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A CHARACTERIZATION OF COMPLETENESS OF GENERALIZED METRIC SPACES USING GENERALIZED BANACH CONTRACTION PRINCIPLE

Abstract. In this paper, introducing a contraction principle on generalized metric spaces, a generalization of Banach's fixed point theorem is obtained under the completeness condition of the space. Moreover, it is established that, using such contraction principle, completeness of the generalized metric space can be characterized.

1. Introduction and prerequisites

Several generalizations of the celebrated Banach's fixed point theorem [1] were studied by various eminent mathematicians of different times, e.g., [3], [4], [9], [10] and others. It was shown by Connell [5] that there exists some metric space which is not complete but every contraction on it has a fixed point. This establishes the fact that "completeness" of a metric space is only sufficient for achieving a unique fixed point of Banach's contraction. In [7], Kannan introduced another contraction principle, called Kannan contraction, and established a fixed point theorem in presence of completeness condition of a metric space. It was Subrahmanyam [10], who proved that Kannan's theorem actually characterizes metric completeness. However, Kannan's contraction principle is not a generalization of Banach's contraction principle. In [11], Suzuki has shown that metric completeness can be characterized by a family of functions satisfying a generalized Banach's contraction principle.

Recently, a generalization of metric space was introduced and studied by Branciari [2], after replacing the triangle inequality of a metric space by a more general inequality involving four points instead of three points. In the same paper [2], Branciari gave an example of a generalized metric

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space which is not a metric space. Following his definition, some fixed point theorems on such generalized metric space was also obtained by several researchers, such as, Lahiri and Das [8], Das [6], et. al.

In this paper, we have introduced a contraction principle and obtained a fixed point theorem for functions obeying such principle on a complete generalized metric space. We have also established that Banach's fixed point theorem for generalized metric space [2] follows as a corollary to our theorem. Moreover, in the last section of this paper, we have shown that completeness of generalized metric space can be achieved precisely via the existence of a unique fixed point of functions belonging to a family obeying such generalized Banach contraction principle.

We begin with a couple of definitions which we require in the sequel. Throughout this paper, we denote the set of natural numbers by $\mathbb{N}$, the set of all positive real numbers by $\mathbb{R}^+$ and use the abbreviation GMS for generalized metric spaces.

Definition 1.1. [2] Let X be a set and $d : X \times X \to \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct point $\xi, \eta \in X$, each is different from x and y, the following hold:

1. $d(x,y) = 0 \Leftrightarrow x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,y) \leq d(x,\xi) + d(\xi,\eta) + d(\eta,y)$

Then (X,d) is called a generalized metric space (in short, GMS).

Definition 1.2. [2] Let (X,d) be a GMS. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a Cauchy sequence if for any $\epsilon > 0$ there exists a natural number $n_\epsilon \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}, n \geq n_\epsilon$ one has $d(x_n, x_{n+m}) < \epsilon$.
Further, a GMS (X,d) is called complete if every Cauchy sequence in X converges.

Throughout this paper, for a mapping $T : X \to X$ and $x \in X$, we shall use the following notations:

- Tx will stand for $T(x)$ (the image of x under the map T)
- T^2x will denote the element $T(T(x))$ and hence, for each $n \in \mathbb{N}$, $T^n x = T(T^{n-1}(x))$.

2. Generalized Banach's contraction on GMS

Theorem 2.1. *Let (X,d) be a complete GMS, and T a mapping on X. Define a non-increasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r) = \begin{cases} 1, & if\ 0 \leq r \leq 1/3 \\ \frac{2(1-r)}{3r(r+1)}, & if\ 1/3 \leq r < 1. \end{cases}$$

Assume that, there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \Rightarrow \ d(Tx, Ty) \le rd(x, y) \ for \ all \ x, y \in X.$$

Then there exists a unique fixed point z of T, and $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Proof. Since $\theta(r) \le 1$, then $\theta(r)d(x, Tx) \le d(x, Tx)$ for all $x \in X$. Then by our hypothesis,

$$d(Tx, T^2x) \le rd(x, Tx) \ for \ all \ x \in X. \tag{1}$$

Now, fix $u \in X$, and construct a sequence $\{u_n\}$ such that, $u_n = T^n u$. Then by (1),

$$d(u_n, u_{n+1}) = d(T^n u, T^{n+1} u) \le r^n d(u, Tu).$$

Hence, $\{u_n\}$ is a Cauchy sequence. Since the space X is complete, the Cauchy sequence $\{u_n\}$ converges to some point $z \in X$. Now, we show that

$$d(Tx, z) \le rd(x, z), \quad \text{for all } x \in X - \{z\}. \tag{2}$$

For $x \in X - \{z\}$, there exists $v \in \mathbb{N}$ such that

$$d(u_n, z) \le d(x, z)/5 \quad \text{for all } n \in \mathbb{N} \ and \ n \ge v.$$

