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CONVERGENCE OF A FINITE DIFFERENCE SCHEME FOR VON FOERSTER EQUATION WITH FUNCTIONAL DEPENDENCE

Abstract. We analyse a finite difference scheme for von Foerster–McKendrick type equations with functional dependence forward in time and backward with respect to one dimensional spatial variable. Some properties of solutions of a scheme are given. Convergence of a finite difference scheme is proved. The presented theory is illustrated by a numerical example.

Introduction

Von Foerster–McKendrick type models are well known models of mathematical biology, describing a population with a structure of its members, given for example by their age [3], size [1] or level of maturation of individuals [7]. Existence, uniqueness and other properties of solutions for above mentioned models are studied in the literature. We are interested in some class of initial problems, originating in [7], which presents erytroid production model, based on a continuous maturation-proliferation mechanism. Far-reaching generalization of this problem is presented in [9]. In this paper we deal with the problem considered in [9] with one dimensional spatial variable.

Let $T > 0$, $\tau_0, \tau_1 \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, +\infty)$. Let us introduce

$$I_0 = [-\tau_0, 0], \quad I = [0, T], \quad B = [-\tau_0, 0] \times [-\tau_1, \tau_1],$$
$$E_0 = [-\tau_0, 0] \times \mathbb{R}_+, \quad E = [0, T] \times \mathbb{R}_+.$$

For a given function $q\colon I_0 \cup I \to \mathbb{R}$, $t \in I$ define the Hale operator $q_t\colon I_0 \to \mathbb{R}$ by

$$q_t(s) = q(t + s), \quad s \in I_0,$$

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see [4]. Denote $\alpha_+ = \max\{0, \alpha\}$. Given a function $w\colon E_0 \cup E \to \mathbb{R}$ and a point $(t,x) \in E$ define a function $w_{(t,x)}\colon B \to \mathbb{R}$ by

$$w_{(t,x)}(s,y) = w(t+s, (x+y)_+), \quad (s,y) \in B.$$

Our definition of $w_{(t,x)}$ differs from the definition of the Hale operator given in [6] since negative values of the spatial coordinate x do not have biological interpretation.

By $\mathbf{F}(X)$ denote the class of all real functions defined on X, where X is an arbitrary set. For any metric space Y we denote by $C(Y)$ the class of all continuous real functions on Y, whereas $C_+(Y)$ denotes the class of all nonnegative continuous real functions on Y. For $q \in C(I_0)$, $w \in C(B)$ we define

$$\|q\|_{C(I_0)} = \max\{|q(t)|\colon t \in I_0\}, \quad \|w\|_{C(B)} = \max\{|w(t,x)|\colon (t,x) \in B\}.$$

Let $\Omega_0 = E \times C(I_0)$, $\Omega = E \times C(B) \times C(I_0)$. Suppose that $v\colon E_0 \to \mathbb{R}$ is a given bounded and continuous initial function and

$$c\colon \Omega_0 \to \mathbb{R}_+, \qquad \lambda\colon \Omega \to \mathbb{R}.$$

Consider the differential functional equation

$$\partial_t u(t,x) + c(t,x,z_t)\,\partial_x u(t,x) = u(t,x)\lambda\left(t,x,u_{(t,x)},z_t\right) \tag{1}$$

with the initial condition

$$u(t,x) = v(t,x), \quad (t,x) \in E_0, \tag{2}$$

where

$$z(t) = \int_0^{+\infty} u(t,y)\,dy, \quad t \in [-\tau_0, T]. \tag{3}$$

We assume that $c(t,0,q) = 0$ for $t \in I$, $q \in C(I_0)$. This condition implies that the boundary condition is not necessary for the problem, cf. [2, 9]. Differential equations, equations with deviated argument, differential integral equations can be derived from (1) by specializing the operators c, λ, see examples in [9].

This paper extends our previous results concerning approximation of solutions to the above problem without functional dependence by finite difference schemes, see [8].

Solutions to (1)–(2) exist on unbounded domain. However, due to computational constraints solutions of the problem can be approximated only on a finite mesh. The paper is devoted to a difference method for approximation of infinite domain solutions to (1)–(2) by a difference scheme with a finite number of knots. We give conditions on the size of a finite rectangular mesh, which enable to obtain expected error of approximation of (3) for a prescribed initial data and a discretization parameter.

The partial derivative $\partial_x u$ is approximated by the backward difference quotient, since $c \geq 0$. We consider only nonnegative approximation of solutions of the problem. Therefore, the Courant–Friedrich's–Levy stability condition (cf. [5], p. 274) is replaced by a modified stability condition which implies also nonnegativity of solutions of our scheme. However, the presented theory, under slight modifications, remains valid if solutions of the problem are negative. Values of $z(t)$ are approximated by a sufficiently large finite rectangular rule. We introduced some class of initial functions to be assured that the rule approximating $z(t)$ is well defined. Interpolating operators are defined as functions c, λ are dependent on continuous functional argument.

The paper is organized as follows:

(i) a finite difference scheme is introduced and some properties of its solutions are given;
(ii) convergence of the scheme is proved;
(iii) results of numerical experiments illustrating the presented theory are given.

