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REMARKS ON THE TOPOLOGIES IN THE LEBESGUE MEASURABLE SETS

Abstract. This paper deals with the abstract density topologies in the family of Lebesgue measurable sets generated by an operator similar to the density lower operator defined in the family of measurable sets.

In the theory of density topologies on the real line, in the family of Lebesgue measurable sets the density topology is introduced usually by an operator having desired properties and defined on the family of Lebesgue measurable sets. In that way we get the classical density topology, ψ-density topology, f-density topology, $\langle s\rangle$-density topology (see [8], [6], [5], [3], [7], [1], [2]).

Let us assume that $\mathbb{R}$ is the set of reals, ℓ is the Lebesgue measure on $\mathbb{R}$, $\mathcal{L}$ is the family of Lebesgue measurable sets on $\mathbb{R}$ and $\mathbb{L}$ is the family of ℓ-null sets. By $\mathcal{T}_{\mathrm{nat}}$ and $\mathcal{T}_d$ we will denote the natural topology and the classical density topology on $\mathbb{R}$, respectively. The fact that $\ell(A\triangle B)=0$ for any sets $A,B\in\mathcal{L}$ will be denoted by $A\sim B$.

We shall say that an operator $\Phi:\mathcal{L}\to\mathcal{L}$ is a lower density operator if the following conditions are satisfied:

(I) $\Phi(\emptyset)=\emptyset$, $\Phi(\mathbb{R})=\mathbb{R}$,
(II) $\Phi(A\cap B)=\Phi(A)\cap\Phi(B)$ for any $A,B\in\mathcal{L}$,
(III) $A\sim B\Rightarrow\Phi(A)=\Phi(B)$ for any $A,B\in\mathcal{L}$,
(IV) $\Phi(A)\sim A$ for any $A\in\mathcal{L}$.

Many examples of such operators are known. One of well-known examples is the classical density operator denoted by Φ_d. It is worth adding that if

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we consider two lower density operators Φ_1 and Φ_2 then $\Phi_1(A) \sim \Phi_2(A)$ for any $A \in \mathcal{L}$.

In this paper we will consider operator defined as follows.

Definition 1. Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator such that, for any sets $A, B \in \mathcal{L}$ we infer that

1^0 $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,
2^0 $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,
3^0 $A \sim B \Rightarrow \Phi(A) = \Phi(B)$,
4^0 $\ell(\Phi(A) \setminus A) = 0$.

We will call this operator an almost density operator. It is clear that the condition 2^0 implies the monotonicity of Φ i.e if $A \subset B$ then $\Phi(A) \subset \Phi(B)$ for any $A, B \in \mathcal{L}$.

Obviously if an operator Φ is a lower density operator, then it satisfies conditions $1^0 - 4^0$ above. There exists an operator $\Phi : \mathcal{L} \to \mathcal{L}$ fullfiling conditions $1^0 - 4^0$ which is not a lower density operator. Indeed, let

$$\Phi_0(A) = \begin{cases} \mathbb{R}, & A \sim \mathbb{R}, \\ \emptyset, & \neg(A \sim \mathbb{R}) \end{cases} \tag{1}$$

for any $A \in \mathcal{L}$. It is easy to verify that conditions 1^0–4^0 are fullfiled and condition (IV) is not fullfiled.

Moreover, if Φ is a lower density operator and Φ^* is an operator satisfying conditions $1^0 - 4^0$, then

$$\begin{aligned} \ell(\Phi(A) \setminus \Phi^*(A)) &= \ell((\Phi(A) \cap A \cup \Phi(A) \setminus A) \setminus \Phi^*(A)) \\ &\leq \ell(A \setminus \Phi^*(A)) + \ell(\Phi(A) \setminus A) = 0 \end{aligned}$$

for any $A \in \mathcal{L}$.

