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# $L$-APPROACH MEROTOPIES AND THEIR CATEGORICAL PERSPECTIVE

**Abstract.** In the present paper, we have made a category theoretic and lattice theoretic study of some nearness-like structures in the $L$-approach theory. Using $L$-grills, the notion of $L$-approach grill structure is introduced as a characterization of $L$-approach grill merotopy on $X$; their categorical perspectives and implications are also investigated. A number of illustrative examples are included.

## 1. Introduction

A topological space is the result of axiomatization of the concept of nearness between a set and a point. Nearness-like structures on a non-empty set, such as proximity [19, 20], uniformity [29], merotopy [8], contiguity [7] and generalizations and variations of these concepts have been created to handle problems of a 'topological' nature. Merotopic spaces for which a relationship between the near collection and the closure operator induced by the merotopy exists were termed as nearness spaces by Herrlich [6] in 1974. He observed that the three concepts micromeric, farness and nearness are equivalent. Nearness spaces generalize proximity spaces, uniform spaces and (symmetric) topological spaces, and are convenient tools for the study of extensions of topological spaces from a categorical viewpoint (see also [13]). The applications of near sets in the field of computer science can be seen in [21, 22, 23]. To measure the degree of nearness between a set and a point, the notion of approach structure was introduced (see [16]). The notion of distance in approach spaces is closely related to the notion of nearness. Moreover, proximity and nearness concepts arising naturally in the context of approach spaces can be seen in [18]. Lowen et al. employed the theory of associated merotopy in the completion of approach spaces (see [18]). Thus experimenting with the nearness-like concepts in approach theory became

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mandatory. Keeping this problem in mind, approach merotopic spaces were introduced by Lowen and Lee [17]. An approach merotopy measures to what degree a collection of sets contains small members. It has been shown by H. Herrlich [6] that the concepts *collections of sets containing arbitrarily small members* and *nearness of collection of sets* are equivalent. Khare and Singh [12] studied merotopic spaces in $L$-fuzzy theory (see also [11, 10, 14, 24, 25, 26, 27]).

The main aim of this paper is to introduce and make category theoretic and lattice theoretic study of the notions of $L$-approach premerotopy, $L$-approach merotopy, $L$-approach contiguity and $L$-approach grill merotopy on a non-empty set $X$. Prerequisites for the present paper are collected in Section 2. In Section 3, we introduce and study the first three structures category theoretically; the interrelations between them are also established. Various nontrivial examples of the above structures are included. We consider here $L$-merotopic spaces involving near members, while Lowen and Lee considered in [17] merotopic spaces involving micromeric collections. $L$-merotopy, if defined using micromeric collections, is not equivalent to the form of merotopy described in Khare and Singh [11, 12]. This is due to ambiguous behavior of $L$-fuzzy points. In Section 4, it is shown that the categories **LMER**, **LCON** and **LPMER** can be embedded as full subcategories in **LAMER**, **LACON** and **LAPMER** respectively. The order structures of the above mentioned near families on $X$ are also discussed. By adding an $L$-grill generated axiom in Section 5, we define $L$-approach grill merotopy on $X$ and we introduce its equivalent characterization – $L$-approach grill structure, by restricting $\nu$ to the family of all $L$-grills on $X$. The category **LAGRL**, with objects as $L$-approach grill merotopic spaces along with contractions, is shown to be a topological construct. In Section 6, we embed the category **LGRL** of $L$-grill merotopic spaces and $L$-merotopic maps in **LAGRL** both bireflectively and bicoreflectively. The theory of $L$-approach premerotopy and $L$-approach contiguity developed here contributes to the classical theory of approach merotopic structures on $X$.

## 2. Preliminaries and basic results

Throughout this paper, let $X$ be a non-empty ordinary set and $(L, \leq)$ be a completely distributive complete lattice with order reversing involution $' : L \to L$, largest element 1 and smallest element 0. We denote by $J$ an arbitrary index set, by $\aleph_0$ the first infinite cardinal number, and by $|\mathcal{A}|$ the cardinality of $\mathcal{A}$, where $\mathcal{A} \subseteq L^X$. The definitions of an $L$-fuzzy subset, an $L$-fuzzy point, an $L$-fuzzy mapping, and related concepts are found in [15]. The set of all (finite) subsets of $L^X$ is denoted by $\mathcal{P}(L^X)$ ($\mathfrak{F}(L^X)$). For any $t \in L$, the mapping which sends each $x \in X$ to $t$ is denoted by $\mathbf{t}$. In particular, $\mathbf{0} \in L^X$ sends each $x \in X$ to the smallest element 0 of $L$. For $\mathcal{A}$, $\mathcal{B}$ subsets of $L^X$, we say $\mathcal{A} \vee \mathcal{B} \equiv \{f \vee g : f \in \mathcal{A},\, g \in \mathcal{B}\}$; $\mathcal{A}$ *corefines* $\mathcal{B}$ (written as $\mathcal{A} \prec \mathcal{B}$) if and only if for all $f \in \mathcal{A}$ there exists $g \in \mathcal{B}$ such that $g \leq f$. An *atom* in $L$ is a non-zero element $a \in L$ such that there is no other element $b \in L$ with $b \neq 0$ and $b \leq a$ (the theory of atoms in the more generalized setting of effect algebras can be found in [3, 9]). Note that each singleton in $(\mathcal{P}(X), \subseteq)$ is an atom. A non-zero element $m \in L$ is called a *molecule* of $L$ if for every $a, b \in L$, $m \leq a \vee b \Longrightarrow m \leq a$ or $m \leq b$. The set of all molecules of $L$ is denoted by $M(L)$. Note that $m$ is a molecule (atom) of $L$ if and only if $x_m$ is a molecule (atom) of $L^X$, for each $x \in X$. Further every atom of $L$ is a molecule and each fuzzy point in $I^X$, where $I \equiv [0,\, 1]$, is a molecule (see [15]). If $L$ is a completely distributive lattice, then each element in $L$ is the supremum of some molecules in $L$ (see [5]).

For $\mathcal{A} \subseteq L^X$, $stack(\mathcal{A}) = \{f \in L^X : g \leq f \text{ for some } g \in \mathcal{A}\}$. An *L-filter* on $X$ is a non-empty subset $\mathcal{F}$ of $L^X$ satisfying: $\mathbf{0} \notin \mathcal{F}$; if $f \in \mathcal{F}$ and $f \leq g$, then $g \in \mathcal{F}$; and if $f \in \mathcal{F}$ and $g \in \mathcal{F}$, then $f \wedge g \in \mathcal{F}$. A maximal $L$-filter on $X$ is called an *L-ultrafilter* on $X$. Note that for any $x \in X$ and an atom $p$ of $L$, $[x_p] = \{f \in L^X : x_p \in f\}$ is an $L$-ultrafilter on $X$. An *L-grill* on $X$ is a subset $\mathcal{G}$ of $L^X$ satisfying: $\mathbf{0} \notin \mathcal{G}$; if $f \in \mathcal{G}$ and $f \leq g$, then $g \in \mathcal{G}$; and if $f \vee g \in \mathcal{G}$, then $f \in \mathcal{G}$ or $g \in \mathcal{G}$. Further if $p \in M(L)$, then $[x_p]$ is an $L$-grill on $X$ (see [15]). Let $\mathfrak{G}(X)$ denote the set of all $L$-grills on $X$.

**2.1** [11, 12]. Let $\xi \subseteq \mathcal{P}(L^X)$. Then $\xi$ is called an *L-merotopy* (*L-contiguity*) on $X$ provided, for $\mathcal{A}$, $\mathcal{B} \in \mathcal{P}(L^X)$ (with $|\mathcal{A}| < \aleph_0$ and $|\mathcal{B}| < \aleph_0$),

(M1) $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \in \xi \Longrightarrow \mathcal{A} \in \xi$,
(M2) $\bigwedge \mathcal{A} \neq \mathbf{0} \Longrightarrow \mathcal{A} \in \xi$,
(M3) $\emptyset \neq \xi \neq \mathcal{P}(L^X)$,
(M4) $\mathcal{A} \vee \mathcal{B} \in \xi \Longrightarrow \mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$.

The pair $(X,\, \xi)$ is called an *L-merotopic* (*L-contiguity*) *space.* If $\xi \subseteq \mathcal{P}(L^X)$ satisfies (M1), (M2) and (M3), then $\xi$ is an *L-premerotopy* on $X$ and $(X, \xi)$ is called an *L-premerotopic space.* For an $L$-merotopic ($L$-contiguity) space $(X,\, \xi)$, we define: $cl_\xi(f) = \bigvee\{x_p \in L^X : \{x_p,\, f\} \in \xi\}$, $f \in L^X$. Then $cl_\xi$ is an $L$-Čech closure operator (see [15]) on $X$.

