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ON PINCHING THEOREMS FOR COMPACT PSEUDO-UMBILICAL SUBMANIFOLD

Abstract. Consider submanifolds in the nested space. For a compact pseudo-umbilical submanifold with parallel mean curvature vector of a Riemannian submanifold with constant curvature immersed in a quasi-constant curvature Riemannian manifold, two sufficient conditions are given to let the pseudo-umbilical submanifold become a totally umbilical submanifold.

1. Introduction

Suppose M_2^{n+p+q} is an $(n+p+q)$-dimensional quasi-constant curvature Riemannian manifold, $M_1^{n+p}(c)$ is an $(n+p)$-dimensional submanifold with constant curvature c in M_2^{n+p+q}, and M^n is a compact pseudo-umbilical submanifold with parallel mean curvature vector in $M_1^{n+p}(c)$. Then we know that M^n is a submanifold in M_2^{n+p+q}. Now, we use σ and σ' to denote the norm of second fundamental form of M^n in M_2^{n+p+q} and $M_1^{n+p}(c)$ respectively. According to our discussion in the sequel, we can get two sufficient conditions to make the compact pseudo-umbilical submanifold M^n be a totally umbilical submanifold in $M_1^{n+p}(c)$ as follows.

Theorem 1. *Suppose $M_1^{n+p}(c)$ is a submanifold with constant curvature c in a quasi-constant curvature space M_2^{n+p+q} and M^n is a compact pseudo-umbilical submanifold with parallel mean curvature vector in $M_1^{n+p}(c)$, where $n, p \geq 2$, if the infimum Q of Ricci curvature R_{ii} on M^n satisfies*

$$Q \geq \frac{2n(p-2)(a+H^2)+(p-2)(n+1)(b+|b|)}{2[2+n(p-1)]} + (n-2)(c+H^2) - \frac{n-2}{n(n-1)}\tau',$$

then M^n is a totally umbilical submanifold in $M_1^{n+p}(c)$.

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Theorem 2. *Suppose $M_1^{n+p}(c)$ is a submanifold with constant curvature c in a quasi-constant curvature space M_2^{n+p+q} and M^n is a compact pseudo-umbilical submanifold with parallel mean curvature vector in $M_1^{n+p}(c)$, where $n, p \geq 2$, if the infimum Q of Ricci curvature R_{ii} on M^n satisfies one of the following conditions*

$$\text{(i)} \qquad Q \geq (n-2)(c+H^2) - \frac{n-2}{n(n-1)}\tau' + \mu(\sigma - nH^2),$$

$$\text{(ii)} \qquad Q \geq (n-2)(c+H^2) - \frac{n-2}{n(n-1)}\tau' + \mu(\sigma' - nH^2),$$

where $\mu = \min\left\{\frac{1}{2}, \frac{p-2}{n(p-1)}\right\}$, then $M^n$is a totally umbilical submanifold in $M_1^{n+p}(c)$.

2. Local formulas

We choose a local orthogonal frame field $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}, e_{n+p+1}, \ldots, e_{n+p+q}\}$ on M_2^{n+p+q}, such that $\{e_1, \ldots, e_n\}$ is a tangent frame field and $\{e_{n+1}, \ldots, e_{n+p}\}$ is a normal frame field on M^n when $M_1^{n+p}(c)$ restricts to M^n. While M_2^{n+p+q} restricts to M^n, $\{e_1, \ldots, e_n\}$ is a tangent frame field and $\{e_{n+1}, \ldots, e_{n+p}, e_{n+p+1}, \ldots, e_{n+p+q}\}$ is a normal frame field on M^n. We fix the range of indices as follows

$$1 \leq i, j, k, \ldots, \leq n; \qquad 1 \leq A, B, \ldots, \leq n+p+q,$$
$$n+1 \leq \alpha_1, \beta_1, \ldots \leq n+p; \qquad n+1 \leq \alpha_2, \beta_2, \ldots \leq n+p+q.$$

Since M_2^{n+p+q} is a quasi-constant curvature Riemannian manifold, its curvature tensor field satisfies

$$\text{(1)} \qquad \begin{aligned} K_{ABCD}^{M_2} = a(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + b(\delta_{AC}\lambda_B\lambda_D \\ + \delta_{BD}\lambda_A\lambda_C - \delta_{AD}\lambda_B\lambda_C - \delta_{BC}\lambda_A\lambda_D), \end{aligned}$$

where $\sum_A \lambda_A^2 = 1$ and a, b, λ_A are smooth functions on M_2^{n+p+q}.