Remark 2. *There exists an operator Φ satisfying conditions $1^0 - 4^0$ such that for any lower density operator Φ^* and for every set A of positive measure such that $\neg(A \sim \mathbb{R})$, we have that $\ell(\Phi^*(A) \setminus \Phi(A)) > 0$.*

Proof. Let Φ be as in (1). Obviously it satisfies conditions $1^0 - 4^0$. Let Φ^* be a lower density operator. Then for any set A of positive measure and such that $\neg(A \sim \mathbb{R})$ we have

$$\begin{aligned} \ell(\Phi^*(A) \setminus \Phi(A)) &= \ell(\Phi^*(A) \cap A) + \ell(\Phi^*(A) \setminus A) \\ &= \ell(\Phi^*(A) \cap A) + \ell(A \setminus \Phi^*(A)) = \ell(A) > 0. \ \blacksquare \end{aligned}$$

From the above remark we obtain immediately that there exist operators Φ_1 and Φ_2 satisfying conditions $1^0 - 4^0$ such that $\neg(\Phi_1(A) \sim \Phi_2(A))$ for some $A \in \mathcal{L}$.

Let us introduce the following notation

$$\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\},$$

for an arbitrary operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$. If the family $\mathcal{T}_\Phi$ forms a topology on $\mathbb{R}$ we shall say that $\mathcal{T}_\Phi$ is a topology generated by the operator Φ.

Theorem 3. (cf. Th. 6 [4]) *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 4^0$. Then the family $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ forms a topology on $\mathbb{R}$.*

Proof. By conditions 1^0 and 2^0 it is clear that the family $\mathcal{T}_\Phi$ contains $\emptyset$, $\mathbb{R}$ and is closed under finite intersections. Let $\{A_t\}_{t\in\mathbb{T}} \subset \mathcal{T}_\Phi$. We show that $\bigcup_{t\in\mathbb{T}} A_t \in \mathcal{T}_\Phi$.

Let B be a measurable kernel of the set $\bigcup_{t\in\mathbb{T}} A_t$. Then for every $t \in \mathbb{T}$ we get that $B \cap A_t \sim A_t$. Hence, by condition 3^0, we obtain that

$$B \subset \bigcup_{t\in\mathbb{T}} A_t \subset \bigcup_{t\in\mathbb{T}} \Phi(A_t) = \bigcup_{t\in\mathbb{T}} \Phi(B \cap A_t) \subset \Phi(B).$$

By condition 4^0 we have that $\ell(\Phi(B) \setminus B) = 0$. Thus $\bigcup_{t\in\mathbb{T}} A_t \in \mathcal{L}$ and obviously $\bigcup_{t\in\mathbb{T}} A_t \subset \Phi(\bigcup_{t\in\mathbb{T}} A_t)$. ∎

Theorem 4. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 3^0$. Then the family $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ forms a topology on $\mathbb{R}$ if and only if*

$$(*) \qquad \underset{A\in\mathcal{L}}{\forall} (A \subset \Phi(A) \implies \ell(\Phi(A) \setminus A) = 0).$$

Proof. The proof of the sufficient condition runs similarly like the proof of Theorem 3.

Necessity. Let us suppose that

$$\underset{A\in\mathcal{L}}{\exists} (A \subset \Phi(A) \wedge (\Phi(A) \setminus A) \notin \mathbb{L}).$$

Then there exists a nonmeasurable set $B \subset \Phi(A) \setminus A$. It is clear that $A \in \mathcal{T}_\Phi$ and the condition 3^0 implies that $A \cup \{b\} \in \mathcal{T}_\Phi$ for every $b \in B$. As the family $\mathcal{T}_\Phi$ forms topology, we get that $\bigcup_{b\in B}(A \cup \{b\}) = A \cup B \in \mathcal{T}_\Phi$. It implies that $A \cup B \in \mathcal{L}$ and finally $B \in \mathcal{L}$. This contradiction ends the proof. ∎

It is worth noting that the condition 3^0 plays a part in the above proofs. The following example demonstrates that this condition is essential.

Example 5. Let $\Phi(A) = A$ for any $A \in \mathcal{L}$. Obviously this operator satisfies conditions $1^0, 2^0$ and 4^0 but it does not satisfy the condition 3^0 and the family $\mathcal{T}_\phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ does not form a topology.