An $L$-merotopy $\xi$ on $X$ is called an *L-nearness* if the following condition is satisfied:

(M5) $\{cl_\xi(f),\, cl_\xi(g)\} \in \xi \Longrightarrow \{f,\, g\} \in \xi$.

Then $cl_\xi$ is an $L$-fuzzy Kuratowski closure operator on $X$. Note that $\xi_d = \{\mathcal{A} \subseteq L^X : \mathbf{0} \notin \mathcal{A}\}$ is the largest $L$-merotopy ($L$-premerotopy) on $X$, called the *discrete* $L$-merotopy ($L$-premerotopy) and $\xi_i = \{\mathcal{A} \subseteq L^X : \bigwedge \mathcal{A} \neq \mathbf{0}\}$ is the smallest $L$-merotopy ($L$-premerotopy) on $X$, called the *indiscrete* $L$-

merotopy ($L$-premerotopy) on $X$. Similarly, $\xi_d$ and $\xi_i$ are defined in an $L$-contiguity space by taking $|\mathcal{A}| < \aleph_0$ (see [11]). We sometimes refer to the above mentioned topological structures as *nearness-like structures.*

## 3. $L$-approach merotopic structures

In this section, we axiomatize $L$-approach premerotopy, $L$-approach merotopy and $L$-approach contiguity on $X$ and investigate their categorical properties. One reason why we have taken $L$ to be a completely distributive complete lattice is that there are enough molecules in such a lattice (see [5]). For standard definitions in the the theory of categories we refer to [1], and for lattices see [5, 15].

**Definition 3.1.** A function $\nu : \mathcal{P}(L^X) \to [0, \infty]$ ($\nu : \mathfrak{F}(L^X) \to [0,\infty]$) is called an $L$-approach merotopy ($L$-approach contiguity) on $X$ if for any $\mathcal{A}$, $\mathcal{B} \in \mathcal{P}(L^X)$ ($\mathfrak{F}(L^X)$),

(LAM1) $\mathcal{A} \prec \mathcal{B} \Longrightarrow \nu(\mathcal{A}) \leq \nu(\mathcal{B})$,
(LAM2) $\bigwedge \mathcal{A} \neq \boldsymbol{0} \Longrightarrow \nu(\mathcal{A}) = 0$,
(LAM3) $\boldsymbol{0} \in \mathcal{A} \Longrightarrow \nu(\mathcal{A}) = \infty$,
(LAM4) $\nu(\mathcal{A} \vee \mathcal{B}) \geq \nu(\mathcal{A}) \wedge \nu(\mathcal{B})$.

The pair $(X, \nu)$ is called an $L$-approach merotopic ($L$-approach contiguity) space. For an $L$-approach merotopic ($L$-approach contiguity) space $(X, \nu)$, we define: $cl_\nu(f) = \bigvee\{x_p \in L^X : \nu(\{x_p, f\}) < \infty\}$, $f \in L^X$. Then $cl_\nu$ is an $L$-Čech closure operator on $X$.

An $L$-approach merotopy $\nu$ on $X$ is called an $L$-approach nearness on $X$ if the following condition is satisfied:

(LAM5) $\nu(\{cl_\nu(f), cl_\nu(g)\}) \geq \nu(\{f, g\})$.

In this case, $cl_\nu$ is an $L$-fuzzy Kuratowski closure operator on $X$.

If a function $\nu : \mathcal{P}(L^X) \longrightarrow [0, \infty]$ satisfies (LAM1), (LAM2) and (LAM3), then $\nu$ is called an $L$-approach premerotopy on $X$ and $(X, \nu)$ is called an $L$-approach premerotopic space.

**Remark 3.1.** When $L = \{0, 1\}$, that is the classical case, condition (LAM5) is taken as

(LAM5′) $\nu(\{cl_\nu(A) : A \in \mathcal{A}\}) \geq \nu(\mathcal{A})$.

In the generalized $L$-fuzzy case, when $L \neq \{0, 1\}$, (LAM5′) is replaced by (LAM5) otherwise, as noted by Artico and Moresco [2], the generalized $L$-fuzzy case coincides with the classical case.

The following observations are obvious:

(1) if $\nu$ is an $L$-approach merotopy on $X$, then $\nu|_{\mathfrak{F}(L^X)}$ is an $L$-approach contiguity on $X$ and their closures coincide;

(2) if $\nu$ is an $L$-approach premerotopy or $L$-approach merotopy ($L$-approach contiguity) on $X$, then $\nu(\emptyset) = 0$ (here we have assumed that $\bigwedge \emptyset = 1$) and $\nu(\mathcal{A}) = \nu(\text{stack}(\mathcal{A}))$, for any $\mathcal{A} \in \mathcal{P}(L^X)$ $(\mathcal{A}, \text{stack}(\mathcal{A}) \in \mathfrak{F}(L^X))$.

**Examples 3.1.**

(i) Let $(X, cl)$ be an $L$-closure space, $r, r_1 \in (0, \infty)$ and $\mathcal{B} \subseteq L^X$ such that $\bigwedge\{cl(f) : f \in \mathcal{B}\} = \mathbf{0}$ and $\mathbf{0} \notin \mathcal{B}$. Define $\nu_{r,r_1} : \mathcal{P}(L^X) \to [0, \infty]$ as follows for $\mathcal{A} \subseteq L^X$,

$$\nu_{r,r_1}(\mathcal{A}) = \begin{cases} 0, & \text{if } \bigwedge\{cl(f) : f \in \mathcal{A}\} \neq \mathbf{0}, \\ \infty, & \text{if } \mathbf{0} \in \mathcal{A}, \\ r, & \text{if } \mathcal{A} \prec \mathcal{B} \text{ and } \bigwedge\{cl(f) : f \in \mathcal{A}\} = \mathbf{0}, \\ r_1, & \text{otherwise.} \end{cases}$$

Then $\nu_{r,r_1}$ is an $L$-approach premerotopy which is not an $L$-approach merotopy on $X$ in general when $r < r_1$. (For instance, consider $X$ to be the set $\mathbb{R}$ of all real numbers with the usual topology on it, $\mathcal{B} = \{[1, 2], \{3\}\}, \mathcal{A} = \{[1, 1.5], \{3\}\}$ and $\mathcal{C} = \{[1.5, 2], \{3\}\}$. Then $\nu_{r,r_1}(\mathcal{A}) \wedge \nu_{r,r_1}(\mathcal{C}) = r_1$ and $\nu_{r,r_1}(\mathcal{A} \vee \mathcal{C}) = r$.) If $r = r_1$, then $\nu_{r,r}$ is an $L$-approach merotopy on $X$. One may observe that when $\nu_{r,r_1}$ takes values only 0 or $\infty$, either $\nu_{r,r_1}$ is reduced to an $L$-merotopy on $X$, or it is not even an $L$-premerotopy on $X$. So this example uses behavior of $L$-approach theory purely.

(ii) Let $(X, cl)$ be a closure space, $\nu : \mathcal{P}^2(X) \to [0, \infty]$ be an approach seminearness structure [17] (i.e., a $\{0, 1\}$-approach merotopy) or an approach premerotopy on $X$ and $m^*$ denote an outer measure on $X$. Define $\mu : \mathcal{P}^2(X) \to [0, \infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$, $\mu(\mathcal{A}) = \nu(\mathcal{A})$, if $\inf\{m^*(cl(A)) : A \in \mathcal{A}\} > 0$ or $\bigcap\{cl(A) : A \in \mathcal{A}\} \neq \emptyset$; and $\mu(\mathcal{A}) = \infty$, otherwise. Then $\mu$ is an approach premerotopy which is not an approach seminearness structure on $X$ in general.

(iii) Let $\nu$ be an $L$-approach premerotopy on $X$. Then $\tilde{\nu} : \mathcal{P}(L^X) \to [0, \infty]$ defined as, for $\mathcal{A} \subseteq L^X$,

$$\tilde{\nu}(\mathcal{A}) = \inf\{\nu(\mathcal{F}) : \mathcal{A} \prec \mathcal{F}, \text{ where } \mathcal{F} \text{ is an } L\text{-filter on X}\}$$

is an $L$-approach premerotopy on $X$ in general.

