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NOTE ON THE EXISTENCE OF A Ψ-BOUNDED SOLUTION FOR A LYAPUNOV MATRIX DIFFERENTIAL EQUATION

Abstract. In this paper, we give a necessary and sufficient condition for the existence of at least one Ψ-bounded solution of a linear nonhomogeneous Lyapunov matrix differential equation. In addition, we give a result in connection with the asymptotic behavior of the Ψ-bounded solutions of this equation.

1. Introduction

The importance of the boundedness of the solutions for systems of ordinary differential equations and their applications to different areas of science and technology are well known.

This work is concerned with linear nonhomogeneous Lyapunov matrix differential equation

$$X' = A(t)X + XB(t) + F(t) \tag{1}$$

where A, B and F are continuous n×n matrix-valued function on $\mathbb{R}_+ = [0, \infty)$.

The basic problem under consideration is the determination of the necessary and sufficient condition for the existence of a solution with some specified boundedness condition.

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations was studied in many papers, as e.g. [1, 3–9]. Here, the function Ψ is a continuous scalar function in [1, 3, 7, 8] as it is a continuous matrix function in [4–6, 9].

In [9], the authors studied the problem of Ψ-boundedness of the solutions for the corresponding Kronecker product system (2) associated with (1) (i.e. a linear nonhomogeneous differential system of the form $x' = G(t)x + f(t)$) in

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the hypothesis that the free term f of the system is a Lebesgue Ψ-integrable function defined on $\mathbb{R}_+$. But the obtained results in [9] are particular cases of our general results stated in [5]. Indeed, if in Theorems 2.1 and 2.2 ([5]), the fundamental matrix Y is replaced with the fundamental matrix Z⊗Y of the linear system (6), the Theorems 1 and 2 ([9]) follow. In addition, in Theorems 1 and 2 ([9]) there are a few mistakes in connection with the matrix Ψ.

Recently, a necessary and sufficient condition for the existence of at least one Ψ-bounded solution on $\mathbb{R}_+$ of (1) for every Lebesgue Ψ-integrable matrix function F has been obtained in our paper [6].

The purpose of the present paper is to give a necessary and sufficient condition so that the linear nonhomogeneous Lyapunov matrix differential equation (1) have at least one Ψ-bounded solution on $\mathbb{R}_+$ for every continuous and Ψ-bounded matrix function F on $\mathbb{R}_+$, with additional hypotheses on A, B, Ψ.

Here, as in [4], [5] and [6], Ψ will be a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

In order to be able to solve our problem, we use the technique of Kronecker product of matrices which has been successfully applied in various fields of the matrix theory.

The results of the present paper extend the results from [4], [5], [9] and include our results [4] as a particular case, when $B(t) = O_n$.

2. Preliminaries

In this section we present some basic definitions and results which are useful later on.

Let $\mathbb{R}^n$ be the Euclidean n - dimensional space. For $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ be the norm of x ($^{\mathrm{T}}$ denotes transpose).

Let $\mathbb{M}_{m\times n}$ be the linear space of all m×n real valued matrices.

For a $n \times n$ real matrix $A = (a_{ij})$, we define the norm $|A| = \sup\limits_{\|x\|\leq 1} \|Ax\|$.

It is well-known that $|A| = \max\limits_{1\leq i\leq n} \{\sum\limits_{j=1}^{n} |a_{ij}|\}$.

Definition 1. [2] Let $A = (a_{ij}) \in \mathbb{M}_{m\times n}$ and $B = (b_{ij}) \in \mathbb{M}_{p\times q}$. The Kronecker product of A and B, written $A\otimes B$, is defined to be the partitioned block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 1. *The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions are defined:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2). $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$;
3). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
4). $(A \otimes B) \cdot (C \otimes D) = AC \otimes BD$;
5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
6). $(A + B) \otimes C = A \otimes C + B \otimes C$;
7). $I_p \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8). $(A(t) \otimes B(t))' = A'(t) \otimes B(t) + A(t) \otimes B'(t)$; (here, $'$ denotes derivative $\frac{d}{dt}$).