The next theorem shows that the condition 3^0 can be replaced by some property.

Property 6. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions 1^0 and 2^0. If there exists an operator $\Phi^* : \mathcal{L} \to 2^{\mathbb{R}}$ such that $\Phi(A) \subset \Phi^*(A)$ for any $A \in \mathcal{L}$ and $\mathcal{T}_{\Phi^*} = \{A \in \mathcal{L} : A \subset \Phi^*(A)\}$ forms a topology on $\mathbb{R}$, then the family $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ forms a topology on $\mathbb{R}$ as well.*

Proof. By condition 1^0 and 2^0 it is clear that the family $\mathcal{T}_\Phi$ contains $\emptyset$, $\mathbb{R}$ and is closed under finite intersections. Let $\{A_t\}_{t\in\mathbb{T}} \subset \mathcal{T}_\Phi$. Since $\{A_t\}_{t\in\mathbb{T}} \subset \mathcal{T}_{\Phi^*}$ then it is clear that $\bigcup_{t\in\mathbb{T}} A_t \in \mathcal{L}$. Moreover the condition 2^0 implies that $\bigcup_{t\in\mathbb{T}} A_t \subset \Phi(\bigcup_{t\in\mathbb{T}} A_t)$. Hence we get that $\bigcup_{t\in\mathbb{T}} A_t \in \mathcal{T}_\Phi$. ∎

Example 7. Let

$$\Phi(A) = \begin{cases} \mathbb{R}, & A = \mathbb{R}, \\ \mathbb{R} \setminus \{1\}, & A \sim \mathbb{R} \wedge A \neq \mathbb{R}, \\ \emptyset, & \neg(A \sim \mathbb{R}) \end{cases} \tag{3}$$

for any $A \in \mathcal{L}$. It is easy to verify that conditions $1^0, 2^0$ are fulfiled and the condition 3^0 is not satisfied. However $\Phi(A) \subset \Phi_0(A)$ for any $A \in \mathcal{L}$, so the family $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ forms a topology on $\mathbb{R}$.

The following observation proves that a nonempty measurable set for which the result of operator Φ is null set can be omitted in generated topology $\mathcal{T}_\Phi$.

Observation 8. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 3^0$ and let $\mathcal{N}_0 = \{A \in \mathcal{L} : \ell(\Phi(A)) = 0\}$. Then*

$$\{A \in \mathcal{L} : A \subset \Phi(A)\} = \{A \in \mathcal{L} \setminus \mathcal{N}_0 : A \subset \Phi(A)\} \cup \{\emptyset\}.$$

Remark 9. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 4^0$ and let $Z \subset \mathbb{R}$. Then the operator $\Phi_Z : \mathcal{L} \to 2^{\mathbb{R}}$ defined as follows*

$$\Phi_Z(A) = \begin{cases} \mathbb{R}, & A \sim \mathbb{R}, \\ \Phi(A) \cap Z, & \neg(A \sim \mathbb{R}) \end{cases}$$

for any $A \in \mathcal{L}$ also satisfies conditions $1^0 - 4^0$ and

$$\mathcal{T}_{\Phi_Z} = \{A \subset \mathbb{R} : A \sim \mathbb{R}\} \cup (\mathcal{T}_\Phi \cap 2^Z)$$

forms a topology. Moreover if the inner Lebesgue measure of Z is equal to 0 then $\mathcal{T}_{\Phi_Z} = \{A \subset \mathbb{R} : A = \emptyset \vee A \sim \mathbb{R}\}$.

The following examples show that there is no connection in the sense of inclusion between the topologies generated by operators satisfying conditions $1^0 - 4^0$.