(iv) Let $(X, cl)$ be an $L$-closure space and $r \in (0, \infty]$. Define $\nu^r : \mathcal{P}(L^X) \to [0, \infty]$ as follows: for $\mathcal{A} \subseteq L^X$,

$$\nu^r(\mathcal{A}) = \begin{cases} 0, & \text{if for each } \mathcal{B} \subseteq \mathcal{A} \text{ with } |\mathcal{B}| < \aleph_0, \ \wedge\{cl(f) : f \in \mathcal{B}\} \neq \mathbf{0}, \\ \infty, & \text{if } \mathbf{0} \in \mathcal{A}, \\ r, & \text{otherwise.} \end{cases}$$

Then $\nu^r$ is an $L$-approach merotopy on $X$.

**Remark 3.2.** Let $X = \{1, 2, 3, 4\}$ with the discrete topology and $\mathcal{A} = \{\{1, 4\}, \{2, 3\}\}$. Then $\nu_{r,r_1}(\mathcal{A}) = r_1$ but $\tilde{\nu}_{r,r_1}(\mathcal{A}) = \infty$, where $\tilde{\nu}_{r,r_1}$ is the $L$-approach premerotopy on $X$ defined in Example 3.1(iii) when $\nu = \nu_{r,r_1}$. Thus $\tilde{\nu} \neq \nu$ in Example 3.1(iii) in general. ∎

More non-trivial examples of $L$-approach merotopies can be seen in Examples 5.1. We now present methods of obtaining an $L$-approach merotopy from an $L$-approach premerotopy and an $L$-approach contiguity on $X$.

**Proposition 3.1.** *Let $(X, \nu)$ be an $L$-approach premerotopic space. Define $\tilde{\nu} : \mathcal{P}(L^X) \to [0, \infty]$ as, for $\mathcal{A} \subseteq L^X$*

$$\tilde{\nu}(\mathcal{A}) = \begin{cases} 0, & \text{if there does not exist } \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n \text{ with } \nu(\mathcal{A}_i) = \infty \\ & \text{for all } i\,(1 \leq i \leq n) \text{ and } \mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}, \\ \infty, & \text{otherwise.} \end{cases}$$

*Then $\tilde{\nu}$ is an $L$-approach merotopy on $X$.*

**Proof.** The conditions (LAM1) and (LAM3) are obvious. Condition (LAM4) follows by noting that if $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}$ and $\mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_m \prec \mathcal{B}$, then $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \vee \mathcal{B}_1 \vee \mathcal{B}_2 \vee \cdots \vee \mathcal{B}_m \prec \mathcal{A} \vee \mathcal{B}$. Finally let $\bigwedge \mathcal{A} \neq \mathbf{0}$ but $\tilde{\nu}(\mathcal{A}) \neq 0$. Then there exist $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ with $\nu(\mathcal{A}_i) = \infty$ for all $i$, $1 \leq i \leq n$, and $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}$. Consequently, there exists $i \leq n$ such that $\bigwedge \mathcal{A}_i \neq \mathbf{0}$ implying $\nu(\mathcal{A}_i) = 0$, which is a contradiction. Therefore if $\bigwedge \mathcal{A} \neq \mathbf{0}$, then $\tilde{\nu}(\mathcal{A}) = 0$, and (LAM2) is satisfied. ∎

**Remark 3.3.** There is yet another method of converting an $L$-approach premerotopy to an $L$-approach merotopy as we may define $\hat{\nu}$ to be, $\hat{\nu}(\mathcal{A}) = \inf\{\nu(\mathcal{G}) : \mathcal{G} \in \mathfrak{G}(X) \text{ and } \mathcal{A} \prec \mathcal{G}\}$, for all $\mathcal{A} \subseteq L^X$. Then $\hat{\nu}$ is an $L$-approach merotopy on $X$ : the conditions (LAM1) and (LAM3) are obvious; and (LAM4) follows by noting that for $\mathcal{G} \in \mathfrak{G}(X)$ and $\mathcal{A}, \mathcal{B} \subseteq L^X, \mathcal{A} \vee \mathcal{B} \prec \mathcal{G}$ if and only if $\mathcal{A} \prec \mathcal{G}$ or $\mathcal{B} \prec \mathcal{G}$. For (LAM2), let $\bigwedge \mathcal{A} \neq \mathbf{0}$. Then there exist $p \in M(L)$ and $x \in X$ such that $x_p \in \bigwedge \mathcal{A}$. Thus $\mathcal{A} \subseteq [x_p]$ and consequently $\hat{\nu}(\mathcal{A}) = 0$. ∎

Let $\hat{\tilde{\nu}}_{r,r_1}$ be the $L$-approach merotopy obtained from $\tilde{\nu}_{r,r_1}$ of Remark 3.2 by the method given in Remark 3.3. Then $\hat{\tilde{\nu}}_{r,r_1}(\mathcal{A}) = \infty \neq r_1 = \nu_{r,r_1}(\mathcal{A})$. Hence from a given $L$-approach premerotopy we can obtain an $L$-approach merotopy on $X$ different from it.

**Proposition 3.2.** *Let $\nu$ be an $L$-approach contiguity on $X$. Define $\tilde{\nu} : \mathcal{P}(L^X) \to [0, \infty]$ as follows*

$$\tilde{\nu}(\mathcal{A}) = \bigvee \{\nu(\mathcal{B}) : \mathcal{B} \subseteq \mathcal{A} \text{ and } |\mathcal{B}| < \aleph_0\}, \text{ for every } \mathcal{A} \in \mathcal{P}(L^X).$$

*Then $\tilde{\nu}$ is an $L$-approach merotopy on $X$.*

**Proof.** Straightforward. ∎

**Definition 3.2.** Let $(X, \nu)$ and $(Y, \nu')$ be any $L$-approach premerotopic or $L$-approach merotopic ($L$-approach contiguity) spaces. An $L$-fuzzy mapping $T^{\rightarrow} : X \to Y$ is called a contraction if $\nu'(T^{\rightarrow}(\mathcal{A})) \leq \nu(\mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}(L^X)$ $(\mathcal{A} \in \mathfrak{F}(L^X))$.

**Remark 3.4.** For any $L$-approach premerotopic or $L$-approach merotopic ($L$-approach contiguity) spaces $(X, \nu)$ and $(Y, \nu')$, $T^{\rightarrow} : X \to Y$ is a contraction if and only if $\nu'(T^{\leftarrow}(\mathcal{A})) \geq \nu(\mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}(L^Y)$ $(\mathcal{A} \in \mathfrak{F}(L^Y))$.

Let **LAPMER**, **LAMER** and **LACON** denote the categories of $L$-approach premerotopies, $L$-approach merotopies and $L$-approach contiguities respectively and their contractions. Then clearly **LAMER** is a full subcategory of **LAPMER**. In view of Remark 3.4, the following propositions are routine to verify.

**Proposition 3.3.** *The category* **LACON** *is a bireflective full subcategory of* **LAMER**.

**Proposition 3.4.** *The category* **LAMER** *is a bicoreflective full subcategory of* **LAPMER**.

**Theorem 3.1.** *The category* **LAMER** (**LACON**) *is a topological construct.*

**Proof.** The category **LAMER** is clearly concrete. Consider a source $(T_j^{\rightarrow} : X \to X_j)_{j \in J}$ in **LAMER** for any family $((X_j, \nu_j))_{j \in J}$ of **LAMER**-objects. Define $\nu : \mathcal{P}(L^X) \to [0, \infty]$ by

$$\nu(\mathcal{A}) = \sup \left\{ \inf_{i=1}^{n} \sup_{j \in J} \nu_j(T_j^{\rightarrow}(\mathcal{A}_i)) : (\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A}) \right\}, \text{ for every } \mathcal{A} \in \mathcal{P}(L^X),$$