Meanwhile, we can get the following conclusions from [1]

$$\text{(2)} \qquad \sigma = \sum_{\alpha_2=n+1}^{n+p+q} \mathtt{tr} H_{\alpha_2}^2, \qquad \sigma' = \sum_{\alpha_1=n+1}^{n+p} \mathtt{tr} H_{\alpha_1}^2, \qquad \sigma \geq \sigma'.$$

Let

$$\text{(3)} \quad \tau = \sigma - \mathtt{tr} H_{n+1}^2 = \sum_{\alpha_2=n+2}^{n+p+q} \mathtt{tr} H_{\alpha_2}^2, \quad \tau' = \sigma' - \mathtt{tr} H_{n+1}^2 = \sum_{\alpha_1=n+2}^{n+p} \mathtt{tr} H_{\alpha_1}^2,$$

then by properly adjusting the normal frames so that the mean curvature vector of M^n is in the direction of e_{n+1}, this adjustment can be assured by a degenerated transformation, we can get the following relations

$$\tau = \sigma - nH^2, \qquad \tau' = \sigma' - nH^2, \qquad \tau \geq \tau', \tag{4}$$

and for any $A \in R$ the equality

$$\begin{aligned}
&\sum_{\alpha_1=n+2}^{n+p} \sum_{i,j} h_{ij}^{\alpha_1} \Delta h_{ij}^{\alpha_1} \\
&= (1+A) \sum_{\alpha_1=n+2}^{n+p} \sum_{i,j,k,m} h_{ij}^{\alpha_1} \left(h_{km}^{\alpha_1} R_{mijk} + h_{mi}^{\alpha_1} R_{mkjk}\right) \\
&\quad - An(a+H^2)\tau' - Ab\Big(\sum_k \lambda_k^2\Big)\tau' - Anb \sum_{\alpha_1=n+2}^{n+p} \sum_i \Big(\sum_m \lambda_m h_{im}^{\alpha_1}\Big)^2 \\
&\quad + A \sum_{\alpha_1=n+2}^{n+p} \sum_{\alpha_2=n+2}^{n+p+q} [\mathtt{tr}(H_{\alpha_1} H_{\alpha_2})]^2 \\
&\quad + A \sum_{\alpha_1=n+2}^{n+p} \sum_{\alpha_2=n+2}^{n+p+q} \left[\mathtt{tr}(H_{\alpha_1}^2 H_{\alpha_2}^2) - \mathtt{tr}(H_{\alpha_1} H_{\alpha_2})^2\right] \\
&\quad - A \sum_{\alpha_1=n+2}^{n+p} \sum_{\beta_1=n+2}^{n+p} \left[\mathtt{tr}(H_{\alpha_1}^2 H_{\beta_1}^2) - \mathtt{tr}(H_{\alpha_1} H_{\beta_1})^2\right],
\end{aligned} \tag{5}$$

always holds.

Since $0 \leq \sum_k \lambda_k^2 \leq 1$, by Schwarz inequality, we can obtain

$$\begin{aligned}
0 &\leq \sum_{\alpha_1=n+2}^{n+p} \sum_i \Big(\sum_m \lambda_m h_{im}^{\alpha_1}\Big)^2 \\
&\leq \sum_{\alpha_1=n+2}^{n+p} \sum_i \Big(\sum_m \lambda_m^2\Big) \sum_m (h_{im}^{\alpha_1})^2 \leq \sum_{\alpha_1=n+2}^{n+p} \sum_{i,m} (h_{im}^{\alpha_1})^2 = \tau'.
\end{aligned} \tag{6}$$

