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ON REAL-VALUED HOMOMORPHISMS IN COUNTABLY GENERATED DIFFERENTIAL STRUCTURES

Abstract. Real valued homomorphisms on the algebra of smooth functions on a differential space are described. The concept of generators of this algebra is emphasized in this description.

1. Introduction

When all the real homomorphisms defined on an algebra of real functions defined on a space are evaluations then we say that the space is smoothly real-compact. There are many articles stating this property for various spaces. In [11], [4] it is shown that the spaces of real continuous functions on \mathbb{R} and \mathbb{R}^n are smoothly real-compact. In [2] this property has been shown for the spaces of functions of class C^k ($k = 1, \dots, \infty$) on separable Banach spaces. Much information about this topic can be found in [9]. The most important, from the point of view of Sikorski spaces, is the article [6] since it discusses smooth real-compactness of smooth spaces which are a wider category than the Sikorski spaces. Many conditions for those spaces to be smoothly real-compact are given there. In [1] and [7] many important results were obtained for a very wide class of algebras. In our article we emphasize the concept of generators of a differential space. We use techniques suitable for Sikorski spaces. Real valued homomorphisms are classified by their values on generators.

2. Basic concepts and definitions

Let M be a nonempty set and \mathcal{C} a set of real functions on M . We introduce on M a topology $\tau_{\mathcal{C}}$, the weakest topology in which the functions from

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\mathcal{C} are continuous. We say that the set \mathcal{C} is *closed with respect to superposition* if all functions of a form $\omega \circ (f_1, \dots, f_n)$ where $f_1, \dots, f_n \in \mathcal{C}$, $\omega \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, are in \mathcal{C} . Adding to \mathcal{C} all the functions of this form we obtain what we will call the superposition closure of \mathcal{C} , denoted by $sc\mathcal{C}$ following Waliszewski [15]. For any $A \subseteq M$ by \mathcal{C}_A we denote the set of all functions f on A such that for any $p \in A$ there exists an open neighborhood $U \in \tau_{\mathcal{C}}$ of p and a function $g \in \mathcal{C}$ such that $f|_{U \cap A} = g|_{U \cap A}$. If $\mathcal{C} = \mathcal{C}_M$ then we say that \mathcal{C} is *closed with respect to localization* [13]. We call the set of real functions \mathcal{C} on a nonempty set M a *differential structure* if it is:

- i) closed with respect to superposition, $\mathcal{C} = sc\mathcal{C}$,
- ii) closed with respect to localization, $\mathcal{C} = \mathcal{C}_M$.

A differential structure is always an algebra with unity and contains all constant functions.

DEFINITION 1. A pair (M, \mathcal{C}) is said to be a differential space if M is a nonempty set and \mathcal{C} a differential structure on it.

We define a *differential subspace* of a differential space (M, \mathcal{C}) to be any pair (A, \mathcal{C}_A) where $A \subseteq M$, $A \neq \emptyset$.

DEFINITION 2. The differential structure \mathcal{C} is generated by a set of functions \mathcal{C}_0 if $\mathcal{C} = (sc\mathcal{C}_0)_M$.

Thus \mathcal{C} is the smallest differential structure that contains \mathcal{C}_0 . Sometimes we write $\mathcal{C} = \text{Gen } \mathcal{C}_0$. If $\mathcal{C} = \text{Gen } \mathcal{C}_0$ then for any $f \in \mathcal{C}$ and any point $p \in M$ there exists an open neighborhood $U \in \tau_{\mathcal{C}}$ of p and functions $f_1, \dots, f_n \in \mathcal{C}_0$, $\omega \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$ such that $f|_U = \omega \circ (f_1, \dots, f_n)|_U$. We say that the differential space (M, \mathcal{C}) is *finitely generated* if it is generated by a finite set of a real functions. A differential space is *countably generated* if it is generated by a countable set of real functions but it is not finitely generated.

We denote by $(\mathbb{R}^I, \varepsilon_I)$ the differential space with the structure ε_I generated by the set of projections $\mathcal{C}_0 = \{\pi_i : i \in I\}$, where $\pi_i : \mathbb{R}^I \rightarrow \mathbb{R}$ is defined by $\pi_i(x) = x_i$ for $x = (x_i)_{i \in I}$. This is a generalization of the Cartesian space $(\mathbb{R}^n, \varepsilon_n)$ where $\varepsilon_n = C^\infty(\mathbb{R}^n)$.

The *spectrum* of an algebra \mathcal{C} is the set

$$\text{Spec } \mathcal{C} = \{\chi : \mathcal{C} \rightarrow \mathbb{R}\},$$

where χ is a homomorphism that preserves unity.

Evaluation of the algebra \mathcal{C} at a point $p \in M$ is the homomorphism $\chi \in \text{Spec } \mathcal{C}$ given by

$$(1) \quad \chi(f) = f(p) \quad \forall f \in \mathcal{C}.$$

We will denote it by ev_p . We define the mapping $\text{ev} : M \rightarrow \text{Spec } \mathcal{C}$ by the formula:

$$(2) \quad \text{ev}(p) = \text{ev}_p.$$

DEFINITION 3. [6] We say that a differential space (M, \mathcal{C}) is smoothly real-compact if any $\chi \in \text{Spec } \mathcal{C}$ is evaluation at some point $p \in M$.