where $\mathfrak{C}(\mathcal{A}) = \{(\mathcal{A}_i)_{i=1}^n : \bigvee_{i=1}^n \mathcal{A}_i \prec \mathcal{A}, n \in \mathbb{N}\}$; here $\mathbb{N}$ denotes the set of all natural numbers. Then $\nu$ satisfies (LAM1) and (LAM3) obviously. Let $\bigwedge \mathcal{A} \neq \mathbf{0}$. Then $\bigwedge(\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n) \neq \mathbf{0}$, for every $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A})$. Thus $\nu(\mathcal{A}) = 0$. Finally let $\mathcal{A}, \mathcal{B} \in \mathcal{P}(L^X)$, $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A})$ and $(\mathcal{B}_k)_{k=1}^m \in \mathfrak{C}(\mathcal{B})$. Then $(\mathcal{A}_i)_{i=1}^n \bigcup (\mathcal{B}_k)_{k=1}^m \in \mathfrak{C}(\mathcal{A} \vee \mathcal{B})$. Since $\mathfrak{C}(\mathcal{A}) \vee \mathfrak{C}(\mathcal{B}) = \{(\mathcal{A}_i)_{i=1}^n \bigcup (\mathcal{B}_k)_{k=1}^m : (\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A}) \text{ and } (\mathcal{B}_k)_{k=1}^m \in \mathfrak{C}(\mathcal{B})\}$, therefore $\mathfrak{C}(\mathcal{A}) \vee \mathfrak{C}(\mathcal{B}) \subseteq \mathfrak{C}(\mathcal{A} \vee \mathcal{B})$. Now let $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A} \vee \mathcal{B})$. Then $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A} \vee \mathcal{B} \prec \mathcal{A}, \mathcal{B}$. Thus $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A}) \vee \mathfrak{C}(\mathcal{B})$. Hence $\mathfrak{C}(\mathcal{A}) \vee \mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\mathcal{A} \vee \mathcal{B})$. Also $\emptyset \in \mathfrak{C}(\mathcal{A})$ and $\emptyset \in \mathfrak{C}(\mathcal{B})$ yields $\nu(\mathcal{A} \vee \mathcal{B}) \geq \nu(\mathcal{A}) \wedge \nu(\mathcal{B})$. Thus $\nu$ is an $L$-approach merotopy on $X$. Since $\sup_{j \in J} \nu_j(T_j^{\rightarrow}(\mathcal{A})) \leq \nu(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{P}(L^X)$, $T_j^{\rightarrow} : X \to X_j$ is a contraction for each $j \in J$. To show that $\nu$ is initial, let $(Y, \nu')$ be any $L$-approach merotopic space and $U^{\rightarrow} : Y \to X$ be an $L$-fuzzy mapping such that $T_j^{\rightarrow} \circ U^{\rightarrow} : Y \to X_j$ is a contraction for each $j \in J$. Then for any $\mathcal{A} \in \mathcal{P}(L^Y)$ and $j \in J$, $\nu_j(T_j^{\rightarrow}(U^{\rightarrow}(\mathcal{A}))) \leq \nu'(\mathcal{A})$ and consequently $\sup_{j \in J} \nu_j(T_j^{\rightarrow}(U^{\rightarrow}(\mathcal{A}))) \leq \nu'(\mathcal{A})$. So for any $\mathcal{B} \in$

$\mathcal{P}(L^X)$, we have $\sup_{j\in J}\nu_j(T_j^{\rightarrow}(\mathcal{B})) \leq \nu'(U^{\leftarrow}(\mathcal{B}))$. Let $\nu'(\mathcal{A}) < \nu(U^{\rightarrow}(\mathcal{A}))$ for some $\mathcal{A}\in\mathcal{P}(L^Y)$. Then there is $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(U^{\rightarrow}(\mathcal{A}))$ such that $\nu'(\mathcal{A}) < \inf_{i=1}^n \sup_{j\in J}\nu_j(T_j^{\rightarrow}(\mathcal{A}_i))$ and therefore we get

$$\begin{aligned}\nu'(\mathcal{A}) < \inf_{i=1}^{n}\nu'(U^{\leftarrow}(\mathcal{A}_i)) &\leq \nu'(U^{\leftarrow}(\mathcal{A}_1)\vee U^{\leftarrow}(\mathcal{A}_2)\vee\cdots\vee U^{\leftarrow}(\mathcal{A}_n))\\ &= \nu'(U^{\leftarrow}(\mathcal{A}_1\vee\mathcal{A}_2\vee\cdots\vee\mathcal{A}_n))\\ &\leq \nu'(U^{\leftarrow}U^{\rightarrow}(\mathcal{A})) \leq \nu'(\mathcal{A}),\end{aligned}$$

which is absurd. Therefore $\nu(U^{\rightarrow}(\mathcal{A})) \leq \nu'(\mathcal{A})$ for any $\mathcal{A}\in\mathcal{P}(L^Y)$, and hence $U^{\rightarrow}: Y\to X$ is a contraction implying that $\nu$ is the initial structure. It may be observed that this $\nu$ is unique. For, let $\tilde{\nu}$ be any initial $L$-approach merotopy on $X$, and $\mathcal{A}\in\mathcal{P}(L^X)$. Then $\tilde{\nu}(\mathcal{A}) \leq \nu(\mathcal{A})$. Suppose that $\tilde{\nu}(\mathcal{A}) < \nu(\mathcal{A})$. Therefore there exists $(\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A})$ such that $\tilde{\nu}(\mathcal{A}) < \inf_{i=1}^n\sup_{j\in J}\nu_j(T_j^{\rightarrow}(\mathcal{A}_i))$. Thus for all $i=1,2,\ldots,n$, we have $\tilde{\nu}(\mathcal{A}) < \sup_{j\in J}\nu_j(T_j^{\rightarrow}(\mathcal{A}_i)) \leq \tilde{\nu}(\mathcal{A}_i)$. Consequently $\tilde{\nu}(\mathcal{A}) < \inf_{i=1}^n\tilde{\nu}(\mathcal{A}_i)$ and hence $(\mathcal{A}_i)_{i=1}^n \notin \mathfrak{C}(\mathcal{A})$, which is a contradiction. Thus $\tilde{\nu}(\mathcal{A}) = \nu(\mathcal{A})$. As a resultant, **LAMER** is a topological construct. The proof for **LACON** follows in the same manner by replacing $\mathcal{P}(L^X)$ and $\mathcal{P}(L^Y)$ by $\mathfrak{F}(L^X)$ and $\mathfrak{F}(L^Y)$ respectively. ∎

It has now been proved that the categories **LAMER** and **LACON** are topological categories. So we construct here their exact final structures.

**Proposition 3.5.** *For any family* $((X_j,\nu_j))_{j\in J}$ *of* **LAMER** (**LACON**)-*objects and a sink* $(T_j^{\rightarrow}: X_j\to X)_{j\in J}$, *the map* $\nu:\mathcal{P}(L^X)\to[0,\infty]$ $(\nu:\mathfrak{F}(L^X)\to[0,\infty])$ *defined by, for* $\mathcal{A}\in\mathcal{P}(L^X)$

$$\begin{aligned}\nu(\mathcal{A}) &= \begin{cases}0, & \text{if } \bigwedge\mathcal{A}\neq\mathbf{0},\\ \inf_{j\in J}\nu_j(T_j^{\leftarrow}(\mathcal{A})), & \text{otherwise}\end{cases}\\ &= \begin{cases}0, & \text{if } \bigwedge\mathcal{A}\neq\mathbf{0},\\ \inf\{\nu_j(\mathcal{A}_j):\mathcal{A}_j\in\mathcal{P}(L^{X_j}) \text{ for some}\\ j\in J \text{ such that } \mathcal{A}\subseteq\operatorname{stack}(T_j^{\rightarrow}(\mathcal{A}_j))\}, & \text{otherwise}\end{cases}\end{aligned}$$

*is the final $L$-approach merotopy ($L$-approach contiguity) on $X$.*

**Proof.** Firstly, we prove that the above two definitions are equivalent. Let $\mathcal{A}\in\mathcal{P}(L^X)$ such that $\bigwedge\mathcal{A}=\mathbf{0}$, $j\in J$ and $\mathcal{A}_j\in\mathcal{P}(L^{X_j})$. Since $T_j^{\leftarrow}(\mathcal{A}) \subseteq T_j^{\leftarrow}(\operatorname{stack}(T_j^{\rightarrow}(\mathcal{A}_j))) \subseteq \operatorname{stack}(T_j^{\leftarrow}(T_j^{\rightarrow}(\mathcal{A}_j)))$, we get $T_j^{\leftarrow}(\mathcal{A}) \prec T_j^{\leftarrow}(T_j^{\rightarrow}(\mathcal{A}_j)) \prec \mathcal{A}_j$. Thus $\nu_j(T_j^{\leftarrow}(\mathcal{A})) \leq \nu_j(\mathcal{A}_j)$ and consequently

$$\inf_{j\in J}\nu_j(T_j^{\leftarrow}(\mathcal{A})) \leq \inf\{\nu_j(\mathcal{A}_j):\mathcal{A}_j\in\mathcal{P}(L^{X_j})$$