Proof. See in [2]. ■

Definition 2. The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^{\mathrm{T}},$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

Lemma 2. *The vectorization operator $\mathcal{V}ec : \mathbb{M}_{n\times n} \longrightarrow \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ and $\mathcal{V}ec^{-1}$ are continuous operators.*

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for $A = (a_{ij}) \in \mathbb{M}_{n\times n}$, we have

$$\|\mathcal{V}ec(A)\| = \max_{1\le i,j\le n}\{|a_{ij}|\} \le \max_{1\le i\le n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\} = |A|.$$

Thus, the vectorization operator is continuous and $\|\mathcal{V}ec\| \le 1$.

In addition, for $A = I_n$ (identity $n\times n$ matrix) we have $\|\mathcal{V}ec(I_n)\| = |I_n|$ and then, $\|\mathcal{V}ec\| = 1$.

The inverse of the vectorization operator, $\mathcal{V}ec^{-1} : \mathbb{R}^{n^2} \longrightarrow \mathbb{M}_{n\times n}$, is defined by

$$\mathcal{V}ec^{-1}(u) = \begin{pmatrix} u_1 & u_{n+1} & \cdots & u_{n^2-n+1} \\ u_2 & u_{n+2} & \cdots & u_{n^2-n+2} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & u_{2n} & \cdots & u_{n^2} \end{pmatrix},$$

where $u = (u_1, u_2, \cdots, u_{n^2})^{\mathrm{T}} \in \mathbb{R}^{n^2}$.

We have

$$|\mathcal{V}ec^{-1}(u)| = \max_{1\le i\le n}\left\{\sum_{j=0}^{n-1}|u_{nj+i}|\right\} \le n \cdot \max_{1\le i\le n^2}\{|u_i|\} = n\cdot\|u\|.$$

Thus, $\mathcal{V}ec^{-1}$ is a continuous operator. ■

Remark. Obviously, if F is a continuous matrix function on $\mathbb{R}_+$, then $f = \mathcal{V}ec(F)$ is a continuous vector function on $\mathbb{R}_+$ and reciprocally.

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations (see in [9]):

Lemma 3. *If $A, B, M \in \mathbb{M}_{n\times n}$, then*

1). $\mathcal{V}ec(AMB) = (B^{\mathrm{T}} \otimes A)\cdot\mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^{\mathrm{T}} \otimes I_n)\cdot\mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n \otimes A)\cdot\mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^{\mathrm{T}} \otimes A)\cdot\mathcal{V}ec(I_n)$.

Proof. It is a simple exercise.

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0,\infty)$, $i = 1, 2, \ldots, n$, be continuous functions and

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \cdots \Psi_n].$$

Definition 3. [4] A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ is said to be Ψ-bounded on $\mathbb{R}_+$ if Ψf is bounded on $\mathbb{R}_+$ (i.e. $\sup_{t\ge 0}\|\Psi(t)f(t)\| < +\infty$).

Below we extend this definition for matrix functions.

Definition 4. A matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is said to be Ψ-bounded on $\mathbb{R}_+$ if the matrix function ΨM is bounded on $\mathbb{R}_+$ (i.e. $\sup_{t\ge 0}|\Psi(t)M(t)| < +\infty$).

We shall assume that A, B and F are continuous $n\times n$-matrices on $\mathbb{R}_+$.

By a solution of (1), we mean a continuous differentiable $n\times n$-matrix function X satisfying the equation (1) for all $t \ge 0$.

The following lemmas play a vital role in the proof of main result.

The first Lemma is done in [9]. Because the proof was incomplete, we presented it as Lemma 7, [6], with a complete proof.

Lemma 4. [6] *The matrix function $X(t)$ is a solution of* (1) *on the interval $J \subset \mathbb{R}$ if and only if the vector valued function $x(t) = \mathcal{V}ec(X(t))$ is a solution of the differential system*

$$x' = (I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)x + f(t), \tag{2}$$

where $f(t) = \mathcal{V}ec(F(t))$, on the same interval J.