1. Finite difference scheme

We approximate solutions of (1)–(2) on a sufficiently large bounded area since practical computations cannot be performed on unbounded domain. Let $N_0, N_1 \in \mathbb{N}$, $h_0 = \dfrac{\tau_0}{N_0}$, $h_1 = \dfrac{\tau_1}{N_1}$ and $m = \dfrac{h_0}{h_1}$, $h = (h_0, h_1)$. There is $N \in \mathbb{N}$ such that $Nh_0 < a \leq (N+1)h_0$. Define

$$I_{0.h} = \{t^{(i)} \colon i = -N_0, \ldots, 0\}, \quad I_h = \{t^{(i)} \colon i = 0, \ldots, N\},$$

where $t^{(i)} = ih_0$. For a given discretization parameter h_0 define $N_h \in \mathbb{N}$ such that $h_0 N_h \to \infty$ as $h_0 \to 0$. Define $M^{(i)} = N_h + N_1 N$ for $i = -N_0, \ldots, 0$ and $M^{(i)} = N_h + N_1(N-i)_+$ for $i = 0, \ldots, N+1$. Let

$$E_{0.h} = \{(t^{(i)}, x^{(j)}) \colon i = -N_0, \ldots, 0,\ j = 0, \ldots, M^{(0)}\},$$
$$E_h = \{(t^{(i)}, x^{(j)}) \colon i = 0, \ldots, N,\ j = 0, \ldots, M^{(i)}\},$$

where $x^{(j)} = jh_1$, be the finite meshes on some bounded parts of E_0 and E, respectively. Define

$$E'_h = \{(t^{(i)}, x^{(j)}) \in E_h \colon (t^{(i+1)}, x^{(j)}) \in E_h\}, \quad I'_h = \{t^{(i)} \in I_h \colon t^{(i+1)} \in I_h\}.$$

For discrete functions $u\colon E_{0.h} \cup E_h \to \mathbb{R}$, $z\colon I_{0.h} \cup I_h \to \mathbb{R}$ we write $u^{(i,j)} = u(t^{(i)}, x^{(j)})$, $z^{(i)} = z(t^{(i)})$. Let $E^* = (\tilde{E}_0 \cup \tilde{E}) \cap (E_0 \cup E)$, where

$$\tilde{E}_0 = \{(t,x) \in E_0 \colon x \leq M^{(0)} h_1\},$$
$$\tilde{E} = \{(t,x) \in E \colon t \in [t^{(i)}, t^{(i+1)}],\ x \in [0, M^{(i+1)} h_1],\ i = 0, \ldots, N\}.$$

Introduce the interpolating operator $T_h\colon \mathbf{F}\,(E_{0.h}\cup E_h)\to C(E^*)$. Define $S=\{(\alpha,\beta)\colon \alpha,\beta\in\{0,1\}\}$. Two cases will be considered.

I. Suppose that $(t,x)\in\tilde{E}_0$. There are $(t^{(i)},x^{(j)})$, $(t^{(i+1)},x^{(j+1)})\in E_{0.h}$ such that $t^{(i)}\le t\le t^{(i+1)}$ and $x^{(j)}\le x\le x^{(j+1)}$. Then

$$(4)\qquad (T_hu)(t,x)=\sum_{(\alpha,\beta)\in S} u^{(i+\alpha,i+\beta)}\left(\frac{t-t^{(i)}}{h_0}\right)^{\alpha}\left(1-\frac{t-t^{(i)}}{h_0}\right)^{1-\alpha}\times$$
$$\times\left(\frac{x-x^{(j)}}{h_1}\right)^{\beta}\left(1-\frac{x-x^{(j)}}{h_1}\right)^{1-\beta},$$

provided that $0^0=1$.

II. Suppose that $(t,x)\in\tilde{E}$ and $t\le h_0N$. There are $i,j\in\mathbb{N}$ such that $[t^{(i)},t^{(i+1)}]\times[x^{(j)},x^{(j+1)}]\subset\tilde{E}$ and $t^{(i)}\le t\le t^{(i+1)}$, $x^{(j)}\le x\le x^{(j+1)}$. Then $(T_hu)(t,x)$ is given by (4). If $(t,x)\in\tilde{E}$ and $Nh_0<t\le T$, then $(T_hu)(t,x)=(T_hu)(Nh_0,x)$.

Note that T_hu is a continuous function on E^*. The definition of T_h is based on the definition of the interpolating operator given in [6], page 86.

Define the interpolating operator $\tilde{T}_{h_0}\colon \mathbf{F}(I_{0.h}\cup I_h)\to C(I_0\cup I)$ by

$$(\tilde{T}_{h_0}z)(t)=\Big(1-\frac{t-t^{(i)}}{h_0}\Big)z^{(i)}+\frac{t-t^{(i)}}{h_0}z^{(i+1)},$$

$t\in[t^{(i)},t^{(i+1)}]$, $-N_0\le i\le N-1$ and $(\tilde{T}_{h_0}z)(t)=(\tilde{T}_{h_0}z)(Nh_0)$ for $t\in(Nh_0,T]$.

Given discrete functions $u\colon E_{0.h}\cup E_h\to\mathbb{R}$, $z\colon I_{0.h}\cup I_h\to\mathbb{R}$ denote

$$c^{(i,j)}[z]=c\big(t^{(i)},x^{(j)},(\tilde{T}_{h_0}z)_{t^{(i)}}\big),$$
$$\lambda^{(i,j)}[u,z]=\lambda\big(t^{(i)},x^{(j)},(T_hu)_{(t^{(i)},x^{(j)})},(\tilde{T}_{h_0}z)_{t^{(i)}}\big).$$

Introduce the difference operators δ_0, δ_1 :

$$\delta_0u^{(i,j)}=(u^{(i+1,j)}-u^{(i,j)})/h_0,\quad \delta_1u^{(i,j)}=(u^{(i,j)}-u^{(i,j-1)})/h_1.$$