Lemma 1. (see [2]) *Suppose $M_1^{n+p}(c)$ is a submanifold of M_2^{n+p+q}, M^n is a submanifold of $M_1^{n+p}(c)$, τ and τ' are given by* (3), *then*

$$\tau\tau' \geq \sum_{\alpha_1=n+2}^{n+p} \sum_{\alpha_2=n+2}^{n+p+q} [\mathtt{tr}(H_{\alpha_1} H_{\alpha_2})]^2 \geq \frac{1}{p-1}(\tau')^2.$$

Lemma 2. (see [2]) *Suppose $M_1^{n+p}(c)$ is a submanifold of M_2^{n+p+q}, M^n is a submanifold of $M_1^{n+p}(c)$, τ and τ' are given by* (3), *then*

(i) $0 \le \sum_{\alpha_1=n+2}^{n+p} \sum_{\beta_1=n+2}^{n+p} \left[\mathtt{tr}(H_{\alpha_1}^2 H_{\beta_1}^2) - \mathtt{tr}(H_{\alpha_1} H_{\beta_1})^2\right] \le \frac{p-2}{p-1}(\tau')^2,$

(ii) $$0 \le \sum_{\alpha_1=n+2}^{n+p} \sum_{\beta_1=n+2}^{n+p} \left[\mathtt{tr}(H_{\alpha_1}^2 H_{\beta_1}^2) - \mathtt{tr}(H_{\alpha_1} H_{\beta_1})^2\right]$$
$$\le \frac{n}{2} \sum_{\alpha_1=n+2}^{n+p} \sum_{\beta_1=n+2}^{n+p} [\mathtt{tr}(H_{\alpha_1} H_{\beta_1})]^2 \le \frac{n}{2}(\tau')^2,$$

(iii) $$0 \le \sum_{\alpha_1=n+2}^{n+p} \sum_{\alpha_2=n+2}^{n+p+q} \left[\mathtt{tr}(H_{\alpha_1}^2 H_{\alpha_2}^2) - \mathtt{tr}(H_{\alpha_1} H_{\alpha_2})^2\right]$$
$$\le \tau\tau' - \sum_{\alpha_1=n+2}^{n+p} \left(\mathtt{tr} H_{\alpha_1}^2\right)^2 \le \tau\tau' - \frac{1}{p-1}\tau'^2 \le \tau\tau'.$$

Lemma 3. (see [2]) *Suppose M^n is a submanifold of $M_1^{n+p}(c)$, K is the infimum of sectional curvatures on M^n, then*

$$\sum_{\alpha_1=n+2}^{n+p} \sum_{i,j,k,m} h_{ij}^{\alpha_1} \left(h_{km}^{\alpha_1} R_{mijk} + h_{mi}^{\alpha_1} R_{mkjk}\right) \ge n\tau' K.$$

Lemma 4. *Suppose M^n is a pseudo-umbilical submanifold with parallel mean curvature vector in a Riemannian manifold $M_1^{n+p}(c)$ with constant curvature ($n \ge 2$), K is the infimum of sectional curvatures R_{ijij}, and Q is the infimum of Ricci curvatures R_{ii}, then*

$$K \ge Q - (n-2)(c + H^2) + \frac{n-2}{n(n-1)}\tau'.$$

Proof. By Gauss equation, we can get the following equality

$$R_{ijij} = c + \sum_{\alpha_1} [h_{ii}^{\alpha_1} h_{jj}^{\alpha_1} - (h_{ij}^{\alpha_1})^2], \tag{7}$$

holds when $i \ne j$. Meanwhile, by the definition of Ricci curvature we can obtain

$$R_{ii} = \sum_{k \ne i} R_{ikik} = (n-1)c + \sum_{\alpha_1} h_{ii}^{\alpha_1} \sum_{k \ne i} h_{kk}^{\alpha_1} - \sum_{\alpha_1} \sum_{k \ne i} (h_{ki}^{\alpha_1})^2, \tag{8}$$

$$R_{jj} = \sum_{k \ne j} R_{jkjk} = (n-1)c + \sum_{\alpha_1} h_{jj}^{\alpha_1} \sum_{k \ne j} h_{kk}^{\alpha_1} - \sum_{\alpha_1} \sum_{k \ne j} (h_{kj}^{\alpha_1})^2. \tag{9}$$