Definition 5. The above system (2) is called 'corresponding Kronecker product system associated with (1)'.

Lemma 5. [6] *The matrix function M(t) is Ψ-bounded on $\mathbb{R}_+$ if and only if the vector function $\mathcal{V}ec(M(t))$ is $I_n \otimes \Psi$-bounded on $\mathbb{R}_+$.*

Proof. From the proof of Lemma 2, it results that

$$\frac{1}{n}|A| \leq \|\mathcal{V}ec(A)\|_{\mathbb{R}^{n^2}} \leq |A|,$$

for every $A \in \mathbb{M}_{n\times n}$.

Setting $A = \Psi(t)M(t)$, $t \geq 0$ and using Lemma 3, we have the inequality

$$\frac{1}{n}|\Psi(t)M(t)| \leq \|(I_n \otimes \Psi(t)) \cdot \mathcal{V}ec(M(t))\|_{\mathbb{R}^{n^2}} \leq |\Psi(t)M(t)|, t \geq 0 \tag{3}$$

for every matrix function $M(t)$.

Now, the Lemma follows immediately. ∎

The next result is Lemma 1 in [9]. Because the proof was incomplete, we presented it as Lemma 6, [6], with a complete proof.

Lemma 6. [6]. *Let $X(t)$ and $Y(t)$ be a fundamental matrices for the systems*

$$x'(t) = A(t)x(t) \tag{4}$$

and

$$y'(t) = y(t)B(t) \tag{5}$$

respectively.

Then, the matrix $Z(t) = Y^{\mathrm{T}}(t) \otimes X(t)$ is a fundamental matrix for the system

$$z'(t) = (I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)z(t). \tag{6}$$

If, in addition, $X(0) = I_n$ and $Y(0) = I_n$, then $Z(0) = I_{n^2}$.

Now, let $Z(t)$ be the above fundamental matrix for the system (6) with $Z(0) = I_{n^2}$.

Let $\widetilde{X}_1$ denote the subspace of $\mathbb{R}^{n^2}$ consisting of all vectors which are values of $I_n \otimes \Psi$-bounded solutions of (6) on $\mathbb{R}_+$ for $t = 0$ and let $\widetilde{X}_2$ an arbitrary fixed subspace of $\mathbb{R}^{n^2}$, supplementary to $\widetilde{X}_1$. Let $\widetilde{P}_1$, $\widetilde{P}_2$ denote the corresponding projections of $\mathbb{R}^{n^2}$ onto $\widetilde{X}_1$, $\widetilde{X}_2$ respectively.

Finally, we recall two theorems which will be used in the proofs of our main results.

Theorem 1. [4] *Let A be a continuous $d \times d$ real matrix on $\mathbb{R}_+$ such that*

$$|\Psi(t)A(t)\Psi^{-1}(t)| \leq L, \text{ for all } t \in \mathbb{R}_+,$$

where L is a positive constant.

Let Ψ be such that

$$|\Psi(t)\Psi^{-1}(s)| \leq M, \text{ for all } t \geq s \geq 0,$$

where M is a positive constant. Let $Y(t)$ be a fundamental matrix for the system $x' = A(t)x$. Then, the system $x' = A(t)x + f(t)$ has at least one Ψ-bounded solution on $\mathbb{R}_+$ for every continuous and Ψ-bounded function f on $\mathbb{R}_+$ if and only if there are two positive constants K and α such that

$$\begin{aligned} &|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \leq Ke^{-\alpha(t-s)}, \quad 0 \leq s \leq t \\ &|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \leq Ke^{-\alpha(s-t)}, \quad 0 \leq t \leq s. \end{aligned} \tag{7}$$