for some $j\in J$ such that $\mathcal{A}\subseteq\operatorname{stack}(T_j^{\rightarrow}(\mathcal{A}_j))\}$. The reverse inequality follows by noting that $\mathcal{A} \subseteq \text{stack}(T_j^{\rightarrow} T_j^{\leftarrow}(\mathcal{A}))$, for any $\mathcal{A} \in \mathcal{P}(L^X)$. Now we will show that $\nu$ is an $L$-approach merotopy on $X$. Clearly $\nu$ satisfies (LAM1), (LAM2) and (LAM3). For (LAM4), let $\mathcal{A}, \mathcal{B} \in \mathcal{P}(L^X)$ with $\bigwedge \mathcal{A} = \mathbf{0}$ and $\bigwedge \mathcal{B} = \mathbf{0}$. Then $\nu(\mathcal{A} \vee \mathcal{B}) = \inf_{j \in J} \nu_j(T_j^{\leftarrow}(\mathcal{A} \vee \mathcal{B})) = \inf_{j \in J} \nu_j(T_j^{\leftarrow}(\mathcal{A}) \vee T_j^{\leftarrow}(\mathcal{B})) \geq \inf_{j \in J}(\nu_j(T_j^{\leftarrow}(\mathcal{A})) \wedge \nu_j(T_j^{\leftarrow}(\mathcal{B}))) = \nu(\mathcal{A}) \wedge \nu(\mathcal{B})$. By Remark 3.4, $T_j^{\rightarrow} : X_j \to X$ is a contraction for each $j \in J$. To show the finality of $\nu$, let $(Y, \nu')$ be any $L$-approach merotopic space and $U^{\rightarrow} : X \to Y$ an $L$-fuzzy mapping such that $U^{\rightarrow} \circ T_j^{\rightarrow} : X_j \to Y$ is a contraction for each $j \in J$ and $\mathcal{A} \in \mathcal{P}(L^X)$ such that $\bigwedge \mathcal{A} = \mathbf{0}$. Then $\nu(\mathcal{A}) = \inf_{j \in J} \nu_j(T_j^{\leftarrow}(\mathcal{A})) \geq \inf_{j \in J} \nu'(U^{\rightarrow}(T_j^{\leftarrow} T_j^{\rightarrow}(\mathcal{A})) \geq \nu'(U^{\rightarrow}(\mathcal{A}))$, for any $\mathcal{A} \in \mathcal{P}(L^X)$. Thus $U^{\rightarrow}$ is a contraction and $\nu$ is the final $L$-approach merotopy on $X$. The proof for the **LACON** case is the same. ∎

## 4. The supercategories of nearness-like structures

In this section, we obtain $L$-approach premerotopy, $L$-approach merotopy and $L$-approach contiguity from $L$-premerotopy, $L$-merotopy and $L$-contiguity respectively and vice versa. Let **LMER**, **LCON** and **LPMER** denote respectively the categories of $L$-merotopic, $L$-contiguity and $L$-premerotopic spaces, along with the respective $L$-merotopic, $L$-contiguity and $L$-premerotopic maps. Then **LMER**, **LCON** and **LPMER** is shown to be bicoreflectively and bireflectively embedded in **LAMER**, **LACON** and **LAPMER** respectively. Lattice structure on the families of all $L$-approach premerotopic, $L$-approach merotopic and $L$-approach contiguity spaces is also discussed. The results in this section also contribute to the classical theory when $\nu$ is an approach contiguity or an approach premerotopy on $X$.

**Definition 4.1.** For any $L$-merotopic or $L$-contiguity or $L$-premerotopic spaces $(X, \xi)$ and $(Y, \xi')$, an $L$-fuzzy map $T^{\rightarrow} : X \to Y$ is called an $L$-merotopic or $L$-contiguity or $L$-premerotopic map respectively if $\mathcal{A} \in \xi \Rightarrow T^{\rightarrow}(\mathcal{A}) \in \xi'$.

**Proposition 4.1.** *Let $(X, \xi)$ be an $L$-premerotopic or $L$-merotopic ($L$-contiguity) space. Then $\nu_\xi : \mathcal{P}(L^X) \to [0, \infty]$ ($\nu_\xi : \mathfrak{F}(L^X) \to [0, \infty]$) defined as*

*for $\mathcal{A} \in \mathcal{P}(L^X)$ ($\mathcal{A} \in \mathfrak{F}(L^X)$),*

$$\nu_\xi(\mathcal{A}) = \begin{cases} 0, & \text{if } \ \mathcal{A} \in \xi, \\ \infty, & \text{otherwise} \end{cases}$$

*is an $L$-approach premerotopy or $L$-approach merotopy ($L$-approach contiguity) respectively on $X$.*

**Proof.** Straightforward. ∎

**Remark 4.1.** Let $(X, \xi)$ and $(Y, \xi')$ be $L$-merotopic or $L$-contiguity or $L$-premerotopic spaces. Then the structure $\nu_\xi$ defined in the above proposition is called the induced $L$-approach merotopy or $L$-approach contiguity or $L$-approach premerotopy on $X$ respectively, and $T^\to : (X, \xi) \to (Y, \xi')$ is an $L$-merotopic or $L$-contiguity or $L$-premerotopic map if and only if $T^\to : (X, \nu_\xi) \to (Y, \nu_{\xi'})$ is a contraction.

Thus **LMER** is embedded as a full subcategory in **LAMER** by the functor $F$ : **LMER** $\to$ **LAMER** such that $F((X, \xi)) = (X, \nu_\xi)$ and $F(T^\to) = T^\to$. Similarly, **LPMER** and **LCON** are embedded as full subcategories in **LAPMER** and **LACON** respectively.

**Proposition 4.2.** *An $L$-approach merotopic or $L$-approach contiguity or $L$-approach premerotopic space $(X, \nu)$ is induced by an $L$-merotopic or $L$-contiguity or $L$-premerotopic space $(X, \xi)$ respectively if and only if $\nu(\mathcal{P}(L^X)) \subseteq \{0, \infty\}$ $(\nu(\mathfrak{F}(L^X)) \subseteq \{0, \infty\}$, where appropriate).*

**Proof.** Let $\nu$ be an $L$-approach merotopy on $X$ such that $\nu(\mathcal{P}(L^X)) \subseteq \{0, \infty\}$, and $\xi_\nu = \{\mathcal{A} \in \mathcal{P}(L^X) : \nu(\mathcal{A}) = 0\}$. Then $\xi_\nu$ is an $L$-merotopy on $X$ and the induced $L$-approach merotopy $\nu_{\xi_\nu}$ coincides with $\nu$. The converse is obvious. The proof for $L$-approach premerotopy ($L$-approach contiguity) on $X$ follows similarly (by replacing $\mathcal{P}(L^X)$ by $\mathfrak{F}(L^X)$). ∎

For any $L$-approach merotopic or $L$-approach contiguity or $L$-approach premerotopic space $(X, \nu)$, the pair $(X, \xi_\nu)$, defined in Proposition 4.2, is an $L$-merotopic or $L$-contiguity or $L$-premerotopic space respectively. Also, for any $L$-approach merotopic or $L$-approach contiguity or $L$-approach premerotopic spaces $(X, \nu)$ and $(Y, \nu')$, if an $L$-fuzzy map $T^\to : (X, \nu) \to (Y, \nu')$ is a contraction, then $T^\to : (X, \xi_\nu) \to (Y, \xi_{\nu'})$ is an $L$-merotopic or $L$-contiguity or $L$-premerotopic map respectively. This, therefore, defines functors $G$ : **LAMER** $\to$ **LMER**, $G$ : **LACON** $\to$ **LCON** and $G$ : **LAPMER** $\to$ **LPMER** by, $G((X, \nu)) = (X, \xi_\nu)$ and $G(T^\to) = T^\to$. ∎

**Theorem 4.1.** *The categories* **LMER**, **LCON** *and* **LPMER** *are bicoreflective subcategories of* **LAMER**, **LACON** *and* **LAPMER**, *respectively.*

**Proof.** For any $L$-approach merotopy ($L$-approach contiguity, $L$-approach premerotopy respectively) $\nu$ on $X$, the $L$-fuzzy identity mapping $1_X^\to : (X, \nu_{\xi_\nu}) \to (X, \nu)$ is an **LMER**-bicoreflection (**LCON**-bicoreflection, **LPMER**-bicoreflection respectively) of $(X, \nu)$ : if $T^\to : (Y, \nu') \to (X, \nu)$ is a contraction, $\mathcal{B} \in \mathcal{P}(L^X)$ and $\nu_{\xi_\nu}(\mathcal{B}) = 0$, then $\nu(\mathcal{B}) = 0$ which in turn yields that $\nu'(T^\leftarrow(\mathcal{B})) = 0$. Hence $T^\to : (Y, \nu') \to (X, \nu_{\xi_\nu})$ is a contraction. ∎