Consider the finite difference scheme corresponding to (1)–(2)

$$(5)\qquad \delta_0u^{(i,j)}+c^{(i,j)}[z]\delta_1u^{(i,j)}=u^{(i,j)}\lambda^{(i,j)}[u,z]\quad\text{on}\quad E_h',\quad j>0,$$

where

$$(6)\qquad z^{(i)}=h_1\sum_{j=0}^{M^{(i)}-1}u^{(i,j)},\quad i=-N_0,\dots,N,$$

with the initial condition

$$(7)\qquad u^{(i,j)}=v^{(i,j)}\quad\text{on}\quad E_{0.h}.$$

It follows from $c(t,0,q)=0$, $t\in[0,a]$, $q\in C(I_0)$, that

$$(8)\qquad \delta_0 u^{(i,0)} = u^{(i,0)}\lambda^{(i,0)}[u,z], \quad i=0,\dots,N-1.$$

There exists exactly one solution of (5)–(8).

Suppose that $\varphi\colon \{x^{(j)}\colon j\in\mathbb{N}\}\to\mathbb{R}$ is a bounded and summable function. Define

$$\|\varphi\| = \sup\{|\varphi^{(j)}|\colon j\in\mathbb{N}\}, \qquad \|\varphi\|_1 = h_1\sum_{j=0}^{\infty}|\varphi^{(j)}|.$$

By L^1 we denote a class of Lebesgue integrable functions defined on $\mathbb{R}_+$ with the standard norm denoted by $\|\cdot\|_{L^1}$. Define the following class of functions.

Definition 1.1. Let $f\colon\mathbb{R}_+\to\mathbb{R}$, $f\in L^1$. The function $f\in L^1_{\mathcal{M}}$ iff there is a decreasing function $g\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $g\in L^1$ and $|f(x)|\le g(x)$ for $x\in\mathbb{R}_+$.

Given $h_1>0$ and $f\colon\mathbb{R}_+\to\mathbb{R}$ let us denote $f_{h_1}=f_{|\{x^{(j)}\colon j\in\mathbb{N}\}}$.

Lemma 1.2. *If $f\colon\mathbb{R}_+\to\mathbb{R}$ and $f\in L^1_{\mathcal{M}}$, then $\|f_{h_1}\|_1<\infty$.*

Proof. One can assume that $h_1\le 1$. There is a decreasing function $F\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $F\in L^1$ and $|f(x)|\le F(x)$, $x\in\mathbb{R}_+$. We have

$$\|F_{h_1}\|_1 = h_1\sum_{j=0}^{\infty}F(x^{(j)}) \le h_1F(0)+\sum_{j=1}^{\infty}\int_{x^{(j-1)}}^{x^{(j)}}F(x)dx \le F(0)+\|F\|_{L^1}.$$

The assertion follows from the inequality $|f(x)|\le F(x)$, $x\in\mathbb{R}_+$. ∎

We make the following assumptions:

Assumption [V]. $v\colon E_0\to\mathbb{R}_+$, there exists a decreasing function $V\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $V\in L^1$ and $v(t,x)\le V(x)$ for $(t,x)\in E_0$.

Assumption [C]. $c\colon\Omega_0\to\mathbb{R}_+$ is continuous, there exist constants L_c, L_c^*, $M_c>0$ such that $c(t,x,q)\le M_c$ and

$$|c(t,x,q)-c(t,\bar{x},\bar{q})|\le L_c|x-\bar{x}|+L_c^*\|q-\bar{q}\|_{C(I_0)}$$

for $(t,x),(t,\bar{x})\in E$, $q,\bar{q}\in C_+(I_0)$.

Assumption [Λ]. $\lambda\colon\Omega\to\mathbb{R}$ is continuous, there exist constants M_λ, L_λ, $L_z>0$ such that $\lambda(t,x,w,q)\le M_\lambda$ and

$$|\lambda(t,x,w,q)-\lambda(t,x,\bar{w},\bar{q})|\le L_\lambda\|w-\bar{w}\|_{C(B)}+L_z\|q-\bar{q}\|_{C(I_0)}$$

for $(t,x)\in E$, $w,\bar{w}\in C_+(B)$, $q,\bar{q}\in C_+(I_0)$.

Assumption [SN]. $c\colon\Omega_0\to\mathbb{R}_+$, $\lambda\colon\Omega\to\mathbb{R}$ and a discretization parameter h satisfy

$$1-\frac{h_0}{h_1}c(t,x,q)+h_0\lambda(t,x,w,q)\ge 0$$

for $(t,x)\in E$, $w\in C_+(B)$, $q\in C_+(I_0)$.

We give some properties of solutions of (5)–(8).

Lemma 1.3. *If Assumptions* $[V]$, $[SN]$ *are satisfied, then any solution of* (5)–(8) *is nonnegative.*

Proof. The proof is by induction on i. For $i = 0$ the assertion holds since $v \geq 0$. Formulas (5) and (8) can be rewritten in the explicit form:

$$u^{(i+1,j)} = u^{(i,j)}\big(1 - \frac{h_0}{h_1}c^{(i,j)}[z] + h_0\lambda^{(i,j)}[u,z]\big) + \frac{h_0}{h_1}c^{(i,j)}[z]u^{(i,j-1)} \tag{9}$$

on E'_h for $j > 0$ and

$$u^{(i+1,0)} = u^{(i,0)}(1 + h_0\lambda^{(i,0)}[u,z]), \quad i = 0,\dots,N-1. \tag{10}$$