Theorem 2. [4] *Suppose that:*

1°. *the matrix functions A and Ψ satisfy the conditions of the above Theorem;*
2°. *the fundamental matrix $Y(t)$ of the system $x' = A(t)x$ satisfies the condition (7), where K and α are positive constants and P_1, P_2 are supplementary projections;*
3°. *the continuous and Ψ-bounded function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ satisfies one of the following conditions:*
   a). $\int_0^\infty \|\Psi(t)f(t)\|dt$ *is convergent,*
   b). $\lim_{t\to+\infty} \int_t^{t+1} \|\Psi(s)f(s)\|ds = 0.$

   Then, every Ψ-bounded solution x of the system $x' = A(t)x + f(t)$ is such that

$$\lim_{t\to+\infty} \|\Psi(t)x(t)\| = 0.$$

Remark. In these Theorems, P_1 and P_2 are supplementary projections as $\widetilde{P}_1$ and $\widetilde{P}_2$, for the system $x' = A(t)x$.

3. The main result

The main result of this paper is the following:

Theorem 3. *Let A and B be a continuous $n \times n$ real matrices such that*

$$|I_n \otimes (\Psi(t)A(t)\Psi^{-1}(t)) + B^{\mathrm{T}}(t) \otimes I_n| \leq L, \text{ for all } t \geq 0,$$

where L is a positive constant. Let Ψ be such that

$$|\Psi(t)\Psi^{-1}(s)| \le M, \text{ for all } t \ge s \ge 0,$$

where M is a positive constant. Let X(t) and Y(t) be a fundamental matrices for the systems (4) *and* (5) *respectively. Then, the equation* (1) *has at least one Ψ-bounded solution on* $\mathbb{R}_+$ *for every continuous and Ψ-bounded matrix function* $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ *if and only if there exist two positive constants K and α such that the functions*

$$\Phi_i(t,s) = \big(Y^{\mathrm{T}}(t) \otimes (\Psi(t)X(t))\big)\,\widetilde{P}_i\big((Y^{\mathrm{T}}(s))^{-1} \otimes (X^{-1}(s)\Psi^{-1}(s))\big),\ i = 1,2$$

satisfy the conditions

$$\begin{aligned} &|\Phi_1(t,s)| \le Ke^{-\alpha(t-s)}, \quad 0 \le s \le t \\ &|\Phi_2(t,s)| \le Ke^{-\alpha(s-t)}, \quad 0 \le t \le s. \end{aligned} \tag{8}$$

Proof. First, we prove the 'only if' part.

Suppose that the equation (1) has at least one Ψ-bounded solution on $\mathbb{R}_+$ for every continuous and Ψ-bounded matrix function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$ be a continuous and $I_n \otimes \Psi$-bounded function on $\mathbb{R}_+$. From Lemma 5, it follows that the matrix function $F(t) = \mathcal{V}ec^{-1}(f(t))$ is continuous and Ψ-bounded on $\mathbb{R}_+$. From the hypothesis, the equation

$$X' = A(t)X + XB(t) + \mathcal{V}ec^{-1}(f(t))$$

has at least one Ψ-bounded solution $X(t)$ on $\mathbb{R}_+$. From Lemma 4 and Lemma 5, it follows that the vector valued function $x(t) = \mathcal{V}ec(X(t))$ is a $I_n \otimes \Psi$-bounded solution on $\mathbb{R}_+$ of the differential system

$$x' = (I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)x + f(t).$$

Thus, this system has at least one $I_n \otimes \Psi$-bounded solution on $\mathbb{R}_+$ for every continuous and $I_n \otimes \Psi$-bounded function f on $\mathbb{R}_+$. From the hypotheses and Lemma 1, we have that

$$\begin{aligned} &(I_n \otimes \Psi(t))(I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)(I_n \otimes \Psi(t))^{-1} \\ &\qquad = (I_n \otimes \Psi(t))(I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)(I_n \otimes \Psi^{-1}(t)) \\ &\qquad = I_n \otimes (\Psi(t)A(t)\Psi^{-1}(t)) + B^{\mathrm{T}}(t) \otimes I_n \end{aligned}$$

and

$$(I_n \otimes \Psi(t))(I_n \otimes \Psi^{-1}(s)) = I_n \otimes (\Psi(t)\Psi^{-1}(s)).$$

Thus, we are in the terms of the Theorem 1 [4] for the above system.