Then by the identities (7), (8) and (9), we get

$$\begin{aligned}
&2R_{ijij} - R_{ii} - R_{jj} + 2(n-2)(c+H^2)\\
&= 2c + 2\sum_{\alpha_1} h_{ii}^{\alpha_1} h_{jj}^{\alpha_1} - \sum_{\alpha_1} 2(h_{ij}^{\alpha_1})^2 - 2c + 2(n-2)H^2\\
&\quad - \sum_{\alpha_1} h_{ii}^{\alpha_1} \sum_{k\neq i} h_{kk}^{\alpha_1} + \sum_{\alpha_1}\sum_{k\neq i}(h_{ki}^{\alpha_1})^2 - \sum_{\alpha_1} h_{jj}^{\alpha_1}\sum_{k\neq j} h_{kk}^{\alpha_1} + \sum_{\alpha_1}\sum_{k\neq j}(h_{kj}^{\alpha_1})^2\\
&= \sum_{\alpha_1\neq n+1}\Big[\sum_{k\neq i}(h_{ki}^{\alpha_1})^2 + \sum_{k\neq j}(h_{kj}^{\alpha_1})^2 + 2h_{ii}^{\alpha_1}h_{jj}^{\alpha_1} - 2(h_{ij}^{\alpha_1})^2\\
&\qquad\qquad - h_{ii}^{\alpha_1}\sum_{k\neq i} h_{kk}^{\alpha_1} - h_{jj}^{\alpha_1}\sum_{k\neq j} h_{kk}^{\alpha_1}\Big]\\
&\quad + 2(n-2)H^2 + \sum_{k\neq i}(h_{ki}^{n+1})^2 + \sum_{k\neq j}(h_{kj}^{n+1})^2 + 2h_{ii}^{n+1}h_{jj}^{n+1} - 2(h_{ij}^{n+1})^2\\
&\quad - h_{ii}^{n+1}\sum_{k\neq i} h_{kk}^{n+1} - h_{jj}^{n+1}\sum_{k\neq j} h_{kk}^{n+1}.
\end{aligned}$$

Furthermore, since M^n is a pseudo-umbilical submanifold ($h_{ij}^{n+p} = H\delta_{ij}$), (3) and (4), we have

$$\begin{aligned}
&2R_{ijij} - R_{ii} - R_{jj} + 2(n-2)(c+H^2)\\
&= \sum_{\alpha_1\neq n+1}\Big[\sum_{k\neq i}\left(h_{ki}^{\alpha_1}\right)^2 + \sum_{k\neq j}(h_{kj}^{\alpha})^2 + 2h_{ii}^{\alpha}h_{jj}^{\alpha} - 2(h_{ij}^{\alpha_1})^2\\
&\qquad\qquad - h_{ii}^{\alpha_1}\sum_{k\neq i} h_{kk}^{\alpha_1} - h_{jj}^{\alpha_1}\sum_{k\neq j} h_{kk}^{\alpha_1}\Big]\\
&= \sum_{\alpha_1\neq n+1}\Big[\sum_{k\neq i}(h_{ki}^{\alpha_1})^2 + \sum_{k\neq j}(h_{kj}^{\alpha_1})^2 + 2h_{ii}^{\alpha_1}h_{jj}^{\alpha_1} - 2(h_{ij}^{\alpha_1})^2 + (h_{ii}^{\alpha})^2 + (h_{jj}^{\alpha})^2\Big]\\
&= \sum_{\alpha_1\neq n+1}\Big[\sum_{k}(h_{ki}^{\alpha_1})^2 + \sum_{k}(h_{kj}^{\alpha_1})^2 + 2h_{ii}^{\alpha_1}h_{jj}^{\alpha_1} - 2(h_{ij}^{\alpha_1})^2\Big].
\end{aligned}$$

That is

$$\begin{aligned}
&2R_{ijij} - R_{ii} - R_{jj} + 2(n-2)(c+H^2)\\
&\qquad = \sum_{\alpha_1\neq n+1}\Big[\sum_{k}(h_{ki}^{\alpha_1})^2 + \sum_{k}(h_{kj}^{\alpha_1})^2 + 2h_{ii}^{\alpha_1}h_{jj}^{\alpha_1} - 2(h_{ij}^{\alpha_1})^2\Big].
\end{aligned}$$