From this definition it follows that the space (M, \mathcal{C}) is smoothly real-compact when the mapping ev is a surjection. For any $f \in \mathcal{C}$ we define the function $\hat{f} : \text{Spec } \mathcal{C} \rightarrow \mathbb{R}$ by

$$(3) \quad \hat{f}(\chi) = \chi(f) \quad \forall \chi \in \text{Spec } \mathcal{C}.$$

The set of all functions of the form \hat{f} will be denoted by $\hat{\mathcal{C}}$. Define $\tau : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ by

$$(4) \quad \tau(f) = \hat{f} \quad \forall f \in \mathcal{C}.$$

The mapping τ is an isomorphism between the algebra \mathcal{C} and the algebra $\hat{\mathcal{C}}$.

3. Main results

LEMMA 1. *The differential space $(\mathbb{R}^n, \varepsilon_n)$ is smoothly real-compact.*

Proof. Let $\chi \in \text{Spec } \varepsilon_n$. We define $p \in \mathbb{R}^n$ by $p_i := \chi(\pi_i)$ for $i = 1, \dots, n$. We will show that $\chi = \text{ev}_p$. It is known that any $f \in \varepsilon_n$ can be represented as

$$(5) \quad f = f(p) + \sum_{i=1}^n g_i(\pi_i - p_i) \quad \text{for } g_1, \dots, g_n \in \varepsilon_n,$$

where the functions g_i satisfy $g_i(p) = \partial_i f(p)$. Then

$$\chi(f) = \chi(f(p)) + \sum_{i=1}^n \chi(g_i)(\chi(\pi_i) - \chi(p_i)) = f(p) + \sum_{i=1}^n \chi(g_i)(p_i - p_i) = f(p).$$

Therefore $\chi(f) = f(p)$ for all $f \in \varepsilon_n$. ■

Now we prove:

LEMMA 2. *Every differential subspace of the differential space $(\mathbb{R}^n, \varepsilon_n)$ is smoothly real-compact.*

Proof. Let (M, \mathcal{C}) be a differential subspace of $(\mathbb{R}^n, \varepsilon_n)$. The inclusion mapping $\iota_M : M \rightarrow \mathbb{R}^n$ is smooth and therefore $\iota_M^* : \varepsilon_n \rightarrow \mathcal{C}$ is a homomorphism. From the definition we know that $\iota_M^*(f) = f|_M$ for all $f \in \varepsilon_n$. For any $\chi \in \text{Spec } \mathcal{C}$ we have $\chi \circ \iota_M^* \in \text{Spec } \varepsilon_n$. From Lemma 1 we know that there exists $p \in \mathbb{R}^n$ such that $\chi \circ \iota_M^*(f) = \chi(\iota_M^*(f)) = \chi(f|_M) = f(p)$ for all $f \in \varepsilon_n$. Suppose that $p \notin M$. Define $\omega \in \varepsilon_n$ by

$$(6) \quad \omega(x_1, \dots, x_n) = (x_1 - p_1)^2 + \dots + (x_n - p_n)^2.$$

Since $\omega|_M > 0$ we have $\frac{1}{\omega|_M} \in \mathcal{C}$. We also know that $\chi((\omega|_M)(\frac{1}{\omega|_M})) = \chi(1) = 1$, and $(\chi \circ \iota_M^*)(\omega) = \chi(\omega|_M) = \omega(p) = 0$. This is a contradiction.

We will show that $\chi = ev_p$. Let $f \in \mathcal{C}$. There exists an open neighborhood $U \in \tau_{\varepsilon_n}$ of p and a function $\kappa \in \varepsilon_n$ such that $f|_{U \cap M} = \kappa|_{U \cap M}$. From [14] we know that there exists a bump function $\phi \in \varepsilon_n$ with $\phi(p) = 1$, $\phi|_{M \cap U} > 0$ and $\phi|_{\mathbb{R}^n - (M \cap U)} = 0$. From these properties it follows that $(f - \kappa|_M)\phi|_M = 0$. Then $\chi((f - \kappa|_M)\phi|_M) = (\chi(f) - \chi(\kappa|_M))\chi(\phi|_M) = 0$. But $\chi(\phi|_M) = (\iota_M \circ \chi)(\phi) = \phi(p) = 1$ so $\chi(f) = \chi(\kappa|_M) = \kappa(p) = f(p)$. We have shown that $\chi(f) = f(p)$ for all $f \in \mathcal{C}$. ■

If the differential structure \mathcal{C} of the differential space (M, \mathcal{C}) is generated by a set of functions \mathcal{C}_0 then we can define a mapping $\phi : M \rightarrow \mathbb{R}^{\mathcal{C}_0}$ by

$$(7) \quad \phi(p)(f) = f(p), \quad f \in \mathcal{C}_0.$$

We will call this mapping the *generator embedding*. We can prove the following:

LEMMA 3. *A differential space (M, \mathcal{C}) with $\mathcal{C} = \text{Gen } \mathcal{C}_0$ is smoothly real-compact iff the differential space $(\phi(M), (\varepsilon_I)_{\phi(M)})$ for $I = |\mathcal{C}_0|$ is smoothly real-compact.*

Proof. If \mathcal{C}_0 separates the points of M then ϕ is a diffeomorphism onto its image so the result is obvious. So assume that \mathcal{C}_0 does not separate points. Then $\bar{\phi} : M \rightarrow \phi(M)$ where $\bar{\phi}(p) = \phi(p)$ is surjective but not injective. Set $F := \bar{\phi}$. We know that $F^* : (\varepsilon_I)_{\phi(M)} \rightarrow \mathcal{C}$ is an isomorphism of algebras. If (M, \mathcal{C}) is smoothly real-compact then for any $\nu \in \text{Spec}(\varepsilon_I)_{\phi(M)}$ there exists $\mu \in \text{Spec } \mathcal{C}$ such that $\mu = \nu \circ (F^*)^{-1}$. Then for any $g \in (\varepsilon_I)_{\phi(M)}$, $\nu(g) = \mu(F^*(g)) = \mu(g \circ F) = g(F(p))$. So if $\mu = ev_p$ then $\nu = ev_{F(p)}$.

If $(\phi(M), (\varepsilon_I)_{\phi(M)})$ is smoothly real-compact then for any $\mu \in \text{Spec } \mathcal{C}$ there exists $\nu \in \text{Spec}(\varepsilon_I)_{\phi(M)}$ defined by $\nu = \mu \circ F^*$, so $\mu = \nu \circ (F^*)^{-1}$. Therefore for any $f \in \mathcal{C}$ we have $\mu(f) = (\nu \circ (F^*)^{-1})(f) = \nu((\phi^*)^{-1}(f)) = ((\phi^*)^{-1}(f))(q) = f(p)$ for any $p \in F^{-1}(q)$. So if $\nu = ev_q$ then $\mu = ev_p$ for all $p \in F^{-1}(q)$. ■

From the last lemma we know that it is sufficient to work on subspaces of Cartesian spaces.

COROLLARY 1. *Let (M, \mathcal{C}) be a differential space with $\mathcal{C} = \text{Gen } \mathcal{C}_0$ for some finite \mathcal{C}_0 . Then (M, \mathcal{C}) is smoothly real-compact.*

Proof. By using the generators \mathcal{C}_0 we can embed (M, \mathcal{C}) into $(\mathbb{R}^{\mathcal{C}_0}, (\varepsilon_{\mathcal{C}_0})_{\phi(M)})$ and then from Lemmas 2, 3 we derive that (M, \mathcal{C}) is smoothly real-compact. ■

From Corollary 1 we obtain:

LEMMA 4. *Let (M, \mathcal{C}) be a differential space. Any $\chi \in \text{Spec } \mathcal{C}$ satisfies the following condition:*

$$(8) \quad \chi(\omega \circ (f_1, \dots, f_n)) = \omega(\chi(f_1), \dots, \chi(f_n)),$$

for all $\omega \in \varepsilon_n$ and $f_1, \dots, f_n \in \mathcal{C}$, $n \in \mathbb{N}$.

Proof. Let $\beta_1, \dots, \beta_n \in \mathcal{C}$ be arbitrary functions. We define the mapping $F : (M, \mathcal{C}) \rightarrow (\mathbb{R}^n, \varepsilon_n)$ by:

$$F(p) = (\beta_1(p), \dots, \beta_n(p)), \quad p \in M.$$

This mapping is smooth and it is onto its image. Therefore the mapping $F^* : (\varepsilon_n)_{F(M)} \rightarrow \mathcal{C}$ is a homomorphism. For any $\chi \in \text{Spec } \mathcal{C}$ we have $\chi \circ F^* \in \text{Spec}((\varepsilon_n)_{F(M)})$. From Corollary 1 we know that there exists $q \in F(M)$ such that $\chi \circ F^* = ev_q$ for some $q \in F(M)$. Also there exists $p \in M$ such that

$$(\chi \circ F^*)(\omega|_{F(M)}) = ev_{F(p)}(\omega|_{F(M)}) \quad \forall \omega \in \varepsilon_n.$$

We can rewrite this in the form

$$\chi(\omega \circ F) = \omega(F(p)) = \omega(\beta_1(p), \dots, \beta_n(p)) \quad \forall \omega \in \varepsilon_n.$$

By setting $\omega = \pi_i$, $i = 1, \dots, n$, we obtain $\chi(\beta_i) = \chi(\pi_i \circ F) = \pi_i(F(p)) = \beta_i(p)$ and finally $\chi(\omega \circ (\beta_1, \dots, \beta_n)) = \omega(\chi(\beta_1), \dots, \chi(\beta_n))$ for all $\omega \in \varepsilon_n$. ■

Now we prove the following:

LEMMA 5. *Let (M, \mathcal{C}) be a differential space such that $\mathcal{C} = \text{Gen } \mathcal{C}_0$ and let $\chi \in \text{Spec } \mathcal{C}$. If $\chi|_{\mathcal{C}_0} = ev_p|_{\mathcal{C}_0}$ then $\chi = ev_p$.*

Proof. First we will show that if $f \in sc\mathcal{C}_0$ then $\chi(f) = f(p)$. From Lemma 4 we know that $\chi(\omega \circ (\beta_1, \dots, \beta_n)) = \omega(\chi(\beta_1), \dots, \chi(\beta_n))$ for $\omega \in \varepsilon_n$ and $\beta_1, \dots, \beta_n \in \mathcal{C}_0$. We also know that $\chi(\beta_i) = ev_p(\beta_i) = \beta_i(p)$. We can write $\chi(\omega \circ (\beta_1, \dots, \beta_n)) = \omega(\beta_1(p), \dots, \beta_n(p)) = \omega \circ (\beta_1, \dots, \beta_n)(p) = ev_p(\omega \circ (\beta_1, \dots, \beta_n))$. So we see that $\chi|_{sc\mathcal{C}_0} = ev_p|_{sc\mathcal{C}_0}$.

Now let $f \in \mathcal{C}$ be an arbitrary function. We know that for every $p \in M$ there exists an open neighborhood $U \in \tau_{\mathcal{C}}$, functions $\beta_1, \dots, \beta_n \in \mathcal{C}_0$ and a function $\omega \in \varepsilon_n$ such that $f|_U = \omega \circ (\beta_1, \dots, \beta_n)|_U$. There also exists a bump function ψ which separates the point p in the set U . This function is constructed by compositing some function from ε_n with some generators from \mathcal{C}_0 . We know that the homomorphism χ equals evaluation at p on this function, so $\chi(\psi) = \psi(p) = 1$. Now the following equality holds: $\psi \cdot (f - \omega \circ (\beta_1, \dots, \beta_n)) = 0$. By applying the homomorphism χ to this equality we obtain $\chi(\psi) \cdot \chi(f - \omega \circ (\beta_1, \dots, \beta_n)) = \chi(f) - \chi(\omega \circ (\beta_1, \dots, \beta_n)) = 0$ so $\chi(f) = \chi(\omega \circ (\beta_1, \dots, \beta_n)) = ev_p(\omega \circ (\beta_1, \dots, \beta_n)) = f(p) = ev_p(f)$. We see that $\chi(f) = f(p)$ for all $f \in \mathcal{C}$. ■

From Lemma 5 we get:

COROLLARY 2. *The differential space $(\mathbb{R}^I, \varepsilon_I)$ is smoothly real-compact.*

Proof. Let $\chi \in \text{Spec } \varepsilon_I$ be any homomorphism. Define $p \in \mathbb{R}^I$ by $p_i = \chi(\pi_i)$ for $i \in I$. Then $\chi(\pi_i) = \pi_i(p)$ so $\chi(\pi_i) = \text{ev}_p(\pi_i)$. Since the structure ε_I is generated by the set $\{\pi_i : i \in I\}$ we see that χ is evaluation at p on the generators. From the last Lemma we derive that χ is an evaluation on the whole ε_I . ■

By using the whole \mathcal{C} as the set of generators we can embed M in $\mathbb{R}^{\mathcal{C}}$. We denote this embedding by ι so $\iota : M \rightarrow \mathbb{R}^{\mathcal{C}}$, $\iota(p)_f = f(p)$. This is a special case of a generator embedding. We can also map $\text{Spec } \mathcal{C}$ into $\mathbb{R}^{\mathcal{C}}$ using the mapping $\kappa : \text{Spec } \mathcal{C} \rightarrow \mathbb{R}^{\mathcal{C}}$ defined by $\kappa(\chi)_f = \hat{f}(\chi) = \chi(f)$. It is obvious that $\iota = \kappa \circ \text{ev}$. In [6] Kriegel, Michor and Schachermayer have shown that $\iota(M)$ is dense in $\kappa(\text{Spec } \mathcal{C})$ in the Tikhonov topology of $\mathbb{R}^{\mathcal{C}}$. Since the mapping κ is a homeomorphism one can easily see:

COROLLARY 3. *$\text{ev}(M)$ is dense in $\text{Spec } \mathcal{C}$ in the topology $\tau_{\hat{\mathcal{C}}}$.*

This property will allow us to prove an interesting fact about the space $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$.

LEMMA 6. *If (M, \mathcal{C}) is a differential space then $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ is a differential space.*

Proof. To prove that $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ is a differential space, we have to show that the set $\hat{\mathcal{C}}$ is closed with respect to superposition with smooth functions from ε_n and is closed with respect to localization.

Let $g = \omega \circ (\hat{f}_1, \dots, \hat{f}_n)$ for some $\omega \in \varepsilon_n$ and $\hat{f}_1, \dots, \hat{f}_n \in \hat{\mathcal{C}}$. From Lemma 4 we know that

$g(\chi) = \omega \circ (\hat{f}_1, \dots, \hat{f}_n)(\chi) = \tau(\omega \circ (f_1, \dots, f_n))(\chi) \quad \forall \chi \in \text{Spec } \mathcal{C}$. We have shown that $g \in \hat{\mathcal{C}}$ so $\hat{\mathcal{C}}$ is closed with respect to superposition.

