DEMONSTRATIO MATHEMATICA
Vol. XLV No 3 2012

10.1515/dema-2013-0389

Marzena Szajewska

GAUSSIAN CURVATURE OF THE BERGMAN METRIC WITH WEIGHTED BERGMAN KERNEL ON THE UNIT DISC

Abstract. In the paper Gaussian curvature of Bergman metric on the unit disc and the dependence of this curvature on the weight function has been studied.

1. Introduction

The object of this paper is to study the sign of Gaussian curvature for the Bergman metric defined by a weighted Bergman kernel.

S. Dragomir [1] studied weighted Bergman kernels of bounded domains. He proved that the weighted Bergman metric

$$B_\mu(z,z) = \sum_{1\le i,j\le n} \left(\frac{\partial}{\partial z_i} \frac{\partial}{\partial \overline{z}_j} \log \mathcal{K}_\mu(z,z) \right) dz_i \otimes d\overline{z}_j \tag{1}$$

is well defined.

S. G. Krantz and J. Yu in [3] considered some invariants of the Bergman metric. They explore the boundary behavior of the Ricci curvature and scalar curvature. They gave formulas expressing the Bergman kernel, the Bergman metric and its holomorphic sectional curvature in terms of some classical minimum integrals. Using their results we investigate a sign of Gaussian curvature for the Bergman metric defined on the unit disc.

In section 2 we give the most important notions and facts concerning weighted Bergman kernels. In section 3 we consider examples of Bergman kernels. We calculate Gaussian curvature for these kernels. The main results are given in section 3.

2000 *Mathematics Subject Classification*: 53D35, 53A35, 53C21.
Key words and phrases: Bergman metric, weighted Bergman kernel, Gaussian curvature.

2. Preliminaries

Let $\Omega := \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$ be an open unit disc, and μ a positive continuous weight function on the unit disc. The weight μ is supposed to be an admissible (see [4]).

The weighted Bergman space (μ-Bergman space) $A^2(\Omega, \mu)$ is the set of all μ-square integrable holomorphic functions on Ω.

The weighted Bergman kernel $\mathcal{K}_\mu$ associated to Ω is given by

$$\mathcal{K}_\mu(z, w) = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}, \ \forall z, w \in \Omega, \tag{2}$$

where $\{\varphi_n\}$ is an orthonormal basis for $A^2(\Omega, \mu)$ with respect to the L^2-inner product

$$\langle f, g\rangle_\mu = \int_\Omega f(z)\overline{g(z)}\mu(z)d\rho(z),$$

where $d\rho$ denotes the Lebesgue measure on $\mathbb{R}^2$ and the norm of $f \in A^2(\Omega, \mu)$ is defined by $||f||_\mu = \langle f, f\rangle_\mu^{1/2}$.

It is known that the Bergman kernel $\mathcal{K}_\mu$ is independent of the choice of orthonormal basis (see [4]) and determined by two properties:

- $\underset{w\in\Omega}{\forall}\ K_\mu(\cdot, w) \in A^2(\Omega, \mu)$,
- $\underset{w\in\Omega}{\forall}\ \underset{f\in A^2(\Omega,\mu)}{\forall}\ f(w) = \langle f, K_\mu(\cdot, w)\rangle$, (the reproducing property).

The Bergman metric is determined by $\mathcal{K}_\mu$ on Ω as follows:

$$h_\mu(z) = \frac{\partial^2}{\partial z \partial \overline{z}} \log \mathcal{K}_\mu(z, z) dz \otimes d\overline{z}. \tag{3}$$

It has been proved in [1] that this Hermitian form is defined positive on the unit disc Ω. This form is covariant with K_μ with respect to biholomorphic maps between bounded domains.