Summing from 1 to n with respect to index i ($i \neq j$), and using the identity (3), we can obtain

$$\begin{aligned}
&\sum_{i\neq j} 2R_{ijij} - \sum_{i\neq j} R_{ii} - (n-1)R_{jj} + 2(n-1)(n-2)(c+H^2) \\
&= \sum_{\alpha_1\neq n+1}\Big[\sum_{i,k}(h_{ki}^{\alpha_1})^2 - \sum_k (h_{kj}^{\alpha_1})^2 + (n-1)\sum_k (h_{kj}^{\alpha_1})^2 \\
&\qquad\qquad + \sum_{i\neq j} 2h_{ii}^{\alpha_1}h_{jj}^{\alpha_1} - 2\sum_{i\neq j}(h_{ij}^{\alpha_1})^2\Big] \\
&= \sum_{\alpha_1\neq n+1}\sum_{i,k}(h_{ki}^{\alpha_1})^2 + (n-2)\sum_{\alpha_1\neq n+1}\sum_k (h_{kj}^{\alpha_1})^2 + 2\sum_{\alpha_1\neq n+1}\Big(\sum_i h_{ii}^{\alpha_1} - h_{jj}^{\alpha_1}\Big)h_{jj}^{\alpha} \\
&\quad - 2\sum_{\alpha_1\neq n+1}\Big[\sum_i (h_{ij}^{\alpha_1})^2 - (h_{jj}^{\alpha_1})^2\Big] \\
&= \tau' + (n-4)\sum_{\alpha_1\neq n+1}\sum_k (h_{kj}^{\alpha_1})^2,
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{i\neq j} 2R_{ijij} = \sum_{i\neq j} R_{ii} + (n-1)R_{jj} - 2(n-1)(n-2)(c+H^2) \\
+ \tau' + (n-4)\sum_{\alpha_1\neq n+1}\sum_k (h_{kj}^{\alpha_1})^2.
\end{aligned}$$

Since $R_{ii} \geq Q, R_{jj} \geq Q$, we get

$$\begin{aligned}
\sum_{i\neq j} 2R_{ijij} \geq 2(n-1)Q - 2(n-1)(n-2)(c+H^2) \\
+ \tau' + (n-4)\sum_{\alpha_1\neq n+1}\sum_k (h_{kj}^{\alpha_1})^2.
\end{aligned}$$

Because the choice of index j is arbitrary, we can sum over the index j from 1 to n and get

$$\sum_{i\neq j}\Big(\sum_j 2R_{ijij}\Big) \geq 2n(n-1)Q - 2n(n-1)(n-2)(c+H^2) + 2(n-2)\tau'.$$

Since K is the infimum of sectional curvatures, we have

$$K \geq Q - (n-2)(c+H^2) + \frac{n-2}{n(n-1)}\tau',$$

which completes the proof of the Lemma. ∎

3. Proofs of theorems

Proof of Theorem 1. Let $A \geq 0$, we can derive the following results from identities (4), (5), (6), Lemma 1, Lemma 3 and Lemma 2(ii).

(i) When $b \geq 0$

$$\sum_{\alpha_1=n+2}^{n+p}\sum_{i,j} h_{ij}^{\alpha_1}\Delta h_{ij}^{\alpha_1}$$
$$\geq (1+A)n\tau' K - An(a+H^2)\tau' - Ab\tau' - Anb\tau' + A\frac{1}{p-1}(\tau')^2 - \frac{n}{2}(\tau')^2$$
$$= \tau'\left\{(1+A)nK - An(a+H^2) - A(n+1)b + \frac{A}{p-1}\tau' - \frac{n}{2}\tau'\right\}.$$