Let $\epsilon = d(x, z)/5$. Then, since $\{u_n\}$ is a Cauchy sequence and $d(u_n, z) \le d(x, z)/5$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\theta(r)d(u_n, Tu_n) &\le d(u_n, u_{n+1}) \\
&\le d(u_n, z) + d(z, u_m) + d(u_m, u_{n+1}), \ m \ge n \ge v \\
&< 3d(x, z)/5 \\
&= d(x, z) - d(x, z)/5 - d(x, z)/5 \\
&\le d(x, z) - d(u_n, u_{n+1}) - d(u_{n+1}, z) \\
&\le d(u_n, x).
\end{aligned}$$

Therefore $\theta(r)d(u_n, Tu_n) \le d(u_n, x)$. Hence, by our hypothesis,

$$d(u_{n+1}, Tx) \le rd(u_n, x) \ for \ n \ge v.$$

Then $n \to \infty$, we get

$$d(Tx, z) \le rd(x, z).$$

This completes the proof of (2). Now, if possible, let $T^j z \neq z$, for all $j \in \mathbb{N}$. Hence from (2),

$$d(T^{j+1}z, z) \le r^j d(Tz, z) \ for \ j \in \mathbb{N}. \tag{3}$$

We consider the following two cases:

1. $0 \le r \le 1/3$
2. $1/3 \le r < 1$.

In case 1, we note $3r^3 + r^2 + r < 1$ and $r^2 + 2r^3 < 1$. If we assume, $d(T^3z, z) < d(T^4z, T^3z)$ then we have,

$$\begin{aligned} d(z,Tz) &\leq d(z,T^3z) + d(T^3z,T^4z) + d(T^4z,Tz) \\ &< 2d(T^3z,T^4z) + d(T^4z,T^3z) + d(T^3z,T^2z) + d(T^2z,Tz) \\ &< 3r^3d(z,Tz) + r^2d(z,Tz) + rd(z,Tz) \\ &< 2r^2d(z,Tz) + r^2d(Tz,z) \\ &= (3r^3 + r^2 + r)d(z,Tz) < d(z,Tz) \qquad \text{a contradiction.} \end{aligned}$$

So, $d(T^3z, z) \geq d(T^4z, T^3z) = \theta(r)d(T^4z, T^3z)$, since in this case $\theta(r) = 1$. Again by our hypothesis and equation (3) we have,

$$\begin{aligned} d(z,Tz) &\leq d(z,T^3z) + d(T^3z,T^4z) + d(T^4z,Tz) \\ &\leq r^2d(z,Tz) + r^3d(z,Tz) + r^3d(Tz,z) \\ &= 2r^2d(z,Tz) + rd(z,Tz) \\ &= (2r^3 + r^2)d(z,Tz) < d(z,Tz) \qquad \text{a contradiction.} \end{aligned}$$

In case 2, we note that for $x, y \in X$, either $\theta(r)d(x,Tx) < d(x,y)$ or $\theta(r)d(Tx,T^2x) < d(Tx,y)$ holds. Indeed, if $\theta(r)d(x,Tx) \geq d(x,y)$ and $\theta(r)d(Tx,T^2x) \geq d(Tx,y)$, then

$$\begin{aligned} d(x,Tx) &\leq d(x,T^2x) + d(T^2x,y) + d(y,Tx) \\ &< rd(x,Tx) + \theta(r)d(T^2x,T^3x) + \theta(r)d(Tx,T^2x) \\ &< rd(x,Tx) + \theta(r)r^2d(x,Tx) + \theta(r)rd(x,Tx) \\ &< (r + \theta(r)r^2 + \theta(r)r)d(x,Tx) \\ &= (r + \theta(r)r(r+1))d(x,Tx) \\ &= r(1 + \theta(r)(r+1))d(x,Tx) < d(x,Tx). \end{aligned}$$