Let a function $f : \text{Spec } \mathcal{C} \rightarrow \mathbb{R}$ satisfy the localization condition in the space $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$. For any open subset $\hat{U} \in \text{Spec } \mathcal{C}$ there is $\hat{g} \in \hat{\mathcal{C}}$ such that $f|_{\hat{U}} = \hat{g}|_{\hat{U}}$. We can uniquely define a function $h : M \rightarrow \mathbb{R}$ by the condition $h(p) = f(\text{ev}_p)$ for all $p \in M$. For any open set $\hat{U} \in \text{Spec } \mathcal{C}$ the set $U = \{p \in M : \text{ev}_p \in \hat{U}\}$ is open. From the definitions of h and U we know that $h|_U = g|_U$. Because $g \in \mathcal{C}$ it follows that $h \in \mathcal{C}$. We also know that $\hat{h}|_{\text{ev}M} = f|_{\text{ev}M}$. From Corollary 3 we derive that $f = \hat{h}$. This means that $f \in \hat{\mathcal{C}}$ so $\hat{\mathcal{C}}$ is closed with respect to localization. ■

Now one can prove the following lemmas:

LEMMA 7. *If (M, \mathcal{C}) is a differential space with the structure \mathcal{C} generated by \mathcal{C}_0 then the differential structure $\hat{\mathcal{C}}$ of the differential space $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ is generated by $\hat{\mathcal{C}}_0$.*

Proof. Assume that $\mathcal{C}_0 = \{f_i : i \in I\}$. We know that for any $f \in \mathcal{C}$ there exists an open covering of M such that on each set U of this covering the function f can be expressed in the form $\omega \circ (f_1, \dots, f_n)$ where $f_1, \dots, f_n \in \mathcal{C}$ and $\omega \in \varepsilon_n$. For each open set U of the covering we define $\hat{U} = \{ev_p \in \text{Spec } \mathcal{C} : p \in U\}$. On the set \hat{U} we consider the function $\hat{f} = \tau(\omega \circ (f_1, \dots, f_n))$. The sets of the form \hat{U} might not be a covering of $\text{Spec } \mathcal{C}$ but their union is dense in $\text{Spec } \mathcal{C}$. Therefore we can prolong uniquely this representation of \hat{f} on the whole $\text{Spec } \mathcal{C}$. We have shown that $\hat{\mathcal{C}} = \text{Gen } \hat{\mathcal{C}}_0$. ■

LEMMA 8. *For any differential space (M, \mathcal{C}) the differential space $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ is smoothly real-compact.*

Proof. We need to show that for every homomorphism $\hat{\chi} \in \text{Spec } \hat{\mathcal{C}}$ there exists a homomorphism $\psi \in \text{Spec } \mathcal{C}$ such that $\hat{\chi} = ev_\psi$. Since the algebras \mathcal{C} and $\hat{\mathcal{C}}$ are isomorphic we can define uniquely $\chi \in \text{Spec } \mathcal{C}$ by the formula $\chi(f) = \hat{\chi}(\hat{f})$. We will show that $\hat{\chi} = ev_\chi$. Indeed $ev_\chi(\hat{f}) = \hat{f}(\chi) = \chi(f) = \hat{\chi}(\hat{f})$. ■

LEMMA 9. *Let (M, \mathcal{C}) be a differential space and $\mathcal{C} = \text{Gen } \mathcal{C}_0$. If $\chi_1, \chi_2 \in \text{Spec } \mathcal{C}$ are equal on the generators, $\chi_1|_{\mathcal{C}_0} = \chi_2|_{\mathcal{C}_0}$, then they are equal, $\chi_1 = \chi_2$.*

Proof. Assume that $\chi_1|_{\mathcal{C}_0} = \chi_2|_{\mathcal{C}_0}$ and $\chi_1 \neq \chi_2$. From the last lemma we know that the differential structure $\hat{\mathcal{C}}$ of the differential space $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ is generated by $\hat{\mathcal{C}}_0$. From the condition $\chi_1|_{\mathcal{C}_0} = \chi_2|_{\mathcal{C}_0}$ we derive that $\hat{f}(\chi_1) = \hat{f}(\chi_2)$, for all $\hat{f} \in \hat{\mathcal{C}}_0$. But we know that if the generators do not separate points then all the functions do not separate points so $\forall \hat{f} \in \hat{\mathcal{C}} \hat{f}(\chi_1) = \hat{f}(\chi_2)$ and it follows that $\chi_1(f) = \chi_2(f)$ for all $f \in \mathcal{C}$. This means that $\chi_1 = \chi_2$. ■

LEMMA 10. *If (M, \mathcal{C}) is differential subspace of $(\mathbb{R}^I, \varepsilon_I)$ then any function $f \in \mathcal{C}$ is uniquely continuously prolongable to $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$, where $\tilde{M} = \{p \in \mathbb{R}^I : \exists \chi \in \text{Spec } \mathcal{C} \text{ such that } p_i = \chi(\pi_i|_M) \quad \forall i \in I\}$.*