3. Curvature of the Bergman metric

Using

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad dz = dx + idy$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad d\overline{z} = dx - idy$$

we can write Bergman metric h_μ in real coordinates

$$h = \frac{1}{4}\left[\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right)(dx \otimes dx + dy \otimes dy) + i\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right)(dy \otimes dx - dx \otimes dy)\right],$$

where

$$\Re h_\mu = \frac{1}{4}\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right)(dx \otimes dx + dy \otimes dy)$$

is the real part of h_μ and

$$\Im h_\mu = \frac{1}{4}\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right)(dy \otimes dx - dx \otimes dy)$$

is the imaginary part of h_μ.

It is easy to see that $\Re h_\mu$ is symmetric ($\Re h_\mu(u,v) = \Re h_\mu(v,u)$, $u, v \in T_z D$) and defined positive, while $\Im h_\mu$ is antisymmetric ($\Im h_\mu(u,v) = -\Im h_\mu(v,u)$), and has the closed form

$$\begin{aligned}
d&\left[\frac{1}{4}\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right)(dy \wedge dx - dx \wedge dy)\right] \\
&= d\left[\frac{1}{2}\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right)(dy \wedge dx)\right] \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right) dx \wedge (dy \wedge dx) \\
&\quad + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}\right) dy \wedge (dy \wedge dx) \\
&= \frac{1}{2}\left(\frac{\partial^3 \log \mathcal{K}_\mu(z,z)}{\partial x^3} + \frac{\partial^3 \log \mathcal{K}_\mu(z,z)}{\partial y^2 \partial x}\right)(dx \wedge dy \wedge dx) \\
&\quad + \frac{1}{2}\left(\frac{\partial^3 \log \mathcal{K}_\mu(z,z)}{\partial x^2 \partial y} + \frac{\partial^3 \log \mathcal{K}_\mu(z,z)}{\partial y^3}\right)(dy \wedge dy \wedge dx) = 0.
\end{aligned}$$

Then $\Re h_\mu$ is the metric tensor on Ω. We can write it in the matrix form as

$$g = \frac{1}{4}\begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix},$$

where

$$G := \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial x^2} + \frac{\partial^2 \log \mathcal{K}_\mu(z,z)}{\partial y^2}.$$

Using formula

$$K = -\frac{1}{2\det g}\left[\frac{\partial}{\partial y}\left(\frac{\frac{\partial g_{11}}{\partial y}}{\det g}\right) + \frac{\partial}{\partial x}\left(\frac{\frac{\partial g_{22}}{\partial x}}{\det g}\right)\right]$$

for the Gaussian curvature we can rewrite it as

$$K = -\frac{8}{G}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\log G). \tag{4}$$

Christoffel symbols

$$\Gamma_{ij,k} = \frac{1}{2} g^{kl}\left\{\frac{\partial^2 g_{lj}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{li}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}\right\}$$

for $\Re h_\mu$ are equal

$$\Gamma_{11,1} = \Gamma_{12,2} = \Gamma_{21,2} = -\Gamma_{22,1} = \frac{1}{8}\frac{\partial G}{\partial x},$$
$$\Gamma_{22,2} = \Gamma_{12,1} = \Gamma_{21,1} = -\Gamma_{11,2} = \frac{1}{8}\frac{\partial G}{\partial y},$$

where $[g^{ij}]$ is the inverse matrix to $g = [g_{ij}]$.

We know that $\Gamma^i_{jk} = g^{il}\Gamma_{jk,l}$ then

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \frac{1}{8}\frac{\partial}{\partial x}\log G,$$
$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \frac{1}{8}\frac{\partial}{\partial y}\log G.$$

Recall that the Riemannian curvature tensor of the metric g is given by

$$g^{im}R_{mjkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}.$$

It follows, that

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = \frac{1}{64}K \tag{5}$$

where K is Gaussian curvature of g.

Let's consider some special cases of weighted Bergman kernel and calculate Gaussian curvatures for these kernels.