Suppose that for some $0 < i \leq N-1$ the functions $u^{(k,\cdot)}$, $i - N_0 \leq k \leq i$, are nonnegative. As Assumption [SN] is satisfied it follows from (9), (10) that the assertion holds for $i+1$. ∎

Define an auxiliary function $U\colon E_{0.h} \cup E_h \to \mathbb{R}_+$ by

$$U^{(i,j)} = \begin{cases} V^{(j)}, & (t^{(i)}, x^{(j)}) \in E_{0.h}, \\ (1 + h_0 M_\lambda)^i\, V^{((j-i)_+)}, & (t^{(i)}, x^{(j)}) \in E_h. \end{cases}$$

Lemma 1.4. *If Assumptions* $[V]$, $[\Lambda]$, $[SN]$ *are satisfied, then* $u^{(i,j)} \leq U^{(i,j)}$ *on* $E_{0.h} \cup E_h$.

Proof. The proof is by induction on i. For $i = -N_0,\dots,0$ the assertion follows form Assumption $[V]$. Suppose that $u^{(i,j)} \leq U^{(i,j)}$ for some $0 < i \leq N-1$ and $j = 0,\dots,M^{(i)}$. For $j > 0$ from Assumptions $[V]$, $[SN]$ and (9) we get

$$\begin{aligned} u^{(i+1,j)} &\leq U^{(i,j)}\big(1 - \frac{h_0}{h_1}c^{(i,j)}[z] + h_0\lambda^{(i,j)}[u,z]\big) + \frac{h_0}{h_1}c^{(i,j)}[z]U^{(i,j-1)} \\ &\leq U^{(i,j-1)}\big(1 + h_0 M_\lambda\big) = (1 + h_0 M_\lambda)^{i+1} V^{((j-(i+1))_+)} = U^{(i+1,j)}. \end{aligned}$$

For $j = 0$ we proceed similarly. The proof is completed. ∎

Corollary 1.5. *Under assumptions of Lemma* 1.4 *we have the estimates*

$$\|u^{(i,\cdot)}\| \leq e^{aM_\lambda} V^{(0)}, \quad \|u^{(i,\cdot)}\|_1 \leq e^{aM_\lambda}(V^{(0)}(1 + a/m) + \|V\|_{L^1})$$

for $i = 0,\dots,N$.

2. Convergence

Suppose that $\bar{u}\colon E_0 \cup E \to \mathbb{R}_+$ is a solution of (1)–(2) and $\bar{z}\colon I_0 \cup I \to \mathbb{R}_+$ is given by (3). Denote $\bar{u}_h = \bar{u}_{|E_{0.h} \cup E_h}$ and $\bar{u}_h^{(i,j)} = \bar{u}_h(t^{(i)}, x^{(j)})$. The local discretization error $\xi\colon E_h \to \mathbb{R}$ is defined as follows:

$$\xi^{(i,j)} = \delta_0 \bar{u}_h^{(i,j)} + c^{(i,j)}[\hat{z}_{h_0}]\delta_1 \bar{u}_h^{(i,j)} - \bar{u}_h^{(i,j)}\lambda^{(i,j)}[\bar{u}_h, \hat{z}_{h_0}] \quad \text{on} \quad E'_h,\ j > 0, \tag{11}$$

(12) $$\xi^{(i,0)} = \delta_0 \bar{u}_h^{(i,0)} - u_h^{(i,0)} \lambda^{(i,0)}[\bar{u}_h, \hat{z}_{h_0}], \quad i = 0, \ldots, N-1,$$

where

$$\hat{z}_{h_0}^{(i)} = h_1 \sum_{j=0}^{M^{(i)}-1} \bar{u}_h^{(i,j)}, \quad i = -N_0, \ldots, N.$$

Let $V\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the function given in Assumption $[V]$.

Assumption $[\bar{\mathbf{U}}]$**.** $\bar{u}\colon E_0 \cup E \to \mathbb{R}_+$ is a solution of (1)–(2) of class C^2 and

1) $\bar{u}(t, \cdot) \in L^1_{\mathcal{M}}$, $t \in I_0 \cup I$, and there exists a constant $D_0 > 0$ such that
$$\bar{u}(t,x) \le D_0 V(x) \quad \text{on} \quad E_0 \cup E;$$
2) $\partial_t \bar{u}(t,\cdot), \partial_x \bar{u}(t,\cdot) \in L^1_{\mathcal{M}}$, $t \in I_0 \cup I$, and there exists $D_1 > 0$ such that
$$|\partial_t \bar{u}(t,x)| \le D_1 V(x), \quad |\partial_x \bar{u}(t,x)| \le D_1 V(x) \quad \text{on} \quad E_0 \cup E;$$
3) there exists a constant $C > 0$ such that
$$|\partial_{tt} \bar{u}(t,x)|, \ |\partial_{tx} \bar{u}(t,x)|, \ |\partial_{xx} \bar{u}(t,x)| \le C \quad \text{on} \quad E_0 \cup E;$$
4) $\partial_{tt} \bar{u}(t,\cdot), \partial_{xx} \bar{u}(t,\cdot) \in L^1_{\mathcal{M}}$, $t \in I_0 \cup I$, and there exists $D_2 > 0$ such that
$$|\partial_{tt} \bar{u}(t,x)| \le D_2 V(x), \quad |\partial_{xx} \bar{u}(t,x)| \le D_2 V(x) \quad \text{on} \quad E_0 \cup E;$$
5) there exists a constant $D_3 > 0$ such that
$$|\bar{z}(\bar{t}) - \bar{z}(t)| \le D_3 |\bar{t} - t|, \quad \bar{t}, t \in I_0 \cup I.$$

Lemma 2.1. *If the function $\bar{u}\colon E_0 \cup E \to \mathbb{R}$ satisfies Assumption $[\bar{U}3]$, then*
$$\|(T_h \bar{u}_h)_{(t^{(i)},x^{(j)})} - \bar{u}_{(t^{(i)},x^{(j)})}\|_{C(B)} \le C_{T_h} h_0^2$$
for $(t^{(i)}, x^{(j)}) \in E_h$ such that $(t^{(i)}, x^{(j+N_1)}) \in E_h$, where $C_{T_h} = C(1 + 2m + m^2)$.