This is a contradiction. Since, either $\theta(r)d(u_{2n}, u_{2n+1}) \leq d(u_{2n}, z)$, or $\theta(r)$ $d(u_{2n+1}, u_{2n+2}) \leq d(u_{2n+1}, z)$ holds for every $n \in \mathbb{N}$, $d(u_{2n+1}, Tz) \leq rd(u_{2n}, z)$ or, $d(u_{2n+2}, Tz) \leq rd(u_{2n+1}, z)$ holds for every $n \in \mathbb{N}$. Again, $\{u_n\}$ converges to z implies that the above inequalities ensure that there exists a subsequence of $\{u_n\}$ which converges to Tz. Consequently, $Tz = z$, a contradiction. Therefore, in both cases, there exists $j \in \mathbb{N}$ such that $T^jz = z$. As $\{T^nz\}$ is a Cauchy sequence, we obtain $Tz = z$; i.e., z is a fixed point of T.

Now if possible, let z_1 be another fixed point of T. Then by equation (2), $\forall x \in X$

$$d(Tx, z) \leq rd(x, z).$$

$$\begin{aligned} d(z_1,z) &\leq d(z_1,Tz_1) + d(Tz_1,T^2z_1) + d(T^2z_1,z) \\ &\leq rd(z_1,z_1) + rd(z_1,Tz_1) + rd(Tz_1,z) \\ &\leq r^2d(z,z_1) < d(z,z_1) \qquad \text{a contradiction.} \end{aligned}$$

Thus, the fixed point z of T is unique, which completes the proof. ∎

Banach's fixed point theorem on GMS, proved by Branciari [2], is immediate from the above result.

Corollary 2.1. (Banach's fixed point theorem on GMS) *Let (X, d) be a complete GMS and T a mapping on X such that for each $x, y \in X$, there is some $r \in (0,1)$ and*

$$d(Tx, Ty) \leq rd(x, y),$$

then there exists a unique fixed point $z \in X$ of T and $\lim_{n\to\infty} T^n x = z$, for all $x \in X$.

The following theorem establishes that $\theta(r)$ is the best constant for every $r \in [0, 1)$:

Theorem 2.2. *Define a function $\theta(r)$ from $[0,1)$ onto $(0,1]$ by*

$$\theta(r) = \begin{cases} 1, & if\ 0 \leq r \leq 1/3, \\ \frac{2(1-r)}{3r(r+1)}, & if\ 1/3 \leq r < 1. \end{cases}$$

Then for each $r \in [0, 1)$, there exists a complete GMS (X, d) and a mapping T on X such that T does not have a fixed point and

$$\theta(r)d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) \leq rd(x, y) \quad \textit{for all } x, y \in X.$$

Proof. In the case where $0 \leq r \leq 1/3$, define a subset X of the Euclidean space $\mathbb{R}$ by $X = \{+1, -1\}$. An Euclidean space being a complete metric space is also a complete GMS. Then of course, X (being closed in the Euclidean space) becomes a complete GMS too. Also, define a mapping T on X by $Tx = -x$ for $x \in X$. Then T does not have a fixed point, and

$$\begin{aligned} \theta(r)d(x, Tx) &= 1 \cdot d(1, -1) \\ &= 2 \geq d(x, y) \quad \text{for all } x, y \in X. \end{aligned}$$

Again in the case where $1/3 \leq r < 1$, define a subset X of the Euclidean space $\mathbb{R}$ as follows

$$X = \{0, 1\} \cup \{x_n : n \in \mathbb{N} \cup \{0\}\}$$
$$\text{where } x_n = (1-r)(-r)^n for\ n \in \mathbb{N} \cup \{0\}.$$

It is easy to observe that X in this case is also a complete GMS.

Now, define a mapping T on X by $T0 = 1$, $T1 = x_0$ and $Tx_n = x_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. We prove that T satisfies the conclusion. Consider the following cases:

$$\begin{aligned} d(T0, T1) = d(1, x_0) &= |1 - x_0| \\ &= |1 - 1 + r| = r = rd(0, 1). \end{aligned}$$

It is easy to verify that $d(T0, Tx_n) \leq rd(0, x_n)$ as well as $d(T1, Tx_n) \leq rd(1, x_n)$. Moreover, $d(Tx_n, Tx_m) = rd(x_n, x_m)$, for $m, n \in \mathbb{N} \cup \{0\}$. This completes the proof. ∎

3. Completeness of GMS – a characterization

In this section, we see that completeness of a generalized metric space can be determined precisely by the existence of a unique fixed point of classes of functions satisfying the contraction principle introduced in the previous section.