Example 1. Let $\mu(z) = \frac{1}{\pi}(1-|z|^2)^\alpha$ where $\alpha > -1$ is a parameter. The functions

$$\varphi_m(z) := \sqrt{-\frac{\Gamma(-\alpha)\Gamma(2+m+\alpha)}{\pi \mathrm{Csc}(\pi\alpha)\Gamma(1+m)}}\, z^m, \qquad m \in \mathbb{N}$$

form an orthogonal, complete system in $A^2(\Omega, \mu)$, where

$$\mathrm{Csc}(w) = -\frac{2\sin(w)}{-1+\cos(w)}.$$

$$\langle f, g\rangle_\mu = \int_0^1\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\mu(z)d\theta rdr \qquad (z = re^{i\theta},\ r \le 1)$$

$$\begin{aligned}\langle z^n, z^m\rangle_\mu &= \int_0^1\int_0^{2\pi} r^n e^{in\theta} r^m e^{-im\theta}\frac{1}{\pi}(1-r^2)^\alpha d\theta rdr\\ &= -\frac{\pi\mathrm{Csc}(\pi\alpha)\Gamma(1+\frac{n+m}{2})}{\Gamma(-\alpha)\Gamma(2+\frac{n+m}{2}+\alpha)}\delta_{nm}.\end{aligned}$$

$$\begin{aligned}&\langle\varphi_n(z), \varphi_m(z)\rangle_\mu\\ &\quad= \left\langle\sqrt{-\frac{\Gamma(-\alpha)\Gamma(2+n+\alpha)}{\pi\mathrm{Csc}(\pi\alpha)\Gamma(1+n)}}z^n, \sqrt{-\frac{\Gamma(-\alpha)\Gamma(2+m+\alpha)}{\pi\mathrm{Csc}(\pi\alpha)\Gamma(1+m)}}z^m\right\rangle_\mu\\ &\quad= \sqrt{-\frac{\Gamma(-\alpha)\Gamma(2+n+\alpha)}{\pi\mathrm{Csc}(\pi\alpha)\Gamma(1+n)}}\sqrt{-\frac{\Gamma(-\alpha)\Gamma(2+m+\alpha)}{\pi\mathrm{Csc}(\pi\alpha)\Gamma(1+m)}}\langle z^n, z^m\rangle_\mu\\ &\quad= -\frac{\Gamma(-\alpha)\Gamma(2+\frac{n+m}{2}+\alpha)}{\pi\mathrm{Csc}(\pi\alpha)\Gamma(1+\frac{n+m}{2})}\langle z^n, z^m\rangle_\mu = \delta_{nm}.\end{aligned}$$

Then $\langle\varphi_n, \varphi_m\rangle_\mu = \delta_{nm}$, which means that $\{\varphi_n(z)\}_{n=1}^\infty$ forms an orthonormal system.

Hence the Bergman kernel defined by μ is equal

$$\mathcal{K}_\mu(z, z) = \sum_{n=0}^\infty \varphi_n(z)\overline{\varphi_n(z)} = (1 - z\overline{z})^{-\alpha-2}.$$

The metric tensor g has a form

$$G = \frac{2+\alpha}{(x^2+y^2-1)^2}$$

and using (4) we find Gaussian curvature

$$K = -\frac{64}{2+\alpha}.$$

It's negative on the whole unit disc.

The next example is taken from [4].

Example 2. Let $\mu_k(z) = |z|^k$, $z \in \Omega$ be a weight on Ω for $k \in \mathbb{R}$.

The functions which form an orthogonal, complete system in $A^2(\Omega, \mu_k)$ one can find in [4].