Proof. We define $\tilde{f}(p) = \hat{f}(\chi)$ where $\chi \in \text{Spec } \mathcal{C}$ is such that $\chi(\pi_i) = p_i$ for all $i \in I$. Since a homomorphism is uniquely defined by its value on the generators (Lemma 9) this definition is correct. We see that if $p \in M$ then $\chi = ev_p$ and $\tilde{f}(p) = \hat{f}(ev_p) = f(p)$ so this is indeed a prolongation. This prolongation is continuous since the function \tilde{f} is a realization of the function \hat{f} on the set \tilde{M} which is the image of $\text{Spec } \mathcal{C}$ under the generator embedding using the generators $\tau(\pi_i|_M), i \in I$. Uniqueness follows from the fact that M is dense in \tilde{M} in the topology of \mathbb{R}^I . ■

From Lemma 10 we obtain:

COROLLARY 4. *When (M, \mathcal{C}) is a differential subspace of $(\mathbb{R}^I, \varepsilon_I)$ generated by $\mathcal{C}_0 = \{\pi_i|_M : i \in I\}$ then the mapping $\chi : \mathcal{C}_0 \rightarrow \mathbb{R}$ defined on generators by $\chi(\pi_i|_M) = p_i$ for some $p \in \tilde{M} - M$ can be prolonged to a homomorphism on the whole \mathcal{C} iff all the functions from \mathcal{C} are prolongable to p .*

Let $M = \mathbb{R}^{\mathbb{N}} - \{0\}$ and $\mathcal{C}_M = (\varepsilon_{\mathbb{N}})_M$. Then (M, \mathcal{C}_M) is a differential subspace of $(\mathbb{R}^{\mathbb{N}}, \varepsilon_{\mathbb{N}})$. We will show that this space is smoothly real-compact.

LEMMA 11. *There exists a function $\xi \in \mathcal{C}_M$ which is not prolongable to any continuous function on $\mathbb{R}^{\mathbb{N}}$.*

Proof. We know that there exists a function $\phi \in C^\infty(\mathbb{R})$ satisfying the following properties:

1. $\forall x \in \mathbb{R} \quad \phi(x) \in [0, 1]$
2. $\text{supp}(\phi) \in (-\infty, 1]$
3. $\phi|_{[0, \frac{1}{2}]} = 1$.

For any $k \in \mathbb{N}$ we define $\tilde{\rho}_k : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ by the formula:

$$\tilde{\rho}_k((x_n)) = \sum_{i=1}^k x_i^2$$

for $(x_n) \in \mathbb{R}^{\mathbb{N}}$. Then $\tilde{\rho}_k \in \mathcal{C}^\infty(\mathbb{R}^{\mathbb{N}})$, and $\rho_k = \tilde{\rho}_k|_M \in \mathcal{C}_M$. We define $\xi : M \rightarrow \mathbb{R}$ by:

$$(9) \quad \xi((x_n)) = \sum_{k=1}^{\infty} \phi(k^2 \rho_k((x_n))).$$

We will show that this function belongs to the structure \mathcal{C}_M . For any $k \in \mathbb{N}$ we can define the closed subset $A_k = \{(x_n) \in M : k^2 \rho_k((x_n)) \leq 1\} = \{(x_n) \in M : \rho_k((x_n)) \leq \frac{1}{k^2}\}$. We see that $\text{supp}(\phi \circ (k^2 \rho_k)) \subseteq A_k$. For any $(x_n) \in M$ the sequence $\rho_k((x_n))$ is non-decreasing with respect to k and there exists $k_0 \in \mathbb{N}$ for which $\frac{1}{k^2} < \rho_{k_0}((x_n))$. This means that $(x_n) \notin A_k$. Therefore $\bigcap_{k \in \mathbb{N}} A_k = \emptyset$. We also know that $A_{k+1} \subseteq A_k$. Let us define the family of open subsets $U_k = M - A_k$. Of course $\bigcup_{k \in \mathbb{N}} U_k = M$. If $(x_n) \in U_k$ then $\phi(k^2 \rho_k((x_n))) = 0$. Then for all $m > k$, $x_n \in U_m$ so $\phi(m^2 \rho_m((x_n))) = 0$. This means that only a finite number of elements are non-zero in the sum (9) and therefore

$$\xi((x_n)) = \sum_{j=1}^{k-1} \phi(j^2 \rho_j((x_n))),$$

so $\xi|_{U_k} \in \mathcal{C}_{U_k} = (\mathcal{C}_M)_{U_k}$ for all $k \in \mathbb{N}$. From the localization closedness of the differential structure we derive that $\xi \in \mathcal{C}_M$. Now we will define a sequence

in M convergent to 0 on which the function ξ diverges. Let $z_k = (x_{n,k})$ where

$$x_{n,k} = \begin{cases} \frac{1}{k\sqrt{2}} & \text{for } n = k, \\ 0 & \text{for } n \neq k. \end{cases}$$