For any $L$-approach premerotopic or $L$-approach merotopic ($L$-approach contiguity) space $(X, \nu)$, $\xi^\nu = \{\mathcal{A} \in \mathcal{P}(L^X) : \nu(\mathcal{A}) < \infty\}$ $(\xi^\nu = \{\mathcal{A} \in$

$\mathfrak{F}(L^X) : \nu(\mathcal{A}) < \infty\})$ is also an $L$-premerotopy or $L$-merotopy ($L$-contiguity) on $X$ respectively; and for any $L$-approach merotopic or $L$-approach contiguity or $L$-approach premerotopic spaces $(X, \nu)$ and $(Y, \nu')$, if an $L$-fuzzy map $T^{\rightarrow} : (X, \nu) \rightarrow (Y, \nu')$ is a contraction, then $T^{\rightarrow} : (X, \xi^{\nu}) \rightarrow (Y, \xi^{\nu'})$ is an $L$-merotopic or $L$-contiguity or $L$-premerotopic map respectively. So we have functors $G' :$ **LAMER** $\rightarrow$ **LMER**, $G' :$ **LACON** $\rightarrow$ **LCON** and $G' :$ **LAPMER** $\rightarrow$ **LPMER** defined as: $G'((X, \nu)) = (X, \xi^{\nu})$ and $G'(T^{\rightarrow}) = T^{\rightarrow}$.

**Theorem 4.2.** *The categories* **LMER**, **LCON** *and* **LPMER** *are bireflective subcategories of* **LAMER**, **LACON** *and* **LAPMER** *respectively.*

**Proof.** For any $L$-approach merotopy ($L$-approach contiguity, $L$-approach premerotopy respectively) $\nu$ on $X$, the $L$-fuzzy identity mapping $1_X^{\rightarrow} : (X, \nu) \rightarrow (X, \nu_{\xi^{\nu}})$ is an **LMER**-bireflection (**LCON**-bireflection, **LPMER**-bireflection respectively) of $(X, \nu)$: if $T^{\rightarrow} : (X, \nu) \rightarrow (Y, \nu')$ is a contraction, $\mathcal{B} \in \mathcal{P}(L^Y)$ and $\nu_{\xi^{\nu}}(T^{\leftarrow}(\mathcal{B})) = 0$, then $\nu(T^{\leftarrow}(\mathcal{B})) = 0$. Consequently, $\nu'((T^{\rightarrow} \circ T^{\leftarrow})(\mathcal{B})) = 0$ which in turn yields that $\nu'(\mathcal{B}) = 0$. Hence $T^{\rightarrow} : (X, \nu_{\xi^{\nu}}) \rightarrow (Y, \nu')$ is a contraction. ■

We now concentrate on the ordering in $L$-approach premerotopy, $L$-approach merotopy and $L$-approach contiguity spaces and discuss their lattice structure, giving the exact structure of meets and joins.

**Definition 4.2.** For any set $X$ and $L$-approach merotopies or $L$-approach contiguities or $L$-approach premerotopies $\nu$ and $\nu'$ on $X$, $\nu$ is said to be finer than $\nu'$ or $\nu'$ is said to be coarser than $\nu$, written as $\nu' \leq \nu$, if the map $1_X : (X, \nu) \rightarrow (X, \nu')$ is a contraction.

Note that the indiscrete $L$-merotopy, $L$-contiguity and $L$-premerotopy induce the discrete $L$-approach merotopy, $L$-approach contiguity and $L$-approach premerotopy $\nu_d$ on $X$ respectively and the discrete $L$-merotopy, $L$-contiguity and $L$-premerotopy induce the indiscrete $L$-approach merotopy, $L$-approach contiguity and $L$-approach premerotopy $\nu_i$ on $X$, respectively.

**Proposition 4.3.** *The family of all $L$-approach merotopies ($L$-approach contiguities or $L$-approach premerotopies) on $X$ forms a completely distributive complete lattice with respect to the partial ordering $\leq$. The zero of this lattice is the indiscrete $L$-approach merotopy ($L$-approach contiguity or $L$-approach premerotopy, as appropriate) $\nu_i$ and the unit of this lattice is the discrete $L$-approach merotopy ($L$-approach contiguity or $L$-approach premerotopy, as appropiate) $\nu_d$ on $X$.*

**Proof.** If $\{\nu_j : j \in J\}$ is a family of $L$-approach premerotopies on $X$, then $\bigvee_{j \in J} \nu_j$ and $\bigwedge_{j \in J} \nu_j$ are the join and meet of the given family. Let

$\{\nu_j : j \in J\}$ be a family of $L$-approach merotopies on $X$. Then $\bigwedge_{j\in J} \nu_j$ is the infimum of the family. The supremum of the family is given by $\nu_{\sup} : \mathcal{P}(L^X) \to [0,\infty]$, where for $\mathcal{A} \in \mathcal{P}(L^X)$,

$$\nu_{\sup}(\mathcal{A}) = \sup\{\inf_{i=1}^{n} \sup_{j\in J} \nu_j(\mathcal{A}_i) : (\mathcal{A}_i)_{i=1}^n \in \mathfrak{C}(\mathcal{A})\},$$

here $\mathfrak{C}(\mathcal{A})$ is defined as in the proof of Theorem 3.1. The corresponding supremum and infimum for a family of $L$-approach contiguities on $X$ can be constructed by replacing $\mathcal{P}(L^X)$ by $\mathfrak{F}(L^X)$ in the supremum and infimum of a family of $L$-approach merotopies on $X$. ∎

## 5. The category LAGRL

In this section, we study categorically the theory of $L$-grill generated $L$-merotopies in the context of $L$-approach merotopy. Using this formulation, we obtain a simpler characterization of $L$-approach grill merotopies on $X$. The theory of an $L$-approach contiguity can be studied in this context only when $L^X$ is finite because only then $[x_p] \in \mathfrak{F}(L^X)$, where $x \in X$ and $p \in M(L)$. The theory of grills in near spaces can be seen in [4, 28].

**Definition 5.1.** An $L$-approach merotopy $\nu$ on $X$ is called an $L$-approach grill merotopy if it fulfills

(LAG) $\quad \nu(\mathcal{A}) = \inf\{\nu(\mathcal{G}) : \mathcal{G} \in \mathfrak{G}(X) \text{ and } \mathcal{A} \prec \mathcal{G}\},$ for all $\mathcal{A} \in \mathcal{P}(L^X)$.

By Remark 3.3, the structure defined in (LAG) is clearly an $L$-approach merotopy on $X$. Thus the category **LAGRL** whose objects are $L$-approach grill merotopic spaces forms a full subcategory of **LAMER**. It may be observed that $\nu_i, \nu_d$ and $\nu_{r,r}$ are $L$-approach grill merotopies on $X$, while the $\tilde{\nu}$ from Example 3.1(ii) (when $\nu = \nu_{r,r}$) is an approach seminearness structure [17] but not an $\{0,1\}$-approach grill merotopy on $\mathbb{R}$.

**Theorem 5.1.** *The category* **LAGRL** *is a bicoreflective subcategory of* **LAMER**.

**Proof.** The **LAGRL**-bicoreflection of any $L$-approach merotopic space $(X,\nu)$ is given by $1_X : (X,\tilde{\nu}) \to (X,\nu)$, where the map $\tilde{\nu} : \mathcal{P}(L^X) \to [0,\infty]$ is defined, for $\mathcal{A} \in \mathcal{P}(L^X)$, by

$$\tilde{\nu}(\mathcal{A}) = \inf\{\nu(\mathcal{G}) : \mathcal{G} \in \mathfrak{G}(X) \text{ and } \mathcal{A} \prec \mathcal{G}\}. \ ∎$$

**Theorem 5.2.** *The category* **LAGRL** *is a topological construct.*

**Proof.** Follows from Theorem 5.1 and by noting that any bicoreflective isomorphism-closed full subcategory of a topological construct is a topological construct (see Theorem A.10 of [6]). ∎

Thus an $L$-approach grill merotopy can be characterized by its restriction to the set of all $L$-grills. This being the case, we can present a simpler characterization of $L$-approach grill merotopy as follows:

**Definition 5.2.** A function $\nu : \mathfrak{G}(X) \to [0, \infty]$ is called an $L$-approach grill structure if it satisfies the following conditions:

(LAG1) $\nu([x_p]) = 0$, for all $x \in X$ and $p \in M(L)$;
(LAG2) $\mathcal{G}_1 \prec \mathcal{G}_2 \Longrightarrow \nu(\mathcal{G}_1) \leq \nu(\mathcal{G}_2)$, for every $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}(X)$.