From this Theorem, there exist two positive constants K and α such that the fundamental matrix $Z(t)$ of the system (6) satisfies the conditions

$$\begin{aligned} &|(I_n \otimes \Psi(t))Z(t)\widetilde{P}_1 Z^{-1}(s)(I_n \otimes \Psi(s))^{-1}| \le Ke^{-\alpha(t-s)}, \quad 0 \le s \le t, \\ &|(I_n \otimes \Psi(t))Z(t)\widetilde{P}_2 Z^{-1}(s)(I_n \otimes \Psi(s))^{-1}| \le Ke^{-\alpha(s-t)}, \quad 0 \le t \le s. \end{aligned}$$

By Lemma 6, we have $Z(t) = Y^{\mathrm{T}}(t) \otimes X(t)$. Now the computation shows that (8) holds.

Now, we prove the 'if' part.

Suppose that (8) holds for some $K > 0$ and $\alpha > 0$. Let $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ be a continuous and Ψ-bounded matrix function on $\mathbb{R}_+$. From Lemma 5, it follows that the vector valued function $f(t) = \mathcal{V}ec(F(t))$ is continuous and $I_n \otimes \Psi$-bounded function on $\mathbb{R}_+$. Thus, for the fundamental matrix $Z(t) = Y^{\mathrm{T}}(t) \otimes X(t)$ of the system

$$x' = (I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)x$$

we are in the terms of the Theorem 1 [4] (the 'if' part). From this theorem, it follows that the differential system

$$x' = (I_n \otimes A(t) + B^{\mathrm{T}}(t) \otimes I_n)x + f(t)$$

has at least one $I_n \otimes \Psi$-bounded solution on $\mathbb{R}_+$. Let $x(t)$ be this solution.

From Lemma 4 and Lemma 5, it follows that the matrix function $X(t) = \mathcal{V}ec^{-1}(x(t))$ is a Ψ-bounded solution on $\mathbb{R}_+$ of the equation (1) (because $F(t) = \mathcal{V}ec^{-1}(f(t)))$. Thus, the differential equation (1) has at least one Ψ-bounded solution on $\mathbb{R}_+$ for every continuous and Ψ-bounded matrix function F on $\mathbb{R}_+$.

It completes the proof. ■

Remark. Theorem 3 generalizes Theorem 2.1 [4].

Indeed, in particular case $B(t) = O_n$, the unique solution Z of the non-homogeneous matrix differential equation $Z' = A(t)Z + F(t)$ which takes the value Z_0 for $t = t_0$ is given by

$$Z(t) = X(t)X^{-1}(t_0)Z_0 + \int_{t_0}^{t} X(t)X^{-1}(s)F(s)ds, \quad t \in \mathbb{R}_+.$$

(The proof is similar to the proof of well-known Variation of constants formula for the linear differential system $x' = A(t)x + f(t)$).

If, in addition

$$F(t) = \begin{pmatrix} f_1(t) & f_1(t) & \cdots & f_1(t) \\ f_2(t) & f_2(t) & \cdots & f_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(t) & f_n(t) & \cdots & f_n(t) \end{pmatrix} \text{ and } Z_0 = \begin{pmatrix} z_1^0 & z_1^0 & \cdots & z_1^0 \\ z_2^0 & z_2^0 & \cdots & z_2^0 \\ \vdots & \vdots & \vdots & \vdots \\ z_n^0 & z_n^0 & \cdots & z_n^0 \end{pmatrix},$$

it is easy to see that the solution of the equation $Z' = A(t)Z + F(t)$ is

$$Z(t) = \begin{pmatrix} x_1(t) & x_1(t) & \cdots & x_1(t) \\ x_2(t) & x_2(t) & \cdots & x_2(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_n(t) & x_n(t) & \cdots & x_n(t) \end{pmatrix},$$

where $x = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$ is the solution of the problem

$$x'(t) = A(t)x(t) + f(t), x(t_0) = (z_1^0, z_2^0, \cdots, z_n^0)^{\mathrm{T}},$$

with $f(t) = (f_1(t), f_2(t), \cdots, f_n(t))^{\mathrm{T}}$.