Example 10. Let

$$\Phi(A) = \begin{cases} \mathbb{R}, & A \sim \mathbb{R}, \\ \Phi_d(A) \cap (\mathbb{R} \setminus \mathbb{Q}), & \neg(A \sim \mathbb{R}) \end{cases}$$

for every $A \in \mathcal{L}$, where $\mathbb{Q}$ is the set of all rational numbers. Then according to Remark 9 operator $\Phi : \mathcal{L} \to \mathcal{L}$ satisfies conditions $1^0 - 4^0$ and $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \sim \mathbb{R}\} \cup (\mathcal{T}_d \cap 2^{\mathbb{R}\setminus\mathbb{Q}})$. Thus

$$\{A \subset \mathbb{R} : A = \emptyset \vee A \sim \mathbb{R}\} \subsetneqq \mathcal{T}_\Phi \subsetneqq \mathcal{T}_d$$

and

$$\mathcal{T}_\Phi \cap \mathcal{T}_{nat} = \{A \in \mathcal{T}_{nat} : A = \emptyset \vee A \sim \mathbb{R}\}.$$

Example 11. Let $\Phi : \mathcal{L} \to \mathcal{L}$ be an $\langle s\rangle$-density operator (see [2]). Then this operator satisfies conditions $1^0 - 4^0$ and

$$\mathcal{T}_d \subsetneqq \mathcal{T}_\Phi$$

if only $\langle s\rangle = \{s_n\}_{n\in\mathbb{N}}$ is an unbounded and nondecreasing sequence of positive numbers such that $\liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0$.

The following example presents an operator for which the range contains a nonmeasurable set.

Example 12. Let $Y \notin \mathcal{L}$ and $Y \subset (0,1)$. For every set $A \in \mathcal{L}$ let us put $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ in the folowing manner:

$$\Phi(A) = \begin{cases} \mathbb{R}, & A \sim \mathbb{R}, \\ \Phi_d(A) \cap Y, & \neg(A \sim \mathbb{R}). \end{cases}$$

Obviously the conditions $1^0 - 4^0$ are satisfied and $\Phi((0,1)) = \Phi_d((0,1)) \cap Y = Y \notin \mathcal{L}$. Simultaneously

$$\mathcal{T}_\Phi = \{A \subset \mathbb{R} : A \sim \mathbb{R}\} \cup (\mathcal{T}_d \cap 2^Y).$$

The next theorem shows that we can restrict operator Φ so that the range is the family $\mathcal{L}$.

Theorem 13. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 4^0$. Then there exists a subfamily $\mathcal{S} \subset \mathcal{L}$ containing $\emptyset, \mathbb{R}$ and closed under finite intersections and an operator $\Phi' : \mathcal{S} \to \mathcal{L}$ satisfying conditions $1^0 - 4^0$ such that the family*

$$\mathcal{T}_{\Phi'} = \{A \in \mathcal{S} : A \subset \Phi'(A)\}$$

forms a topology and

$$\mathcal{T}_{\Phi'} = \mathcal{T}_\Phi,$$

where $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$.

Proof. Let $\mathcal{S} = \{A \in \mathcal{L} : \Phi(A) \in \mathcal{L}\}$. Then $\emptyset, \mathbb{R} \in \mathcal{S}$ and $\mathcal{S}$ is closed under finite intersections. Let $\Phi' = \Phi_{|\mathcal{S}}$. It is clear that $\Phi' : \mathcal{S} \to \mathcal{L}$ and conditions $1^0 - 4^0$ are fulfiled. We show that $\mathcal{T}_{\Phi'} = \mathcal{T}_\Phi$. It is sufficient to prove that $\mathcal{T}_\Phi \subset \mathcal{T}_{\Phi'}$. Let $A \in \mathcal{T}_\Phi$. Then $A \subset \Phi(A)$ and $\Phi(A) = A \cup (\Phi(A) \setminus A)$. Since $\ell(\Phi(A) \setminus A) = 0$ we get that $A \in \mathcal{S}$ and $A \subset \Phi(A)$. Thus $A \in \mathcal{T}_{\Phi'}$.

By Theorem 3 the family $\mathcal{T}_\Phi$ forms a topology. Hence $\mathcal{T}_{\Phi'}$ is a topology, as well. ∎

Let us pay attention to some properties of $\mathcal{T}_\Phi$ topologies.