The proof of the Lemma is similar to the proof of Theorem 3.18 in [6].

Lemma 2.2. *Suppose that Assumption $[\bar{U}1,2,5]$ is satisfied. Then there is $\alpha\colon (0,+\infty) \to \mathbb{R}_+$ such that $\lim_{h_0 \to 0} \alpha(h_0) = 0$ and*
$$\|(\tilde{T}_{h_0} \hat{z}_{h_0})_{t^{(i)}} - \bar{z}_{t^{(i)}}\|_{C(I_0)} \le h_0 C_{\tilde{T}_{h_0}} + \alpha(h_0), \quad 0 \le i \le N,$$
where $C_{\tilde{T}_{h_0}} = (1 + \frac{1}{2m}) D_1 \Gamma_V + D_3$, $\Gamma_V = V^{(0)} + \|V\|_{L^1}$.

Proof. Let $s \in [t^{(i-N_0)}, t^{(i)}]$. There is $i - N_0 \le k \le i-1$ such that $s \in [t^{(k)}, t^{(k+1)}]$ and

$$|(\tilde{T}_{h_0} \hat{z}_{h_0})(s) - \bar{z}(s)| \le |\bar{z}(t^{(k)}) - \bar{z}(s)| + |\hat{z}_{h_0}^{(k)} - \bar{z}(t^{(k)})| + \frac{s - t^{(k)}}{h_0} |\hat{z}_{h_0}^{(k+1)} - \hat{z}_{h_0}^{(k)}|.$$

Let

$$P(t^{(k)},h_0)=D_0\int\limits_{x^{(M^{(k)})}}^{\infty}V(x)dx,\qquad R(t^{(k)},h_0)=D_0h_1\sum_{j=0}^{N_1-1}V^{(M^{(k+1)}+j)}.$$

The function $M^{(\cdot)}$ is nonincreasing, hence $x^{(M^{(N)})}\le x^{(M^{(k)})}$ for $-N_0\le k\le N$ and $x^{(M^{(N)})}\to\infty$ as $h_0\to 0$. Therefore

$$P(t^{(k)},h_0)\le\int\limits_{x^{(M^{(N)})}}^{\infty}V(x)\,dx=:\tilde P(h_0)$$

and $\tilde P(h_0)\to 0$ as $h_0\to 0$. Since $V^{(M^{(k+1)}+j)}\le V^{(M^{(N)}+j)}\le V^{(M^{(N)})}$ we have

$$R(t^{(k)},h_0)\le D_0h_1\sum_{j=0}^{N_1-1}V^{(M^{(N)})}\le D_0\tau_1V^{(M^{(N)})}=:\tilde R(h_0).$$

The function V is decreasing, therefore $\tilde R(h_0)\to 0$ as $h_0\to 0$. From Assumption $[\bar U2]$ and Lemma 1.2 we obtain the estimates

$$|\hat z_{h_0}^{(k)}-\bar z(t^{(k)})|\le\frac{D_1}{2}h_1^2\sum_{j=0}^{M^{(k)}-1}V^{(j)}+P(t^{(k)},h_0)\le\frac{D_1}{2}h_1\Gamma_V+\tilde P(h_0),$$

$$|\hat z_{h_0}^{(k+1)}-\hat z_{h_0}^{(k)}|\le D_1h_0h_1\sum_{j=0}^{M^{(k+1)}-1}V^{(j)}+R(t^{(k)},h_0)\le D_1h_0\Gamma_V+\tilde R(h_0).$$

The remaining part of the proof follows from Assumption $[\bar U5]$ and the both above estimates with $\alpha(h_0)=\tilde P(h_0)+\tilde R(h_0)$. ∎

Theorem 2.3. *If Assumptions* $[V]$, $[C]$, $[\Lambda]$, $[\bar U]$ *are satisfied, then*

$$\|\xi^{(i,\cdot)}\|\le V^{(0)}\beta(h_0),\qquad \|\xi^{(i,\cdot)}\|_1\le\Gamma_V\beta(h_0),$$

where $\Gamma_V=V^{(0)}+\|V\|_{L^1}$, $D=D_0L_z+D_1L_c^*$,

$$\beta(h_0)=h_0[C_{\tilde T_{h_0}}D+D_2(1+M_c/m)]+\alpha(h_0)D+D_0L_\lambda C_{T_h}h_0^2.$$

Proof. Let us subtract (1) at the point $(t^{(i)},x^{(j)})\in E_h'$, $j>0$, from (11). Then, by the mean value theorem and Assumptions $[\bar U1,2,4]$, $[C]$, $[\Lambda]$, we have

$$\begin{aligned}|\xi^{(i,j)}|\le{}& h_0D_2V^{(j)}+D_0V^{(j)}(L_\lambda\Delta_h^{(i,j)}+L_z\tilde\Delta_{h_0}^{(i)})\\&+h_1D_2M_cV^{(j-1)}+D_1L_c^*V^{(j)}\tilde\Delta_{h_0}^{(i)},\quad j>0,\end{aligned}$$

where

$\Delta_h^{(i,j)}=\|(T_h\bar u_h)_{(t^{(i)},x^{(j)})}-\bar u_{(t^{(i)},x^{(j)})}\|_{C(B)}$, $\quad\tilde\Delta_{h_0}^{(i)}=\|(\tilde T_{h_0}\hat z_{h_0})_{t^{(i)}}-\bar z_{t^{(i)}}\|_{C(I_0)}$.