We can see that $\lim_{k \rightarrow \infty} z_k = 0 \in \mathbb{R}^{\mathbb{N}}$ and

$$\rho_j((z_k)) = \begin{cases} \frac{1}{2k^2} & \text{for } j \geq k, \\ 0 & \text{for } j < k. \end{cases}$$

For $j \leq k$ we obtain $\phi(j^2 \rho_j((x_k))) = 1$ and therefore

$$\xi((x_k)) = \sum_{j=1}^{\infty} \phi(j^2 \rho_j((x_k))) \geq \sum_{j=1}^k 1 = k.$$

This means that $\lim_{k \rightarrow \infty} \xi((x_k)) = +\infty$. The function ξ is not prolongable to any continuous function in $\mathbb{R}^{\mathbb{N}}$. ■

Now we prove

LEMMA 12. *The differential space (M, \mathcal{C}_M) is smoothly real-compact.*

Proof. From Lemma 9 we know that the set $\text{Spec } \mathcal{C}_M$ may contain only one homomorphism χ_0 which is not an evaluation. This homomorphism would be defined on the generators by the formula $\chi_0(\pi_i|_M) = 0$ for all $i \in I$. So there would be only one point $0 \in \tilde{M} - M$. But it cannot be so since from Corollary 4 we know that all the functions from \mathcal{C}_M are prolongable to the point 0. From the last lemma we know that there exists a function $\xi \in \mathcal{C}_M$ which is not prolongable. ■

One can easily see

COROLLARY 5. *For any $p \in \mathbb{R}^{\mathbb{N}}$ the differential space $(\mathbb{R}^{\mathbb{N}} - \{p\}, (\varepsilon_{\mathbb{N}})_{\mathbb{R}^{\mathbb{N}} - \{p\}})$ is smoothly real-compact.*

Proof. This space is diffeomorphic to (M, \mathcal{C}_M) so it is smoothly real-compact. ■

DEFINITION 4. The disjoint union of differential spaces (M, \mathcal{C}) and (N, \mathcal{D}) where $M \cap N = \emptyset$ is the differential space $(M \cup N, \mathcal{C} \oplus \mathcal{D})$. The structure $\mathcal{C} \oplus \mathcal{D}$ is defined by the property $f \in \mathcal{C} \oplus \mathcal{D} \iff f|_M \in \mathcal{C}$ and $f|_N \in \mathcal{D}$.

We will prove:

LEMMA 13. *If differential spaces (M, \mathcal{C}) and (N, \mathcal{D}) are smoothly real-compact then the differential space $(M \cup N, \mathcal{C} \oplus \mathcal{D})$ is smoothly real-compact.*

Proof. Elements of the algebra $\mathcal{C} \oplus \mathcal{D}$ are pairs (f, g) where $f \in \mathcal{C}$ and $g \in \mathcal{D}$. Let $\chi \in \text{Spec}(\mathcal{C} \oplus \mathcal{D})$. We shall show that it is evaluation at

some point $p \in M \cup N$. From the equalities $(0, 1) + (1, 0) = (1, 1)$ and $(0, 1)(1, 0) = (0, 0)$ we obtain two cases:

- 1) $\chi((1, 0)) = 1$ and $\chi((0, 1)) = 0$,
- 2) $\chi((1, 0)) = 0$ and $\chi((0, 1)) = 1$.

Since every function from $\mathcal{C} \oplus \mathcal{D}$ can be uniquely decomposed as $(f, g) = (f, 0)(1, 0) + (0, g)(0, 1)$ the homomorphism χ acts as follows:

$$\chi((f, g)) = \chi((f, 0))\chi((1, 0)) + \chi((0, g))\chi((0, 1)).$$

In case 1) we will get $\chi((f, g)) = \chi((f, 0))$ and in case 2), $\chi((f, g)) = \chi((0, f))$. The algebra of functions of the form $((f, 0)) \in \mathcal{C} \oplus \mathcal{D}$ is isomorphic to \mathcal{C} . Therefore a homomorphism $\psi \in \text{Spec } \mathcal{C}$ can be extended to a homomorphism from $\mathcal{C} \oplus \mathcal{D}$ by the formula $\bar{\psi}((f, g)) = \psi(f)$. All the homomorphisms in case 1) are of this form. Therefore in case 1) the homomorphism $\chi((f, g)) = \psi(f)$ where $\psi \in \text{Spec } \mathcal{C}$ is such that $\bar{\psi} = \chi$. But since the space (M, \mathcal{C}) is smoothly real-compact there exists a point $p \in M$ such that $\psi = ev_p$. Then we can write $\chi((f, g)) = ev_p((f, g)) = (f, g)(p) = f(p) + g(p)$ for $p \in M \cup N$. We have shown that in case 1) the homomorphism χ is an evaluation. For case 2) the proof is analogous. ■

DEFINITION 5. We denote by $\tilde{\varepsilon}$ the differential structure on $\mathbb{R}^{\mathbb{N}}$ generated by the set $\mathcal{C}_0 = \{\pi_i : i \in \mathbb{N}\} \cup \{\theta_p\}$, where θ_p is the characteristic function of the point $p \in M$.