Theorem 14. *Let $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ be a topology obtained via operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ having the properties $1^0 - 4^0$. Then for every set $E \subset \mathbb{R}$ the following properties hold:*

a) *if $E \in \mathbb{L}$, then E is $\mathcal{T}_\Phi$-nowhere dense,*
b) *$E \in \mathbb{L}$ if and only if E is $\mathcal{T}_\Phi$-closed and $\mathcal{T}_\Phi$-discrete,*
c) *E is compact if and only if E is finite.*

Moreover,

d) *the space $(\mathbb{R}, \mathcal{T}_\Phi)$ is neither Lindelöf space nor the first countable and nor separable,*
e) *$\mathbb{L} \subset K(\mathcal{T}_\Phi)$, where $K(\mathcal{T}_\Phi)$ is the family of meager sets on $\mathbb{R}$ with respect to the topology $\mathcal{T}_\Phi$,*
f) *if $\mathcal{T}_{nat} \subset \mathcal{T}_\Phi$, then $\mathcal{L} = \mathcal{B}(\mathcal{T}_\Phi) = F_\sigma(\mathcal{T}_\Phi)$, where $\mathcal{B}(\mathcal{T}_\Phi)$ and $F_\sigma(\mathcal{T}_\Phi)$ is the family of all Borel sets and F_σ-type sets with respect to the topology $\mathcal{T}_\Phi$, respectively.*

Proof. Conditions a)–d) can be proved in a similar way like Theorems 2.8, 2.10, 2.11 in [8], respectively. The property e) is a consequence of property a). We shall prove condition f).

Let $\mathcal{T}_{\mathrm{nat}} \subset \mathcal{T}_\Phi$ and $A \in \mathcal{L}$. Then $A = \bigcup_{n=1}^{\infty} F_n \cup B$, where F_n is $\mathcal{T}_{\mathrm{nat}}$-closed for $n \in \mathbb{N}$ and $B \in \mathbb{L}$. Hence, by condition b) we get that $A \in F_\sigma(\mathcal{T}_\Phi)$ and $\mathcal{L} \subset F_\sigma(\mathcal{T}_\Phi) \subset B(\mathcal{T}_\Phi)$ and we conclude that $\mathcal{L} = F_\sigma(\mathcal{T}_\Phi) = B(\mathcal{T}_\Phi)$. ∎

Let us observe that condition, that the family $K(\mathcal{T}_\Phi)$ of meager sets with respect to the topology $\mathcal{T}_\Phi$ concides with the family of the Lebesgue null sets, is very important to get the restriction of operator Φ which becomes the lower density operator (cf. [LMZ]). From Theorem 3 and from Theorem 5 in [4] we obtain immediately the following theorems.

Theorem 15. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 4^0$. Then $K(\mathcal{T}_\Phi) = \mathbb{L}$ if and only if there exist a σ-algebra $\mathcal{S} \subset \mathcal{L}$ and a lower density operator $\Phi' : \mathcal{S} \to \mathcal{L}$ such that $\mathcal{T}_\Phi = \mathcal{T}_{\Phi'}$.*

Theorem 16. *Let $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ be an operator satisfying conditions $1^0 - 4^0$. Then $K(\mathcal{T}_\Phi) = \mathbb{L}$ and $Ba(\mathcal{T}_\Phi) = \mathcal{L}$, where $Ba(\mathcal{T}_\Phi)$ is the family of sets having the Baire property with respect to topology $\mathcal{T}_\Phi$ if and only if Φ is a lower density operator.*

The topologies generated by operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ are usually investigated in the aspect of translation and multiplication by coefficient not equal zero (see [1], [2]).