Theorem 3.1. *Let, (X,d) be a generalized metric space and define a function θ as*

$$\theta(r) = \begin{cases} 1, & if\ 0 \le r \le 1/3, \\ \frac{2(1-r)}{3r(r+1)}, & if\ 1/3 \le r < 1. \end{cases}$$

For $r \in [0,1)$ and $\eta \in (0, \theta(r)]$, let $A_{r,\eta}$ be the family of functions T on X satisfying the following:

(a) *For $x, y \in X$,*

$$\eta d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le r d(x, y).$$

Let $B_{r,\eta}$ be the family of mappings T on X satisfying (a) *and the following:*

(b) *$T(X)$ is countably infinite.*
(c) *Every subset of $T(X)$ is closed.*
Then the following are equivalent:

(i) *X is complete.*
(ii) *Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0,1)$.*
(iii) *There exist $r \in [0,1)$ and $\eta \in (0, \theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.*

Proof. By Theorem (2.1), (i)$\Rightarrow$(ii).

Since $B_{r,\eta} \subset A_{r,\theta(r)}$ for $r \in [0,1)$ and $\eta \in (0, \theta(r)]$, (ii)$\Rightarrow$(iii).

To prove (iii)$\Rightarrow$(i). We assume (iii), i.e. there exists $r \in [0,1)$ and $\eta \in (0, \theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point. If possible let X be not complete. So there exists a Cauchy sequence $\{u_n\}$ which does not converge. Define a function from X into $[0,\infty)$ by $f(x) = \lim_{n\to\infty} d(x, u_n)$ for $x \in X$. We note that f is well defined because $\{d(x, u_n)\}$ is a Cauchy sequence for every $x \in X$. Hence, $f(x) - f(y) \le d(x,y) \le f(x) + f(y)$ for $x, y \in X$, $f(x) > 0$ for all $x \in X$ and $f(u_n) = 0$. Define a function T on X as follows: for each $x \in X$, since $f(x) > 0$ and $\lim_{n\to\infty} f(u_n) = 0$, there exists $n_u \in \mathbb{N}$ satisfying

$$f(u_{n_u}) \le (\eta r/(3 + r\eta)) f(x).$$

We put $Tx = u_{n_u}$. Clearly, $f(Tx) \le (\eta r/(3 + r\eta)) f(x)$ and $Tx \in \{u_n : n \in \mathbb{N}\}$ for all $x \in X$. Then $Tx \ne x$ for all $x \in X$ because $f(Tx) < f(x)$, i.e., T does not have a fixed point. Since $T(x) \subset \{u_n : n \in N\}$, (b) holds.

Also it is easy to prove (c). Now, fix $x, y \in X$ with $\eta d(x, Tx) \leq d(x, y)$. In the case where $f(y) > 2f(x)$,

$$\begin{aligned} d(Tx, Ty) &\leq f(Tx) + f(Ty) \\ &\leq (r\eta/(3 + r\eta))(f(x) + f(y)) \\ &\leq (r/3)(f(x) + f(y)) \\ &\leq (r/3)(f(x) + f(y)) + (2r/3)(f(y) - 2f(x)) \\ &= r(f(y) - f(x)) \leq rd(x, y). \end{aligned}$$

In the other case where $f(y) \leq 2f(x)$, we have

$$\begin{aligned} d(x, y) &\geq \eta d(x, Tx) \geq \eta(f(x) - f(Tx)) \\ &\geq \eta(1 - (\eta r/3 + \eta r))f(x) \\ &= (3\eta/3 + \eta r)f(x) \end{aligned}$$

and hence,

$$\begin{aligned} d(Tx, Ty) &\leq f(Tx) + f(Ty) \leq (\eta r/3 + \eta r)(f(x) + f(y)) \\ &\leq (3\eta r/3 + \eta r)f(x) \\ &\leq rd(x, y). \end{aligned}$$

So, (a) holds, that is, $T \in B_{r,\eta}$. By (3) T has a fixed point which yields a contradiction. Hence X is complete. ∎

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