LEMMA 14. *The differential space $(\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$ is smoothly real-compact.*

Proof. We can decompose the space $(\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$ into the direct sum of the spaces $(\mathbb{R}^{\mathbb{N}} - \{p\}, (\varepsilon_{\mathbb{N}})_{\mathbb{R}^{\mathbb{N}} - \{p\}})$ and $(\{p\}, F(p))$ where $F(p)$ is the algebra of all functions on the singleton space. From the definition of $(\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$ it is obvious that $\mathbb{R}^{\mathbb{N}} = \{p\} \cup (\mathbb{R}^{\mathbb{N}} - \{p\})$ and $\tilde{\varepsilon} = (\varepsilon_{\mathbb{N}})_{\mathbb{R}^{\mathbb{N}} - \{p\}} \oplus F(p)$. Both spaces in the direct sum are smoothly real-compact so the space $(\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$ is smoothly real-compact. ■

Now we present the main results:

THEOREM 1. *Any differential subspace of $(\mathbb{R}^{\mathbb{N}}, \varepsilon_{\mathbb{N}})$ is smoothly real-compact.*

Proof. Let $\iota_M : (M, \mathcal{C}) \rightarrow (\mathbb{R}^{\mathbb{N}}, \varepsilon_{\mathbb{N}})$ be the inclusion mapping. For any $\chi \in \text{Spec } \mathcal{C}$, $\chi \circ \iota_M^* \in \text{Spec } (\varepsilon_{\mathbb{N}})$. The space $(\mathbb{R}^{\mathbb{N}}, \varepsilon_{\mathbb{N}})$ is smoothly real-compact so there is $p \in \mathbb{R}^{\mathbb{N}}$ such that $\chi \circ \iota_M^* = ev_p|_{\varepsilon_{\mathbb{N}}}$.

We need to show that $p \in M$. Assume that $p \notin M$. We can treat the space (M, \mathcal{C}) as a differential subspace of $(\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$. Let $\nu_M : (M, \mathcal{C}) \rightarrow (\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$ be the inclusion. Then $\chi \circ \nu_M^* \in \text{Spec } \tilde{\varepsilon}$. Because the space $(\mathbb{R}^{\mathbb{N}}, \tilde{\varepsilon})$ is smoothly real-compact there exists a point $q \in \mathbb{R}^{\mathbb{N}}$ such that $\chi \circ \nu_M^* = ev_q|_{\tilde{\varepsilon}}$. We know that on common generators π_i the equalities $\chi(\pi_i|_M) = ev_p(\pi_i) = p_i$ and

$\chi(\pi_i|_M) = ev_q(\pi_i) = q_i$ holds for all $i \in \mathbb{N}$. This specifies all the coordinates so $p = q$. Therefore we can write $\chi \circ \nu_M^* = ev_p|_{\varepsilon}$. So $(\chi \circ \nu_M^*)(\theta_p) = ev_p(\theta_p) = 1$. We have a contradiction with the fact that $(\chi \circ \nu_M^*)(\theta_p) = \chi(\theta_p|_M) = \chi(0) = 0$. We see that $p \in M$ and $\chi \circ \iota_M^* = ev_p|_{\varepsilon_{\mathbb{N}}}$. So $\chi(\pi_i|_M) = ev_p(\pi_i|_M)$ for all $i \in \mathbb{N}$. The set $\{\pi_i : i \in \mathbb{N}\}$ is the set of generators of the differential space (M, \mathcal{C}) . We derive that $\chi = ev_p$. ■

COROLLARY 6. *Any countably generated differential space is smoothly real-compact.*

Proof. Every countably generated differential space can be treated as a subspace of $(\mathbb{R}^{\mathbb{N}}, \varepsilon_{\mathbb{N}})$. From Theorem 1 we know that all subspaces of this space are smoothly real-compact. ■

4. Conclusion

We have shown how a real-valued homomorphism act on the algebra of smooth functions in a differential space. It is sufficient to examine its value on the generators of the differential structure. A non-prolongable function on a differential space of sequences without zero sequence is constructed. From the existence of this function we deduced that countably generated structural algebra \mathcal{C} of a differential space (M, \mathcal{C}) gives all information about the set M because the set $\text{Spec } \mathcal{C}$ contains only real valued homomorphisms of the form ev_p for $p \in M$. Therefore the geometry of such spaces can be built on their algebras. The choice of generators of the differential structure is not unique so it is important to choose the smallest one. It is also shown that the pair $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ is a differential space for any differential space (M, \mathcal{C}) . For countably generated differential spaces, (M, \mathcal{C}) and $(\text{Spec } \mathcal{C}, \hat{\mathcal{C}})$ are diffeomorphic. Theorem 1 has been obtained as a result of observations about generators. From [1] and [7] it also follows as a conclusion from a much wider theory. In [12] there is an important theorem which gives a sufficient condition for an algebra of functions on an arbitrary topological space to be smoothly real-compact. We could easily show that countably generated differential spaces satisfy this condition. But we have presented our proof using techniques of differential space theory.

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