Similarly, subtracting (1) at the point $(t^{(i)},0)$, $t^{(i)}\in I'_h$, from (12) we get

$$|\xi^{(i,0)}|\le h_0D_2V^{(0)}+D_0V^{(0)}(L_\lambda\Delta_h^{(i,0)}+L_z\tilde{\Delta}_{h_0}^{(i)}).$$

In force of Lemmas 2.1, 2.2 we get inequalities

$$\Delta_h^{(i,j)}\le C_{T_h}h_0^2,\qquad \tilde{\Delta}_{h_0}^{(i)}\le h_0C_{\tilde{T}_{h_0}}+\alpha(h_0),$$

which applied to the above estimates for $|\xi^{(i,j)}|$, $|\xi^{(i,0)}|$ lead to the assertion. ∎

Lemma 2.4. *Let $u_1,u_2\colon E_{0.h}\cup E_h\to\mathbb{R}_+$, $z_1,z_2\colon I_{0.h}\cup I_h\to\mathbb{R}_+$ be arbitrary bounded discrete functions. Then for $t^{(i)}\in I_h$ and $(t^{(i)},x^{(j)})\in E_h$ such that $(t^{(i)},x^{(j+N_1)})\in E_h$ we have*

$$\|(\tilde{T}_{h_0}z_1)_{t^{(i)}}-(\tilde{T}_{h_0}z_2)_{t^{(i)}}\|_{C(I_0)}\le\max_{i-N_0\le k\le i}|z_1^{(k)}-z_2^{(k)}|,$$

$$\|(T_hu_1)_{(t^{(i)},x^{(j)})}-(T_hu_2)_{(t^{(i)},x^{(j)})}\|_{C(B)}\le\max_{\substack{i-N_0\le k\le i,\\ (j-N_1)_+\le l\le j+N_1}}|u_1^{(k,l)}-u_2^{(k,l)}|.$$

Theorem 2.5. *Suppose that Assumptions $[V]$, $[C]$, $[\Lambda]$, $[SN]$, $[\bar{U}]$ are satisfied, there is $\gamma_0\colon(0,+\infty)\to\mathbb{R}_+$ such that $\lim_{h_0\to0}\gamma_0(h_0)=0$ and*

$$|\bar{u}_h^{(i,j)}-u^{(i,j)}|\le\gamma_0(h_0)V^{(j)}\quad on\quad E_{0.h}. \tag{13}$$

Then $\|\bar{u}_h^{(i,\cdot)}-u^{(i,\cdot)}\|$, $\|\bar{u}_h^{(i,\cdot)}-u^{(i,\cdot)}\|_1\to0$ as $h_0\to0$, uniformly with respect to i.

Proof. Denote $\varepsilon^{(i,j)}=\bar{u}_h^{(i,j)}-u^{(i,j)}$. Subtraction of the both sides of (11), (5), and (12), (8) lead to the recurrence error equations

$$\begin{aligned}\varepsilon^{(i+1,j)}&=\varepsilon^{(i,j)}\big(1-\frac{h_0}{h_1}c^{(i,j)}[z]+h_0\lambda^{(i,j)}[u,z]\big)+\frac{h_0}{h_1}\varepsilon^{(i,j-1)}c^{(i,j)}[z]\\&\quad+\frac{h_0}{h_1}C^{(i,j)}\big(\bar{u}_h^{(i,j)}-\bar{u}_h^{(i,j-1)}\big)+h_0\bar{u}_h^{(i,j)}\Lambda^{(i,j)}+h_0\xi^{(i,j)},\quad j\ge1,\end{aligned}$$

$$\varepsilon^{(i+1,0)}=\varepsilon^{(i,0)}(1+h_0\lambda^{(i,0)}[u,z])+h_0\bar{u}_h^{(i,0)}\Lambda^{(i,0)}+h_0\xi^{(i,0)},$$

respectively, where

$$\Lambda^{(i,j)}=\lambda^{(i,j)}[\bar{u}_h,\hat{z}_{h_0}]-\lambda^{(i,j)}[u,z],\qquad C^{(i,j)}=c^{(i,j)}[z]-c^{(i,j)}[\hat{z}_{h_0}].$$

It follows from Assumption $[\bar{U}2]$ that $|\bar{u}_h^{(i,j)}-\bar{u}_h^{(i,j-1)}|\le h_1D_1V^{(j-1)}$. By Assumptions $[SN]$, $[C]$, $[\Lambda]$, $[\bar{U}]$ we conclude that

$$\begin{aligned}|\varepsilon^{(i+1,j)}|&\le|\varepsilon^{(i,j)}|\big(1-\frac{h_0}{h_1}c^{(i,j)}[z]+h_0\lambda^{(i,j)}[u,z]\big)+\frac{h_0}{h_1}c^{(i,j)}[z]|\varepsilon^{(i,j-1)}|\\&\quad+h_0D_1V^{(j-1)}|C^{(i,j)}|+h_0\bar{u}_h^{(i,j)}|\Lambda^{(i,j)}|+h_0|\xi^{(i,j)}|,\quad j\ge1,\end{aligned}\tag{14}$$

$$|\varepsilon^{(i+1,0)}|\le(1+h_0\lambda^{(i,0)}[u,z])|\varepsilon^{(i,0)}|+h_0\bar{u}_h^{(i,0)}|\Lambda^{(i,0)}|+h_0|\xi^{(i,0)}|. \tag{15}$$