In this case, the condition (8) becomes the condition (7), because the solution $x(t)$ is Ψ-bounded on $\mathbb{R}_+$ if and only if the solution $Z(t)$ is Ψ-bounded on $\mathbb{R}_+$. Thus, Theorem 3 generalizes Theorem 2.1 [4].

In the end, we prove a theorem which shows that the asymptotic behavior of the solutions of (1) is determined completely by the asymptotic behavior of $F(t)$ as $t \longrightarrow \infty$.

Theorem 4. *Suppose that:*

1). *the matrices A, B and Ψ satisfy the conditions of the Theorem* 3;
2). *the fundamental matrices* $X(t)$ *and* $Y(t)$ *of* (4) *and* (5) *respectively* ($X(0) = Y(0) = I_n$) *satisfy the condition* (8) *for some* $K > 0$ *and* $\alpha > 0$*;*
3). *the continuous and Ψ-bounded matrix function* $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ *satisfies one of the following conditions:*
   a). $\int_0^\infty |\Psi(t)F(t)|dt$ *is convergent,*
   b). $\lim\limits_{t\to+\infty} \int_t^{t+1} |\Psi(s)F(s)|ds = 0.$

Then, every Ψ-bounded solution X of the equation (1) *is such that*

$$\lim_{t\to+\infty} |\Psi(t)X(t)| = 0.$$

Proof. Let $X(t)$ be a Ψ-bounded solution of the equation (1). From Lemma 4 and Lemma 5, it follows that the function $x(t) = \mathcal{V}ec(X(t))$ is a $I_n \otimes \Psi$-bounded solution on $\mathbb{R}_+$ of the differential system (2), where $f(t) = \mathcal{V}ec(F(t))$.

From Lemma 5, it follows that the function f is continuous and $I_n \otimes \Psi$-bounded on $\mathbb{R}_+$.

From the hypothesis and from the inequality (3), we have

$$\|(I_n \otimes \Psi(t)) \cdot f(t)\|_{\mathbb{R}^{n^2}} \leq |\Psi(t)F(t)|, \quad t \geq 0.$$

It follows that the function f satisfies one of the following conditions
a$'$). $\int_0^\infty \|(I_n \otimes \Psi(t)) \cdot f(t)\|_{\mathbb{R}^{n^2}} dt$ is convergent,
b$'$). $\lim\limits_{t\to+\infty} \int_t^{t+1} \|(I_n \otimes \Psi(s)) \cdot f(s)\|_{\mathbb{R}^{n^2}} ds = 0$
respectively.

From the Theorem 2.2 [4], it follows that

$$\lim_{t\to\infty} \|(I_n \otimes \Psi(t)) \cdot x(t)\|_{\mathbb{R}^{n^2}} = 0.$$

Now, from the inequality (3) again, we have

$$\frac{1}{n}|\Psi(t)X(t)| \leq \|(I_n \otimes \Psi(t)) \cdot x(t)\|_{\mathbb{R}^{n^2}}, t \geq 0.$$

It follows that

$$\lim_{t\to\infty} |\Psi(t)X(t)| = 0.$$

The proof is now complete. ∎

Remark. Theorem 4 generalizes Theorem 2.2 [4].

Remark. If the function F does not fulfill the condition 3 of Theorem 4, then, the Ψ-bounded solution $X(t)$ may be such that $|\Psi(t)X(t)| \nrightarrow 0$ as t $\to \infty$.

This can be seen from the next Example.