The pair $(X, \nu)$ is called an $L$-approach grill space.

It may be observed that for any $L$-approach grill merotopic space $(X, \nu)$, the restriction $\nu|_{\mathfrak{G}(X)}$ is clearly an $L$-approach grill structure on $X$. Conversely for any $L$-approach grill space $(X, \nu)$, the map $\tilde{\nu} : \mathcal{P}(L^X) \to [0, \infty]$ defined in Theorem 5.1 is an $L$-approach grill merotopy on $X$ which coincides with $\nu$ on $\mathfrak{G}(X)$. Moreover for any $L$-approach grill merotopy on $X$, $\nu = (\nu|_{\mathfrak{G}(X)})^{\sim}$. To support the existence of $L$-approach grill structures on $X$, we provide the following examples. Some examples are constructed specifically for the classical theory.

**Examples 5.1.**
(i) Let $cl$ be an $L$-Čech closure on $X$. Then $\nu : \mathfrak{G}(X) \to [0, \infty]$ is an $L$-approach grill structure on $X$, where $\nu$ can be defined in the following ways for $\mathcal{G} \in \mathfrak{G}(X)$,

- $\nu(\mathcal{G}) = 0$, if $\bigwedge cl(\mathcal{G}) \neq \mathbf{0}$; and $\nu(\mathcal{G}) = \sup_{g \in \mathcal{G}} |\text{stack}(cl(g))|$, otherwise.
- $\nu(\mathcal{G}) = 0$, if $\bigwedge cl(\mathcal{G}) \neq \mathbf{0}$; and $\nu(\mathcal{G}) = |cl(\mathcal{G})|$, otherwise.

(ii) Let $(X, cl)$ be a topological space. Then $(X, \nu)$ is a $\{0, 1\}$-approach grill space, where $\nu : \mathfrak{G}(X) \to [0, \infty]$ can be defined in the following ways for $\mathcal{G} \in \mathfrak{G}(X)$ (here $\mathfrak{G}(X)$ is the family of all $\{0, 1\}$-grills on $X$),

- $\nu(\mathcal{G}) = 0$, if $\bigcap cl(\mathcal{G}) \neq \emptyset$; and $\nu(\mathcal{G}) = \sup\{|cl(G)| : G \in \mathcal{G}$ and $|cl(G)| < \aleph_0\}$, otherwise.
- $\nu(\mathcal{G}) = 0$, if $\bigcap cl(\mathcal{G}) \neq \emptyset$; and $\nu(\mathcal{G}) = \inf_{G \in \sec cl(\mathcal{G})} |G|$, otherwise.
- $\nu(\mathcal{G}) = 0$, if $\bigcap cl(\mathcal{B}) \neq \emptyset$ for every finite subset $\mathcal{B}$ of $\mathcal{G}$; and $\nu(\mathcal{G}) = \inf\{|G| : G \in \sec cl(\mathcal{G})\}$, otherwise.

(Recall that $\sec \mathcal{A} \equiv \{A \subseteq X : A \cap B \neq \emptyset$, for all $B \in \mathcal{A}\}$, where $\mathcal{A} \in \mathcal{P}^2(X)$).

- $\nu(\mathcal{G}) = 0$, if $\bigcap cl(\mathcal{G}) \neq \emptyset$ or each element of $\mathcal{G}$ is infinite; and $\nu(\mathcal{G}) = \sup\{|G| : G \in \mathcal{G})$ and $|G| < \aleph_0\}$, otherwise.

(iii) Let $f : (X, cl_X) \to (Y, cl_Y)$ be a closed and continuous map, and $y \in cl_Y(f(x)) \Longrightarrow f(x) \in cl_Y(\{y\})$, for all $y \in Y$ and $x \in X$. Define $\nu : \mathfrak{G}(X) \to [0, \infty]$ as follows: for $\mathcal{G} \in \mathfrak{G}(X)$, $\nu(\mathcal{G}) = 0$, if $\bigcap cl_Y(f(\mathcal{G})) \neq \emptyset$ where $f(\mathcal{G}) =$

$\{f(G) : G \in \mathcal{G}\}$; and $\nu(\mathcal{G}) = \inf_{G \in \sec(cl(\mathcal{G}))} |G|$, otherwise. Then $(X, \tilde{\nu})$ (where $\tilde{\nu}$ is obtained from $\nu$ as in Theorem 5.1) is a $\{0,1\}$-approach nearness space. ∎

Observe that if $\mathcal{G}$ is an $L$-grill on $X$ and $T^{\rightarrow} : X \to Y$ is an $L$-fuzzy map, then $T^{\rightarrow}(\mathcal{G}) \equiv \{f \in L^Y : T^{\rightarrow}(g) \leq f$ for some $g \in \mathcal{G}\}$ is an $L$-grill on $Y$. So, we have the following definition.

**Definition 5.3.** For any $L$-approach grill spaces $(X, \nu)$ and $(Y, \nu')$, an $L$-fuzzy mapping $T^{\rightarrow} : X \to Y$ is called a contraction if $\nu'(T^{\rightarrow}(\mathcal{G})) \leq \nu(\mathcal{G})$, for all $\mathcal{G} \in \mathfrak{G}(X)$.

By Theorem 5.2, **LAGRL** is a topological construct. Therefore we can derive the initial and final structures of **LAGRL** explicitly as follows:

**Proposition 5.1.** *Let $((X_j, \nu_j))_{j \in J}$ be a family of $L$-approach grill spaces and $(T_j^{\rightarrow} : X \to X_j)_{j \in J}$ a source in* **LAGRL**. *The initial $L$-approach grill structure on $X$ is given by $\nu : \mathfrak{G}(X) \to [0, \infty]$ as follows:*

$$\nu(\mathcal{G}) = \sup_{j \in J} \nu_j(T_j^{\rightarrow}(\mathcal{G})), \text{ for all } \mathcal{G} \in \mathfrak{G}(X).$$

**Proof.** Clearly $\nu$ is an $L$-approach grill structure on $X$ and $T_j^{\rightarrow}$ is a contraction for each $j \in J$. Let $(Y, \nu')$ be any $L$-grill space and $U^{\rightarrow} : Y \to X$ be a map such that $T_j^{\rightarrow} \circ U^{\rightarrow} : Y \to X_j$ is a contraction for each $j \in J$. Let $\mathcal{G} \in \mathfrak{G}(Y)$ and $\nu'(\mathcal{G}) < \nu(U^{\rightarrow}(\mathcal{G}))$. Then $\nu'(\mathcal{G}) < \sup_{j \in J} \nu_j(T_j^{\rightarrow}(U^{\rightarrow}(\mathcal{G}))) \leq \nu'(\mathcal{G})$, that is, $\nu'(\mathcal{G}) < \nu'(\mathcal{G})$, which is absurd. Thus $\nu(U^{\rightarrow}(\mathcal{G})) \leq \nu'(\mathcal{G})$ and hence $U^{\rightarrow}$ is a contraction. Consequently $\nu$ is the initial $L$-approach grill structure on $X$. ∎

**Proposition 5.2.** *Let $((X_j, \nu_j))_{j \in J}$ be a family of $L$-approach grill spaces and $(T_j^{\rightarrow} : X_j \to X)_{j \in J}$ a sink in* **LAGRL**. *Then the map $\nu : \mathfrak{G}(X) \to [0, \infty]$ defined as*

*for $\mathcal{G} \in \mathfrak{G}(X)$,*

$$\nu(\mathcal{G}) = \begin{cases} 0, & \text{if } \mathcal{G} \subseteq [x_p] \text{ for some } x \in X \text{ and } p \in M(L), \\ \inf\{\nu_j(\mathcal{G}_j) : \mathcal{G}_j \in \mathfrak{G}(X_j) \text{ for some } j \in J \\ \quad \text{such that } \mathcal{G} \prec T_j^{\rightarrow}(\mathcal{G}_j)\}, & \text{otherwise} \end{cases}$$