Since $|\hat{z}_{h_0}^{(k)} - z^{(k)}| \le \|\varepsilon^{(k,\cdot)}\|_1$, in force of Lemma 2.4 and Assumptions $[C]$, $[\Lambda]$, we obtain

$$|C^{(i,j)}| \le L_c^* \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\|_1,$$
$$|\Lambda^{(i,j)}| \le L_\lambda \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\| + L_z \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\|_1.$$

Note that

$$\sum_{j=1}^{M^{(i)}-1} c^{(i,j)}[z](|\varepsilon^{(i,j-1)}| - |\varepsilon^{(i,j)}|) \le h_1 L_c \sum_{j=1}^{M^{(i)}-2} |\varepsilon^{(i,j)}| \le L_c \|\varepsilon^{(i,\cdot)}\|_1.$$

Summation of (15) and (14) over $j \ge 1$ yields

$$\begin{aligned} \|\varepsilon^{(i+1,\cdot)}\|_1 \le{}& (1 + h_0 M_\lambda + h_0 L_c)\|\varepsilon^{(i,\cdot)}\|_1 \\ &+ h_0 L_c^* D_1 \Gamma_V \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\|_1 \\ &+ h_0 D_0 \Gamma_V \big(L_\lambda \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\| \\ &+ L_z \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\|_1\big) + h_0 \|\xi^{(i,\cdot)}\|_1. \end{aligned} \tag{16}$$

From (14), (15) we obtain

$$\begin{aligned} \|\varepsilon^{(i+1,\cdot)}\| \le{}& (1 + h_0 M_\lambda)\|\varepsilon^{(i,\cdot)}\| + h_0 L_c^* D_1 V^{(0)} \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\|_1 \\ &+ h_0 D_0 V^{(0)} \big(L_\lambda \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\| \\ &+ L_z \max_{i-N_0\le k\le i} \|\varepsilon^{(k,\cdot)}\|_1\big) + h_0 \|\xi^{(i,\cdot)}\|. \end{aligned} \tag{17}$$

By Theorem 2.3 we have $\|\xi^{(i,\cdot)}\|, \|\xi^{(i,\cdot)}\|_1 \le \Gamma_V \beta(h_0)$.

Define an auxiliary comparison function $\Psi\colon I_{0.h} \cup I_h \to \mathbb{R}_+$,

$$\Psi^{(i)} = \gamma_0(h_0)\Gamma_V, \quad -N_0 \le i \le 0,$$
$$\Psi^{(i+1)} = (1 + h_0\Gamma)\Psi^{(i)} + h_0 \Gamma_V \beta(h_0), \quad 0 \le i \le N-1,$$

where $\Gamma = M_\lambda + L_c + \Gamma_V[D_1 L_c^* + D_0(L_\lambda + L_z)]$. It is easy to verify that

$$\Psi^{(i)} \le \gamma_0(h_0) e^{t^{(i)}\Gamma} + \beta(h_0)\Gamma_V t^{(i)} e^{t^{(i)}\Gamma} \qquad \text{for} \quad 0 \le i \le N.$$

We show by induction on i that $\|\varepsilon^{(i,\cdot)}\|$, $\|\varepsilon^{(i,\cdot)}\|_1 \le \Psi^{(i)}$ for $-N_0 \le i \le N$. The assertion for $-N_0 \le i \le 0$ follows from (13). Suppose that the assertion holds for some $0 \le i \le N-1$. Then applying the inductive assumption to (16), (17) we obtain the assertion for $i+1$. The proof is completed. ■

Remark 2.6. Suppose that $H > 0$ is a sufficiently small real number, $h_1 \in (0, H)$. Given a decreasing Lebesgue integrable function $V\colon \mathbb{R}_+ \to \mathbb{R}_+$ and

$\phi\colon (0,H) \to \mathbb{R}_+$ such that $\lim_{h_1\to 0}\phi(h_1) = 0$, we determine N_h satisfying

$$\int\limits_{h_1 N_h}^{\infty} V(x)dx = \phi(h_1).$$

I. If $V(x) = e^{-ax}$, $a > 0$, then $N_h = \left[\frac{1}{ah_1}\ln\frac{1}{a\phi(h_1)}\right]$, where $[x]$ denotes the integral part of $x \in \mathbb{R}$.

II. If $V(x) = a/(1+x^2)$, $a > 0$, then $N_h = \left[\frac{1}{h_1}\tan\left(\frac{\pi}{2} - \frac{\phi(h_1)}{a}\right)\right]$.