Definition 17. We shall say that an operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ is invariant with respect to translation in a set $T \subset \mathbb{R}$ if

$$\underset{A\in\mathcal{L}}{\forall} \ \underset{t\in T}{\forall} \Phi(A+t) = \Phi(A) + t.$$

Definition 18. We shall say that an operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ is invariant with respect to multiplication in a set $T \subset \mathbb{R}$ if

$$\underset{A\in\mathcal{L}}{\forall} \ \underset{\alpha\in T}{\forall} \Phi(\alpha A) = \alpha\Phi(A).$$

Definition 19. We shall say that a family $\mathcal{T}_\Phi$ generated by an operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ is invariant with respect to translation in a set $T \subset \mathbb{R}$ if

$$\underset{A\in\mathcal{T}_\Phi}{\forall} \ \underset{t\in T}{\forall} A + t \in \mathcal{T}_\Phi.$$

Definition 20. We shall say that a family $\mathcal{T}_\Phi$ generated by an operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ is invariant with respect to multiplication in a set $T \subset \mathbb{R}$ if

$$\underset{A\in\mathcal{T}_\Phi}{\forall} \ \underset{\alpha\in T}{\forall} \alpha A \in \mathcal{T}_\Phi.$$

The following property is obvious.

Property 21. *If an operator $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ is invariant with respect to translation (multiplication) in a set $T \subset \mathbb{R}$, then the family $\mathcal{T}_\Phi$ generated by the operator Φ is invariant with respect to translation (multiplication) in the set T.*

It is easy to point out an operator such that the contrary property is not true. The following example presents an operator $\Phi : \mathcal{L} \to \mathcal{L}$, which is invariant with respect to translation in the set $\{0\}$, with respect to multiplication in the set $\{1\}$, and the topology $\mathcal{T}_\Phi$ is invariant with respect to translation in the set $T = \mathbb{R}$ and is invariant with respect to multiplication in the set $T = \mathbb{R} \setminus \{0\}$.

Example 22. Let $x_0 \in \mathbb{R}$ and let $\Phi : \mathcal{L} \to \mathcal{L}$ be defined as follows

$$\Phi(A) = \begin{cases} \mathbb{R}, & A \sim \mathbb{R}, \\ \Phi_d(A) \cap \{x_0\}, & \neg(A \sim \mathbb{R}) \end{cases}$$

for any $A \in \mathcal{L}$. Then

$$\underset{t\in\mathbb{R}\setminus\{0\}}{\forall} \ \underset{A\in\mathcal{L}}{\exists} \Phi(A+t) \neq \Phi(A) + t$$

and

$$\underset{\alpha\in\mathbb{R}\setminus\{1\}}{\forall} \ \underset{A\in\mathcal{L}}{\exists} \Phi(\alpha A) \neq \alpha\Phi(A).$$

Simultaneously, by Remark 9

$$\mathcal{T}_\Phi = \{A \subset \mathbb{R} : A = \emptyset \vee A \sim \mathbb{R}\}$$

and we have that

$$\underset{t\in\mathbb{R}}{\forall}\ \underset{A\in\mathcal{T}_\Phi}{\forall} A+t\in\mathcal{T}_\Phi$$

and

$$\underset{\alpha\in\mathbb{R}\setminus\{0\}}{\forall}\ \underset{A\in\mathcal{T}_\Phi}{\forall} \alpha A\in\mathcal{T}_\Phi.$$

At this moment we present some general properties concerning the invariance of an operator Φ and the topology generated by the operator Φ.

Proposition 23. *Let $\Phi:\mathcal{L}\to 2^{\mathbb{R}}$ be an opeartor generating a topology $\mathcal{T}_\Phi$ which is invariant with respect to translation in the set $\mathbb{R}$. Let $A\in\mathcal{T}_\Phi$ and $\alpha\in\mathbb{R}$. Then the following property holds*

$$\Big(\underset{B\in\mathcal{L}}{\forall}(0\in\Phi(B)\Rightarrow 0\in\Phi(\alpha B))\Big)\Rightarrow(\alpha A\in\mathcal{T}_\Phi).$$

Proof. Let $A\in\mathcal{T}_\Phi$ and $\alpha\neq 0$. Let $y\in\alpha A$. Then $\frac{y}{\alpha}\in A\subset\Phi(A)$. Hence $0\in\Phi(A)-\frac{y}{\alpha}=\Phi(A-\frac{y}{\alpha})$. By the assumption $0\in\Phi(\alpha A-y)=\Phi(\alpha A)-y$. Hence $y\in\Phi(\alpha A)$ and we get that $\alpha A\in\mathcal{T}_\Phi$.