*is the final $L$-approach grill structure on $X$.*

**Proof.** Clearly $\nu$ is an $L$-approach grill structure on $X$ and $T_j^{\rightarrow}$ is a contraction for each $j \in J$. To show that $\nu$ is final, let $(Y, \nu')$ be any $L$-grill space and $U^{\rightarrow} : X \to Y$ be a map such that $U^{\rightarrow} \circ T_j^{\rightarrow} : X_j \to Y$ is a contraction for each $j \in J$. Take any $\mathcal{G} \in \mathfrak{G}(X)$. If $\mathcal{G} \subseteq [x_p]$ for some $x \in X$ and $p \in M(L)$, then $U^{\rightarrow}(\mathcal{G}) \subseteq [(U(x))_p]$. Therefore $\nu'(U^{\rightarrow}(\mathcal{G})) \leq \nu(\mathcal{G})$. If $\mathcal{G} \nsubseteq [x_p]$ for any $x \in X$ and $p \in M(L)$, then there exists $\mathcal{G}_j \in \mathfrak{G}(X_j)$ for some $j \in J$ such that $\mathcal{G} \prec T_j^{\rightarrow}(\mathcal{G}_j)$. Consequently $U^{\rightarrow}(\mathcal{G}) \prec U^{\rightarrow}(T_j^{\rightarrow}(\mathcal{G}_j))$. Thus $\nu'(U^{\rightarrow}(\mathcal{G})) \leq \nu'(U^{\rightarrow} \circ T_j^{\rightarrow}(\mathcal{G}_j)) \leq \nu_j(\mathcal{G}_j)$, for each $j \in J$ such that $\mathcal{G} \prec T^{\rightarrow}(\mathcal{G}_j)$. Therefore $\nu'(U^{\rightarrow}(\mathcal{G})) \leq \inf\{\nu_j(\mathcal{G}_j) : \mathcal{G}_j \in \mathfrak{G}(X_j)$ for some $j \in J$ such that $\mathcal{G} \prec T_j^{\rightarrow}(\mathcal{G}_j)\} = \nu(\mathcal{G})$ for any $\mathcal{G} \in \mathfrak{G}(X)$. Therefore $U^{\rightarrow}$ is a contraction and $\nu$ is the final $L$-approach grill structure on $X$. ∎

## 6. Embedding of LGRL in LAGRL

In this section, we show that an $L$-approach grill space can be characterized completely by $L$-Cauchy grills on $X$; and that **LGRL** (the category of $L$-grill merotopic spaces and $L$-merotopic maps) can be embedded in **LAGRL** (the category of $L$-approach grill spaces and contractions) bireflectively and bicoreflectively.

**Definition 6.1.** Let $(X, \xi)$ be any $L$-merotopic space. An $L$-grill $\mathcal{G}$ on $X$ is called an $L$-Cauchy grill if $\mathcal{G} \in \xi$, and an $L$-merotopic space $(X, \xi)$ is called an $L$-grill merotopic space if it satisfies

(LG) for each $\mathcal{A} \in \xi$, $\mathcal{A} \prec \mathcal{G}$, where $\mathcal{G}$ is an $L$-Cauchy grill.

For example, $(X, \xi)$ where $\xi = \{\mathcal{A} \subseteq L^X : \mathcal{A} \prec \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{G}(X)\}$ is an $L$-grill merotopic space.

The category **LGRL** forms a full subcategory of the category **LMER**.

Let $\mathbb{C} \subseteq \mathfrak{G}(X)$ satisfy

(LC1) $[x_p] \in \mathbb{C}$, for all $x \in X$ and $p \in M(L)$; and
(LC2) if $\mathcal{G} \in \mathbb{C}$ and $\mathcal{F} \prec \mathcal{G}$, then $\mathcal{F} \in \mathbb{C}$.

Then there is exactly one $L$-grill merotopy $\xi_{\mathbb{C}} = \{\mathcal{A} \subseteq L^X : \mathcal{A} \prec \mathcal{G}$ for some $\mathcal{G} \in \mathbb{C}\}$ for which $\mathbb{C}$ is the set of all $L$-Cauchy grills on $X$. Conversely for any $L$-grill merotopy $\xi$ on $X$, the set $\mathbb{C}_\xi$ of all $L$-Cauchy grills fulfills (LC1) and (LC2). Thus an $L$-grill merotopy is characterized completely by the $L$-Cauchy grills.

Let $(X, \xi)$ be an $L$-grill merotopic space. The map $\nu_\xi$ defined in Proposition 4.1 is an $L$-approach grill merotopy on $X$. Thus **LGRL** can be embedded as a full subcategory in **LAGRL** by the functor $H$ : **LGRL** $\rightarrow$ **LAGRL** defined as: $H((X, \xi)) = (X, \nu_\xi)$ and $H(T^{\rightarrow}) = T^{\rightarrow}$.

**Proposition 6.1.** *An $L$-approach grill space $(X, \nu)$ is induced by an $L$-grill merotopic space if and only if $\nu(\mathfrak{G}(X)) \subseteq \{0, \infty\}$.*

**Proof.** Let $\nu(\mathfrak{G}(X)) \subseteq \{0, \infty\}$. Then $\xi_\nu = \{\mathcal{A} \in \mathcal{P}(L^X) : \mathcal{A} \prec \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{G}(X)$ such that $\nu(\mathcal{G}) = 0\}$ is an $L$-grill merotopy on $X$ and the induced $L$-approach grill structure $\nu_{\xi_\nu}$ coincides with $\nu$. The converse is obvious. ∎

It may be noted that for any $L$-approach grill space $(X, \nu)$, $\xi_\nu$ and $\xi^\nu$ defined as $\xi_\nu = \{\mathcal{A} \in \mathcal{P}(L^X) : \mathcal{A} \prec \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{G}(X)$ such that $\nu(\mathcal{G}) = 0\}$

and $\xi^{\nu} = \{\mathcal{A} \in \mathcal{P}(L^X) : \mathcal{A} \prec \mathcal{G}$ for some $\mathcal{G} \in \mathfrak{G}(X)$ such that $\nu(\mathcal{G}) < \infty\}$ are $L$-grill merotopies on $X$. Also for any $L$-approach grill spaces $(X, \nu)$ and $(Y, \nu')$, if an $L$-fuzzy map $T^{\rightarrow} : (X, \nu) \to (Y, \nu')$ is a contraction, then $T^{\rightarrow} : (X, \xi_{\nu}) \to (Y, \xi_{\nu'})$ and $T^{\rightarrow} : (X, \xi^{\nu}) \to (Y, \xi^{\nu'})$ are $L$-merotopic maps. This leads to the definition of functors $G$ : **LAGRL** $\to$ **LGRL** by $G((X,\nu)) = (X, \xi_{\nu})$, $G(T^{\rightarrow}) = T^{\rightarrow}$; and $G'$ : **LAGRL** $\to$ **LGRL** by $G'((X,\nu)) = (X, \xi^{\nu})$, $G'(T^{\rightarrow}) = T^{\rightarrow}$. Thus we have the following theorems.

**Theorem 6.1.** *The category* **LGRL** *is a bicoreflective subcategory of* **LAGRL**.

**Proof.** For any $L$-approach grill structure $\nu$ on $X$, $1_X : (X, \nu_{\xi_{\nu}}) \to (X, \nu)$ is an **LGRL**-bicoreflection of $(X, \nu)$. ∎

**Theorem 6.2.** *The category* **LGRL** *is a bireflective subcategory of* **LAGRL**.

**Proof.** For any $L$-approach grill structure $\nu$ on $X$, $1_X : (X, \nu) \to (X, \nu_{\xi^{\nu}})$ is an **LGRL**-bireflection of $(X, \nu)$. ∎

**Concluding remark.** The present theory is a unified approach to the study of the classical case and the fuzzy case (that is, when $L \equiv [0,1]$). The classical version (that is, when $L = \{0,1\}$) of the theory of $L$-approach premerotopies ($L$-approach contiguities) on $X$ can be established by taking their domains to be the collection of all (finite) subsets of $\mathcal{P}(X)$. So the results on $L$-approach premerotopy and $L$-approach contiguity on $X$ in Sections 3 and 4 also contributes to the classical theory. An $L$-approach contiguity can be obtained from an $L$-approach merotopy $\nu$ on $X$ by restricting $\nu$ to finite subsets of $L^X$. Conversely, as stated in Proposition 3.2, an $L$-approach contiguity induces an $L$-approach merotopy on $X$. Various nontrivial examples of $L$-approach merotopies, $L$-approach contiguities, $L$-approach premerotopies and $L$-approach grill merotopies have been included. Given an $L$-approach merotopy, we can obtain an $L$-approach premerotopy (which is not an $L$-approach merotopy in general) and hence applying Proposition 3.1 or Remark 3.3, a new $L$-approach merotopy on $X$ is obtained in general. Sections 5 and 6 do not include a treatment of $L$-approach contiguity on $X$ (when $L^X$ is not finite) because $[x_p]$ (where $x \in X$ and $p \in M(L)$) may not be finite in general.

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