3. Numerical experiment

Let $E = [0,1] \times \mathbb{R}_+$, $I = [0,1]$, $E_0 = \left[-\frac{1}{10}, 0\right] \times \mathbb{R}_+$, $I_0 = \left[-\frac{1}{10}, 0\right]$. Consider the differential integral equation with delay

$$\begin{aligned}(18)\qquad \partial_t u(t,x) &+ \frac{t\sin^2 x}{1+x}\sin^2(z(t-0.1))\partial_x u(t,x)\\ &= u(t,x)\left\{\frac{1}{1+t} + \frac{f(t)\sin(2x)}{\left(1+\int_{t/2}^{t} z(s)ds\right)(1+x)} - \frac{g(t)x}{1+x}\int\limits_{t-0.1}^{t} u(s,x)ds\right\}\end{aligned}$$

with the initial condition

$$(19)\qquad u(t,x) = (t+1)\sin^2 x/(1+x^2) \quad \text{for} \quad (t,x) \in E_0,$$

where z is given by (3), $A = \pi(1-e^{-2})/4$ and

$$f(t) = \sin^2(A(t+0.9))\left[1 + 0.5At\,(0.75t+1)\right],\ g(t) = \frac{20t\sin^2(A(t+0.9))}{t+0.95}.$$

The function $\bar{u}(t,x) = (t+1)\sin^2 x/(1+x^2)$ is the solution of (18)–(19) and $\bar{z}(t) = A(t+1)$.

Note that there is no deviation with respect to the spatial variable in (18)–(19). Therefore, $M^{(i)} = N_h$, $-N_0 \le i \le N$. We applied the following difference method for (18)–(19):

$$(20)\qquad \delta_0 u^{(i,j)} + t^{(i)}\sin^2(x^{(j)})\sin^2(z^{(i-N_0)})\delta_1 u^{(i,j)} = u^{(i,j)}\lambda^{(i,j)}[u,z],$$

on E'_h for $j > 0$, with the initial condition

$$(21)\qquad u^{(i,j)} = (1+t^{(i)})\sin^2(x^{(j)})/(1+(x^{(j)})^2) \quad \text{on} \quad E_{0.h},$$

where $u^{(i,0)}$, $z^{(i)}$ are given by (8), (6), respectively, and

$$\lambda^{(i,j)}[u,z] = \frac{1}{1+t^{(i)}} + f(t^{(i)})\frac{\sin(2x^{(j)})}{1+Z_{h_0}^{(i)}} - g(t^{(i)})\frac{x^{(j)}}{1+x^{(j)}}U_h^{(i,j)},$$

$$U_h^{(i,j)} = \frac{h_0}{2}(u^{(i-N_0,j)} + u^{(i,j)}) + h_0 \sum_{k=i-N_0+1}^{i-1} u^{(k,j)},$$

$$Z_{h_0}^{(i)} = \int\limits_{t^{(i)}/2}^{t^{(i)}} \big(\tilde{T}_{h_0} z\big)(s)ds$$

$$= \begin{cases} h_0 \displaystyle\sum_{k=\frac{i+1}{2}+1}^{i-1} z^{(k)} + \frac{h_0}{2}\big(z^{(\frac{i-1}{2})} + z^{(i)}\big) + h_0 z^{(\frac{i+1}{2})}, & \text{if } i \text{ is odd,} \\ h_0 \displaystyle\sum_{k=\frac{i}{2}+1}^{i-1} z^{(k)} + \frac{h_0}{2}\big(z^{(\frac{i}{2})} + z^{(i)}\big), & \text{if } i \text{ is even.} \end{cases}$$

Suppose that $u\colon E_{0.h} \cup E_h \to \mathbb{R}_+$ is the solution of (20)–(21) and $z\colon I_{0.h} \cup I_h \to \mathbb{R}_+$ is given by (6). Let $\bar{u}\colon E_0 \cup E \to \mathbb{R}_+$ be the solution of (18)–(19) with $\bar{z}\colon I_0 \cup I \to \mathbb{R}_+$ given by (3) and denote $\bar{u}_h = \bar{u}_{|E_{0.h}\cup E_h}$, $\bar{z}_h = \bar{z}_{|I_{0.h}\cup I_h}$. Let $\varepsilon^{(i,j)} = \bar{u}_h^{(i,j)} - u^{(i,j)}$. We define error of the approximation:

$$\Delta u = \max_{0\le i\le N}\{\|\varepsilon^{(i,\cdot)}\|\}, \quad \Delta_1 u = \max_{0\le i\le N}\{\|\varepsilon^{(i,\cdot)}\|_1\}.$$

Additionally, we define

$$\Delta z = \max_{0\le i\le N}\{|z^{(i)} - \bar{z}_h^{(i)}|\}.$$

The results of computations with N_h defined in Remark 2.6 for $\phi(h) = \sqrt{h}/2$ and $V(x) = 1/(1+x^2)$ are presented in the tables. Estimates of the functions c, λ for the above data are given. During computations we checked that Assumption $[SN]$ was satisfied. The computations were performed by PC.

$h_1 = h_0$, $0 \le c \le 0.38$, $0.06 \le \lambda \le 1$

h_0	$N_h h_1$	Δu	$\Delta_1 u$	Δz
1/50	14.1	$4.64E-03$	$7.51E-03$	$7.24E-02$
1/500	44.7	$1.79E-03$	$2.54E-03$	$2.26E-02$
1/1000	63.2	$1.30E-03$	$1.81E-03$	$1.60E-02$
1/2000	89.4	$9.62E-04$	$1.33E-03$	$1.17E-02$

$h_1 = 2h_0$, $0 \leq c \leq 0.36$, $0.06 \leq \lambda \leq 1$

h_0	$N_h h_1$	Δu	$\Delta_1 u$	Δz
1/50	28.2	$4.21E-03$	$5.32E-03$	$3.88E-02$
1/500	89.4	$7.56E-04$	$1.20E-03$	$1.15E-02$
1/1000	126.5	$5.52E-04$	$8.67E-04$	$8.17E-03$
1/2000	178.9	$4.65E-04$	$6.90E-04$	$6.06E-03$

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