Let $\alpha=0$. Then we conclude that the condition

$$\underset{B\in\mathcal{L}}{\forall}(0\in\Phi(B)\Rightarrow 0\in\Phi(\{0\}))$$

is not true. Indeed, let us suppose that this condition is true. Evidently, putting $B=\mathbb{R}$ we have that $0\in\Phi(\mathbb{R})$. Thus $\{0\}\subset\Phi(\{0\})$ and $\{0\}\in\mathcal{T}_\Phi$. By translation property we obtain that $\{x\}\in\mathcal{T}_\Phi$ for any $x\in\mathbb{R}$. This contradicts the fact that the topology $\mathcal{T}_\Phi$ is included in the family $\mathcal{L}$. In this way the proof is completed. ∎

Proposition 24. *Let $\Phi:\mathcal{L}\to 2^{\mathbb{R}}$ be an opeartor generating a topology $\mathcal{T}_\Phi$ and let Φ be invariant with respect to translation in the set $\mathbb{R}$. Then for every $\alpha\in\mathbb{R}$ the following conditions are equivalent:*

a) $\underset{A\in\mathcal{T}_\Phi}{\forall}(0\in A\Rightarrow 0\in\Phi(\alpha A))$,

b) $\underset{A\in\mathcal{T}_\Phi}{\forall}(\alpha A\in\mathcal{T}_\Phi)$.

Proof. Let $\alpha\neq 0$. We shall prove that a)⇒b).

Let $A\in\mathcal{T}_\Phi$ and $y\in\alpha A$. Then $\frac{y}{\alpha}\in A$. It implies that $0\in A-\frac{y}{\alpha}$ and by the assumption $0\in\Phi(\alpha A-y)=\Phi(\alpha A)-y$. Hence $y\in\Phi(\alpha A)$. Thus $\alpha A\subset\Phi(\alpha A)$. It means that $\alpha A\in\mathcal{T}_\Phi$.

Now, we shall prove b)⇒a).

Let us suppose that

$$\underset{A\in\mathcal{T}_\Phi}{\exists}(0\in A\wedge 0\notin\Phi(\alpha A)).$$

Since $\alpha A\in\mathcal{T}_\Phi$ it follows that $\alpha A\subset\Phi(\alpha A)$. Moreover we have $0\in\Phi(\alpha A)$ because $0\in A$. This contradicts the fact that $0\notin\Phi(\alpha A)$.

Let $\alpha = 0$. Then the both conditions a) and b) are false. Otherwise we obtain that $\{0\} \in \mathcal{T}_\Phi$ and thus $\{x\} \in \mathcal{T}_\Phi$ for every $x \in \mathbb{R}$. It is a contradiction of the fact that the topology $\mathcal{T}_\Phi$ is included in the family $\mathcal{L}$. ■

References

[1] M. Filipczak, T. Filipczak, *A generalization of the density topology*, Tatra Mt. Math. Publ. 34 (2006), 37–47.
[2] M. Filipczak, J. Hejduk, *On topologies associated with the Lebesgue measure*, Tatra Mt. Math. Publ. 28 (2004), 187–197.
[3] J. Hejduk, A. B. Kharazishvili, *On density point with respect to von Neumann's topology*, Real Anal. Exchange 21(1) (1995–96), 278–291.
[4] J. Hejduk, A. Loranty, *On lower and semi lower density operators*, Georgian Math. J. 14 (2007), 661–667.
[5] J. Lukàs, J. Malý, L. Zajiček, *Fine topology methods*, Real Analysis and Potential Theory, Lecture Notes in Math. 1189, Springer, Berlin, 1985.
[6] J. Oxtoby, *Measure and Category*, Springer, New York, 1971.
[7] M. Terepeta, E. Wagner-Bojakowska, *ψ-density topology*, Rend. Circ. Mat. Palermo 48 (1999), 451–476.
[8] W. Wilczyński, *Density Topologies*, Chapter 15 in Handbook of Measure Theory, Ed. E. Pap, Elsevier 2002, 675–702.

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