overline{\text{conv}}f([0, t] \times V([0, t]))) \\ &\leq t \cdot \alpha(f([0, t] \times V([0, t]))) \leq b \cdot c \cdot \alpha(V(I_b))\end{aligned}$$

for each $t \in I_b \subset I_a$. Hence $\alpha(G(V)(t)) \leq b \cdot c \cdot \alpha(V(I_b))$ for each $t \in I_b$. Since $V = \overline{\text{conv}}(\{x\} \cup G(V))$ then

$$\alpha(V(t)) = \alpha(\overline{\text{conv}}(G(V)(t) \cup \{x\})) \leq \alpha(G(V)(t)) \leq b \cdot c \cdot \alpha(V(I_b)).$$

By Lemma 2.7 we have

$$\alpha(V(I_b)) \leq b \cdot c \cdot \alpha(V(I_b)) \leq a \cdot c \cdot \alpha(V(I_b)).$$

Because $0 \leq ca < 1$ so $v(t) = \alpha(V(I_b)) = 0$ and $\alpha(V(t)) = 0$ for each $t \in I_b$. Using Arzela–Ascoli theorem we obtain that V is relatively compact. By Theorem 2.8 the operator G has a fixed point. This means that there exists a Carathéodory solution of the problem (3.1) which is a Carathéodory solution of the problem (1.1).

REMARK 3.2. By a classical solution of (1.1) we understand a function in $C_{rd}(I_a, E)$ such that $x(0) = x_0$ and $x(\cdot)$ satisfies (1.1) for all $t \in I_a$. If we suppose a kind of continuity for f instead of Carathéodory condition we obtain the existence of at least one solution. For such solutions problem (1.1) is equivalent to

$$(3.6) \quad x(t) = x_0 + \int_0^t f(s, x(s))\Delta s, \quad \forall t \in I_a.$$

In this case we need to modify the notion of rd-continuity as follows: the function $f : T \times E \rightarrow E$ is rd-C-continuous provided it is (jointly-) continuous at each point (t, x) for which t is a right-dense point of \mathbb{T} ; and the limits $\lim_{(s,x) \rightarrow (t^-, x)} f(s, x)$ and $\lim_{y \rightarrow x} f(t, y)$ both exist (and are finite) at each (t, x) where t is left-dense and has a left-sided limit at each point. Thus the following theorem holds for classical solutions:

THEOREM 3.3. Suppose that a function $f : I_a \times B_r \rightarrow E$ is an rd-C-continuous function and there exists a constant $c > 0$ satisfying (3.5) for each $X \subset B_r$ and for each subinterval I_b of I_a . Assume that there exist bounded functions $m_1, m_2 \in C_{rd}(I_a, E)$ such that $\|f(t, x(t))\| \leq m_1(t) + m_2(t) \|x(t)\|$ for $(t, x) \in I_a \times B_r$. Then there exists at least one classical solution of the problem (1.1) on I_a .

The proof is similar to the proof of Theorem 3.1 given above.

We finish the paper with some important remarks.

REMARK 3.4. Clearly, for purely discrete time scales (with none accumulation points), the existence of (forward) solutions is trivially given without imposing further compactness assumptions on the right-hand side of the equation (cf. [22] for the continuous case). If a time scale admits at least one right-dense point, then the continuity assumption is not sufficient for the existence of (rd-continuous) solutions of the Cauchy problem (1.1) (for a modified example of Dieudonné see [19]). Nevertheless, we will not distinguish such a discrete case, because some continuity and compactness conditions are necessary to unify the continuous problems and their discretizations.

REMARK 3.5. Let us stress, that we used the Mönch fixed point theorem instead of the Darbo theorem (cf. [22]) to make possible further generalizations of our compactness assumption (3.5). In particular, with no essential changes in the proof we are able to put more general conditions like the Sadovskii condition: $\alpha(F(I \times X)) < \alpha(X)$ whenever $\alpha(X) > 0$ and I is a time scale interval, the Szuffla condition or the other compactness conditions (cf. [23]). The measure α can be replaced even by some axiomatic measures of noncompactness (cf. [13]). We refer the readers to [23] for the details of some differences in the proofs.

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M. Cichoń, I. Kubiaczyk, A. Sikorska-Nowak
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
ul. Umultowska 87
61-614 POZNAŃ, POLAND
E-mails: mcichon@amu.edu.pl
kuba@amu.edu.pl
anetas@amu.edu.pl

A. Yantir
DEPARTMENT OF MATHEMATICS
YASAR UNIVERSITY
IZMIR, TURKEY
E-mail: ahmet.yantir@yasar.edu.tr

Received May 10, 2010; revised version December 27, 2010.