Let $A = \frac{n(p-1)}{2}$, then we have

$$\frac{1}{2}\Delta\tau' = \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j,k}(h_{ijk}^{\alpha_1})^2 + \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j} h_{ij}^{\alpha_1}\Delta h_{ij}^{\alpha_1} \tag{10}$$
$$\geq \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j,k}(h_{ijk}^{\alpha_1})^2$$
$$+ \frac{n\tau'}{2}\{[2+(p-1)n]K - (p-1)n(a+H^2) - (p-1)(n+1)b\}$$
$$\geq \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j,k}(h_{ijk}^{\alpha_1})^2 + \frac{n\tau'}{2}\left\{[2+(p-1)n]\left[Q-(n-2)(c+H^2)+\frac{n-2}{n(n-1)}\tau'\right]\right.$$
$$\left. - (p-1)n(a+H^2) - (p-1)(n+1)b\right\},$$

(ii) When $b < 0$

$$\sum_{\alpha_1=n+2}^{n+p}\sum_{i,j} h_{ij}^{\alpha_1}\Delta h_{ij}^{\alpha_1} \geq (1+A)n\tau' K - An(a+H^2)\tau' + A\frac{1}{p-1}(\tau')^2 - \frac{n}{2}(\tau')^2$$
$$= \tau'\left\{(1+A)nK - An(a+H^2) + \left[\frac{A}{p-1} - \frac{n}{2}\right]\tau'\right\}.$$

Let $A = \frac{n(p-1)}{2}$, then we have

$$\frac{1}{2}\Delta\tau' = \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j,k}(h_{ijk}^{\alpha_1})^2 + \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j} h_{ij}^{\alpha_1}\Delta h_{ij}^{\alpha_1} \tag{11}$$
$$\geq \sum_{\alpha_1=n+2}^{n+p}\sum_{i,j,k}(h_{ijk}^{\alpha_1})^2 + \frac{n\tau'}{2}\left\{[2+(p-1)n]\left[Q-(n-2)(c+H^2)\right.\right.$$
$$\left.\left. + \frac{n-2}{n(n-1)}\tau'\right] - (p-1)n(a+H^2)\right\}.$$

Obviously, the right hand side of inequalities (10) and (11) are nonnegative. By `Hopf` maximum principle, we know that τ' is a constant, then $\Delta\tau' = 0$, so

$$\frac{n\tau'}{2}\Big\{[2+(p-1)n]\Big[Q-(n-2)(c+H^2)+\frac{n-2}{n(n-1)}\tau'\Big] - (p-1)n(a+H^2)-(p-1)(n+1)b\Big\} = 0, \quad b \geq 0, \tag{12}$$

and

$$\frac{n\tau'}{2}\Big\{[2+(p-1)n]\Big[Q-(n-2)(c+H^2)+\frac{n-2}{n(n-1)}\tau'\Big] - (p-1)n(a+H^2)\Big\} = 0, \quad b < 0. \tag{13}$$

When

$$Q > \frac{2n(p-2)(a+H^2)+(p-2)(n+1)(b+|b|)}{2[2+n(p-1)]} + (n-2)(c+H^2) - \frac{n-2}{n(n-1)}\tau'.$$

Using the identities (12) and (13), we can get

$$\tau' = 0.$$

When

$$Q = \frac{2n(p-2)(a+H^2)+(p-2)(n+1)(b+|b|)}{2[2+n(p-1)]} + (n-2)(c+H^2) - \frac{n-2}{n(n-1)}\tau',$$

inequalities in Lemma 2.1 and Lemma 2.2 become equalities, so

$$\sum_{\alpha_1=n+2}^{n+p}\sum_{\beta_1=n+2}^{n+p}[\mathtt{tr}(H_{\alpha_1}^2H_{\beta_1}^2)-\mathtt{tr}(H_{\alpha_1}H_{\beta_1})^2] = \frac{p-2}{p-1}\Big[\sum_{\alpha_1=n+2}^{n+p}\mathtt{tr}H_{\alpha_1}^2\Big]^2. \tag{14}$$

The identity (14) holds if and only if the two following identities hold

$$\Big(\sum_{\alpha_1=n+2}^{n+p}\mathtt{tr}H_{\alpha_1}^2\Big)^2 = (p-1)\sum_{\alpha_1}^{n+p}[\mathtt{tr}H_{\alpha_1}^2]^2, \tag{15}$$

$$(\lambda_i^{\alpha_1}-\lambda_k^{\alpha_1})^2 = 2[(\lambda_i^{\alpha_1})^2+(\lambda_k^{\alpha_1})^2] = 2\sum_j(\lambda_j^{\alpha_1})^2. \tag{16}$$