There are two cases:

1. For $k > -2$ the μ_k-Bergman function $\mathcal{K}_{\mu_k}$ is equal

$$\mathcal{K}_{\mu_k}(z, w) = \frac{1}{\pi(1-z\overline{w})^2} + \frac{k}{2\pi(1-z\overline{w})}. \tag{6}$$

In this case the metric tensor g has a form

$$G=\frac{8+(-1+x^2+y^2)(-6+x^2(-2+k)+y^2(-2+k)-k)k}{(-1+x^2+y^2)^2(-2+(-1+x^2+y^2)k)^2}.$$

Denote $r^2:=x^2+y^2$. Then

$$G=\frac{8+k(6+k-2(2+k)r^2+(-2+k)r^4)}{(-1+r^2)^2(2+k-kr^2)^2}.$$

Gaussian curvature for this case is given by

$$\begin{aligned}K&=-\frac{16(-4-4k+k^2)}{(-2+k)^2}\\&+\frac{768(8+16k+6k^2-4kr^2+k^3r^2)}{(-2+k)^3(8+6k+k^2-4kr^2-2k^2r^2-2kr^4+k^2r^4)}\\&+\frac{8192((2+k)^2(-2(2+k)(4+k)(2+3k)+(4+3k(4+k)(3+2k))r^2))}{(-2+k)^4(8+6k+k^2-4kr^2-2k^2r^2-2kr^4+k^2r^4)^3}\\&+\frac{512((2+k)(200+412k+126k^2-39k^3-12k^4+2(-2+k)(4+3k(4+k)(3+2k))r^2))}{(-2+k)^4(8+6k+k^2-4kr^2-2k^2r^2-2kr^4+k^2r^4)^2}.\end{aligned}$$

Proposition 1. *Let $\mathcal{K}_\mu$ be the Bergman kernel of the form* (6). *Then the sign of the curvature of its Bergman metric depends on the parameter k as follows*

$K>0$ *for* $k\in(-2,2(-2+\sqrt{2}))$ & $0<r<f(k)$;
$K<0$ *for* $k\in(-2,2(-2+\sqrt{2}))$ & $r\in(f(k),1)$ *or* $k=2(-2+\sqrt{2})$ & $r\in(0,1)$ *or* $k>2(-2+\sqrt{2})$ & $r\in\langle 0,1)$;
$K=0$ *for* $k\in(-2,2(-2+\sqrt{2}))$ & $r=f(k)$ *or* $k=2(-2+\sqrt{2})$ & $r=0$,

where a real-valued function $f:\mathbb{R}\to\mathbb{R}$ is a root of the equation $K=0$.

There is only one real-valued function $f\colon(-2,\infty)\to(0,1)$ in Ω.

2. For $k\le -2$, the Bergman function $\mathcal{K}_{\mu_k}$ has a form:

$$\mathcal{K}_{\mu_k}=\frac{(x^2+y^2)^{m_k+1}}{\pi(1-x^2-y^2)^2}+\frac{2m_k+2+k}{2}\frac{(x^2+y^2)^{m_k}}{(1-x^2-y^2)},\tag{7}$$

where $m_k:=\max\left\{j\in\mathbb{Z}\colon j\le -\frac{k}{2}\right\}$.

The metric tensor g is equal:

$$ds^2=\left(\frac{2}{(-1+x^2+y^2)^2}-\frac{(2+k+2m_k)\pi(-2+(2+k+2m_k)\pi)}{((2+k+2m_k)\pi(-1+x^2+y^2)-2(x^2+y^2))^2}\right)\cdot(dx^2+dy^2).$$

Gaussian curvature for this metric has a form:

$$K=-\frac{16(8-16\pi-8k\pi-16m\pi+4\pi^2+4k\pi^2+k^2\pi^2+8m\pi^2+4km\pi^2+4m^2\pi^2)}{(-4+2\pi+k\pi+2m\pi)^2}$$
$$-\frac{65536(2+k+2m)^2\pi^2(-2r^2+(2+k+2m)\pi(-2+3r^2))}{(-4+(2+k+2m)\pi)^4(8r^4+(2+k+2m)^2\pi^2(-1+r^2)^2-2(2+k+2m)\pi(-1-2r^2+3r^4))^3}$$
$$+\frac{8192(2+k+2m)^4\pi^4(-4-3r^2+6(2+k+2m)\pi(-1+r^2))}{(-4+(2+k+2m)\pi)^4(8r^4+(2+k+2m)^2\pi^2(-1+r^2)^2-2(2+k+2m)\pi(-1-2r^2+3r^4))^3}$$
$$+\frac{8192(2+k+2m)\pi(-8r^2+(2+k+2m)\pi(-11+14r^2))}{(-4+(2+k+2m)\pi)^4(8r^4+(2+k+2m)^2\pi^2(-1+r^2)^2-2(2+k+2m)\pi(-1-2r^2+3r^4))^2}$$
$$+\frac{1536(2+k+2m)^3\pi^3(24-8r^2+(2+k+2m)\pi(19-18r^2+4(2+k+2m)\pi(-1+r^2)))}{(-4+(2+k+2m)\pi)^4(8r^4+(2+k+2m)^2\pi^2(-1+r^2)^2-2(2+k+2m)\pi(-1-2r^2+3r^4))^2}$$
$$+\frac{768(2+k+2m)\pi(8(-1+r^2)+(2+k+2m)\pi(6+(-6+(2+k+2m)\pi)r^2))}{(-4+(2+k+2m)\pi)^3(8r^4+(2+k+2m)^2\pi^2(-1+r^2)^2-2(2+k+2m)\pi(-1-2r^2+3r^4))}$$

where $r^2 := x^2 + y^2$.

Proposition 2. *Let $\mathcal{K}_\mu$ be the Bergman kernel of a form (7) then the sign of the curvature of its Bergman metric depends on the parameter k as follows*

$K > 0$ *for* $k \in (-2(l+1), \frac{2\sqrt{2}}{\pi} - \frac{2(1+(l+1)\pi)}{\pi})$ & $0 < r < f(k)$;
$K < 0$ *for* $k \in (-2(l+1), \frac{2\sqrt{2}}{\pi} - \frac{2(1+(l+1)\pi)}{\pi})$ & $r \in (f(k), 1)$ *or*
$k = \frac{2\sqrt{2}}{\pi} - \frac{2(1+(l+1)\pi)}{\pi}$ & $r \in (0, 1)$
or $k \in (\frac{2\sqrt{2}}{\pi} - \frac{2(1+(l+1)\pi)}{\pi}, -2l)$ & $r \in < 0, 1)$;
$K = 0$ *for* $k \in (-2(l+1), \frac{2\sqrt{2}}{\pi} - \frac{2(1+(l+1)\pi)}{\pi})$ & $r = f(k)$ *or*
$k = \frac{2\sqrt{2}}{\pi} - \frac{2(1+(l+1)\pi)}{\pi}$ & $r = 0$,

where $l = 2m_k$ and real-valued function $f\colon (-\infty, -2] \to (0, 1)$ is a root of equation $K = 0$.

And as the previous case there is only one function in Ω.

The last examples show that the Gaussian curvature of Bergman metric is not always negative. There are weight functions μ for which K could be positive or equal zero.

References

[1] S. Dragomir, *On weighted Bergman kernels of bounded domains,* Studia Math. 108 (1994), 149–157.

[2] S. G. Krantz, *Function Theory of Several Complex Variables,* AMS Chelsea Publ., Rhode Island, 1992.

[3] S. G. Krantz, J. Yu, *On the Bergman invariant and curvatures of the Bergman metric*, Illinois J. Math. 40 (1996), 226–244.
[4] Z. Pasternak-Winiarski, *On the dependence of the reproducing kernel on the weight of integration,* J. Funct. Anal. 94 (1990), 110–134.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF BIALYSTOK
Akademicka 2
15-267 BiaŁYSTOK, POLAND
E-mail: marzena@math.uwb.edu.pl

Received June 4, 2010.