Example. Consider the equation (1) with

$$A(t) = \begin{pmatrix} -11 & 8 \\ -12 & 9 \end{pmatrix}, \quad B(t) = \begin{pmatrix} -6 & -10 \\ 3 & 5 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 0 & 0 \\ e^{2t} & 0 \end{pmatrix}.$$

The matrices A, B^{T} have the eigenvalues

$$\lambda_1 = -3,\ \lambda_2 = 1 \quad \text{and} \quad \mu_1 = -1,\ \mu_2 = 0$$

and the Jordan canonical forms

$$L = \operatorname{diag}[-3, 1] \quad \text{and} \quad M = \operatorname{diag}[-1, 0]$$

respectively.

We have $A = ULU^{-1}$, $B^{\mathrm{T}} = VMV^{-1}$, where

$$U = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}.$$

The fundamental matrices for the systems (4) and (5) are

$$X(t) = Ue^{Lt}U^{-1} = \begin{pmatrix} 3e^{-3t} - 2e^{t} & -2e^{-3t} + 2e^{t} \\ 3e^{-3t} - 3e^{t} & -2e^{-3t} + 3e^{t} \end{pmatrix}$$

and

$$Y^{\mathrm{T}}(t) = Ve^{Mt}V^{-1} = \begin{pmatrix} 6e^{-t} - 5 & -3e^{-t} + 3 \\ 10e^{-t} - 10 & -5e^{-t} + 6 \end{pmatrix}$$

respectively.

The fundamental matrix for the system (6) is

$$\begin{aligned} Z(t) &= Y^{\mathrm{T}}(t) \otimes X(t) = Ve^{Mt}V^{-1} \otimes Ue^{Lt}U^{-1} = \\ &= (V \otimes U)(e^{Mt} \otimes e^{Lt}(V^{-1} \otimes U^{-1}) = \\ &= T \cdot diag[e^{-4t}, 1, e^{-3t}, e^{t}] \cdot T^{-1}, \end{aligned}$$

where $T = V \otimes U = \begin{pmatrix} 3 & 6 & 1 & 2 \\ 3 & 9 & 1 & 3 \\ 5 & 10 & 2 & 4 \\ 5 & 15 & 2 & 6 \end{pmatrix}$.

Consider

$$\Psi(t) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

The solutions of the system (6) are

$$z_0(t) = Z(t) \cdot (c_1, c_2, c_3, c_4)^{\mathrm{T}}, \quad t \in \mathbb{R}_+.$$

We note that these solutions are $I_2 \otimes \Psi$ bounded on $\mathbb{R}_+$.

On the other hand,

$$Y^{\mathrm{T}}(t)(Y^{\mathrm{T}}(s))^{-1} = Ve^{M(t-s)}V^{-1}$$

and

$$\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} 3e^{-5(t-s)} - 2e^{-(t-s)} & 2e^{-(t-s)} - 2e^{-5(t-s)} \\ 3e^{-5(t-s)} - 3e^{-(t-s)} & 3e^{-(t-s)} - 2e^{-5(t-s)} \end{pmatrix}.$$

It follows that the condition of Theorem 4 is satisfied with

$$\widetilde{P}_1 = I_4,\ \widetilde{P}_2 = O_4,\ K = 341 \text{ and } \alpha = 1.$$

In addition, the matrices A, B, Ψ satisfy the conditions of Theorem 4.

On the other hand, we have $|\Psi(t)F(t)| = 1$, for all $t \geq 0$. Now, it is easy to see that

$$X_p(t) = \begin{pmatrix} 2e^{-4t} - 4e^{2t} - 2e^{-3t} + 10e^{t} - 6 & \frac{10}{3}e^{-4t} - \frac{28}{3}e^{2t} - 4e^{-3t} + 20e^{t} - 10 \\ 2e^{-4t} - 6e^{2t} - 2e^{-3t} + 15e^{t} - 9 & \frac{10}{3}e^{-4t} - \frac{43}{3}e^{2t} - 4e^{-3t} + 30e^{t} - 15 \end{pmatrix}$$

is a particular Ψ-bounded solution of the system (1) and

$$\lim_{t\to\infty} |\Psi(t)X_p(t)| \neq 0.$$

Remark. This Example shows that the hypothesis 3 of Theorem 4 is a essential condition for the conclusion of Theorem.

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