However, equality (15) is equal to

$$\mathtt{tr}H_{n+2}^2 = \mathtt{tr}H_{n+3}^2 = \cdots = \mathtt{tr}H_{n+p}^2, \tag{17}$$

and we can derive the following identity from (16)

$$\lambda_i^{\alpha_1} = 0, \quad i = 1, \dots, n \tag{18}$$

which implies

$$\mathtt{tr} H_{\alpha_1}^2 = \sum_i (\lambda_i^{\alpha_1})^2 = 0. \tag{19}$$

Combining (17) with equality (19), we obtain $\tau' = 0$. Together with the fact M^n is a pseudo-umbilical submanifold in $M_1^{n+p}(c)$, then we get that M^n is a totally umbilical submanifold in $M_1^{n+p}(c)$. ∎

Proof of Theorem 2. Let $A = 0$, combining (4), (5), Lemma 2.3 with Lemma 2.2 (i), (ii), we can obtain the following inequalities

$$\text{(i)} \quad \sum_{\alpha_1=n+2}^{n+p} \sum_{i,j} h_{ij}^{\alpha_1} \Delta h_{ij}^{\alpha_1} \geq n\tau' K - \frac{p-2}{p-1}(\tau')^2$$
$$= n\tau'[K - \frac{p-2}{n(p-1)}(\sigma' - nH^2)]$$
$$\geq n\tau'[Q - (n-2)(c+H^2) + \frac{n-2}{n(n-1)}\tau - \frac{p-2}{n(p-1)}(\sigma' - nH^2)],$$

$$\text{(ii)} \quad \sum_{\alpha_1=n+2}^{n+p} \sum_{i,j} h_{ij}^{\alpha_1} \Delta h_{ij}^{\alpha_1} \geq n\tau' K - \frac{p-2}{p-1}(\tau')^2$$
$$\geq \tau'[nK - \frac{p-2}{p-1}\tau] = n\tau'[K - \frac{p-2}{n(p-1)}(\sigma - nH^2)]$$
$$\geq n\tau'[Q - (n-2)(c+H^2) + \frac{n-2}{n(n-1)}\tau' - \frac{p-2}{n(p-1)}(\sigma - nH^2)],$$

$$\text{(iii)} \quad \sum_{\alpha_1=n+2}^{n+p} \sum_{i,j} h_{ij}^{\alpha_1} \Delta h_{ij}^{\alpha_1} \geq n\tau' K - \frac{n}{2}(\tau')^2 \geq n\tau'(K - \frac{1}{2}\tau)$$
$$= \frac{n\tau'}{2}[2K - (\sigma - nH^2)]$$
$$\geq \frac{n\tau'}{2}\{2[Q - (n-2)(c+H^2) + \frac{n-2}{n(n-1)}\tau'] - (\sigma - nH^2)\},$$

$$\text{(iv)} \quad \sum_{\alpha_1=n+2}^{n+p} \sum_{i,j} h_{ij}^{\alpha_1} \Delta h_{ij}^{\alpha_1} \geq n\tau' K - \frac{n}{2}(\tau')^2 = n\tau'(K - \frac{1}{2}\tau')$$
$$= n\tau'[K - \frac{1}{2}(\sigma' - nH^2)]$$
$$\geq n\tau'[Q - (n-2)(c+H^2) + \frac{n-2}{n(n-1)}\tau' - \frac{1}{2}(\sigma' - nH^2)].$$

Similar with the proof of Theorem 1, we can get the conclusion that M^n is a totally umbilical submanifold in $M_1^{n+p}(c)$ under the assumption of Theorem 2. ■

Remark 3. The key Lemma 4 has been pointed out in [3] for showing that under suitable assumptions therein a compact pseudo-umbilical submanifold with parallel mean curvature vector in a space form must be a totally umbilical submanifold. Here we would like to give the detailed proof again to emphasize the importance of this lemma in the derivation of Theorem 1 and Theorem 2. The sufficient conditions given here are sharper than those in [4], which implies our conclusions generalize the main results in [4].

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