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SOME WEAKLY CONTRACTIVE MAPPING THEOREMS IN PARTIALLY ORDERED SPACES AND APPLICATIONS

Abstract. The purpose of this paper is to present some fixed point theorems for certain weakly contractive mappings, known as weakly $(\varphi - \psi)$-contractive mappings, in a complete metric space endowed with a partial ordering. Subsequently, we apply our main results to obtain a solution of a first order periodic problem and study the possibility of optimally controlling the solutions of ordinary differential equations via dynamic programming.

1. Introduction

A selfmap f of a metric space (X, d) is called *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$,

$$d(fx, fy) \leq kd(x, y). \tag{1.1}$$

As noted in [2], the above inequality can be expressed in the form

$$d(fx, fy) \leq d(x, y) - qd(x, y), \tag{1.2}$$

where $k = 1 - q$, $q \in [0, 1)$. Therefore the following definition of *weakly contractive maps* due to Alber and Guerre-Delabriere [2] seems to be natural.

A selfmap f of a metric space X is *weakly contractive (or ψ-weakly contractive)* if for all $x, y \in X$,

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \tag{1.3}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\psi(0) = 0$, $\psi(t) > 0$ for all $t \in (0, \infty)$, and $\lim_{t\to\infty} \psi(t) = \infty$.

It is clear that weakly contractive maps are continuous and include contraction maps as a special case for the choice $\psi(t) = (1 - k)t$.

Alber and Guerre-Delabriere [2] obtained certain fixed point theorems in Hilbert spaces for weakly contractive maps and acknowledged that their

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results were true at least for uniformly smooth and uniformly convex Banach spaces. Subsequently, Rhoades [21] extended some of the results appearing in [2] to complete metric spaces under less restricted conditions (see Theorem 1.5 below) and thus establishing that the results obtained in [2] were still valid for arbitrary Banach spaces.

Notice that (1.1) can be expressed as

$$d(fx, fy) \leq (1+q)d(x,y) - (1-q)d(x,y), \tag{1.4}$$

where $k = 2q$ with $q \in [0, \frac{1}{2})$. Therefore we have the following extension of (1.4) to so called weakly $(\varphi - \psi)$-contractive maps in a natural way.

Definition 1.1. A selfmap f of a metric space X is said to be *weakly* $(\varphi - \psi)$*-contractive map of type* (I) if, for all $x, y \in X$,

$$d(fx, fy) \leq \varphi(d(x,y)) - \psi(d(x,y)), \tag{1.5}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous and nondecreasing function and $\psi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous and nonincreasing function satisfying the following conditions:

(C1) $\varphi(0) - \psi(0) = 0$,
(C2) φ and ψ are both positive on $(0,\infty)$, and
(C3) $\varphi(t) - \psi(t) < t$ for all $t > 0$.

Definition 1.2. A selfmap f of a metric space X is said to be *weakly* $(\varphi - \psi)$*-contractive map of type* (II) if for all $x, y \in X$, (1.5) holds, where $\varphi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous and nondecreasing function and $\psi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous and nonincreasing function satisfying the following conditions:

(D1) $\varphi(0) - \psi(0) = 0$,
(D2) φ and ψ are both positive on $(0,\infty)$, and
(D3) $\varphi(t) - \psi(qt) \leq (1-q)t$ and $\varphi(qt) - \psi(0) \leq qt$ for all $t > 0$ and for some $q \in (0,1)$.

Remark 1.3. When $\varphi(t) = t$ for all $t \geq 0$ our definition of weakly $(\varphi - \psi)$-contractive map of type (I) (respectively type (II)) recovers the definition of weakly contractive maps.

Recently, fixed point theorems in partially ordered spaces have been studied among others, in [1, 5, 7, 9–17, 19–20, 23]. The well-known Tarski's theorem [22] has been used in [11] and [13] respectively to study the existence of solutions for fuzzy equations and to prove existence theorems for fuzzy differential equations. Applications to matrix equations and to ordinary differential equations are presented in [19] and [13, 16] respectively. In [5, 7, 9, 23] fixed point theorems for mixed monotone mappings in metric spaces endowed with a partial ordering are proved and, wherein, the authors have applied their results to study the existence and uniqueness of solutions for certain boundary value problems (see [8] and [19]). The usual contraction condition is weakened but at the expense that the operator in question is monotone. The main idea in [13, 20] involves combining the ideas of the well-known contraction mapping principle with those of the monotone iterative techniques (see [6]). Some other references on the topic are [3, 4].

The purpose of this paper is to extend and improve Theorems 1.5 and 1.6 below to the case of weakly $(\varphi - \psi)$-contractive maps of types (I) and (II) in ordered metric spaces under suitable conditions on the domain of maps. The results so obtained may be considered as extensions of those in [21, 10]. In the sequel we apply our main results to obtain a solution of first order periodic problem and study the possibility of optimally controlling the solutions of ordinary differential equations via dynamic programming.

First we recall the following.

Definition 1.4. If $(X, \leq)$ is a partially ordered set and $f : X \to X$, we say that f is monotone nondecreasing (respectively, nonincreasing) if $x, y \in X$, $x \leq y \Rightarrow f(x) \leq f(y)$ (respectively, $f(x) \geq f(y)$). This definition coincides with the notion of a nondecreasing (respectively, nonincreasing) function in the case where $X = \mathbb{R}$ and $\leq$ (respectively, $\geq$) represents the usual total order in $\mathbb{R}$.

The following result is due to Rhoades [21]

Theorem 1.5. *Let (X, d) be a complete metric space and $f : X \to X$ be a weakly contractive map. Then f has a unique fixed point in X.*

We note that the additional condition that $\lim_{t\to\infty} \psi(t) = \infty$ assumed in [2] has been dispensed with in Theorem 1.5 above.

Recently, Harjani and Sadarangani [10] proved the following result.

Theorem 1.6. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(fx, fy) \leq d(x, y) - \psi(d(x, y)) \quad \text{for } x \geq y \quad (x, y \in X) \tag{1.6}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that it is positive on $(0, \infty)$, $\psi(0) = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

2. Main results

In this section we present several fixed point theorems for weakly $(\varphi - \psi)$-contractive maps in a complete metric space endowed with a partial order. Now onwards, $\mathbb{N}$ will denote the set of natural numbers.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous, nondecreasing and weakly $(\varphi - \psi)$-contractive map of type* (I) *satisfying the condition*

$$d(fx, fy) \leq \varphi(d(x, y)) - \psi(d(x, y)) \quad \text{for } x \geq y \quad (x, y \in X)\,. \tag{2.1}$$

If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Proof. If $x_0 = f(x_0)$, then we are done. Suppose that $f(x_0) \neq x_0$. Since $x_0 \leq f(x_0)$ and f is a nondecreasing function, we obtain by induction that

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq f^4(x_0) \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots$$

Put $x_{n+1} = f(x_n)$ for each $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, from (2.1) and, as x_n and x_{n+1} are comparable, we have

$$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) \leq \varphi(d(x_n, x_{n-1})) - \psi(d(x_n, x_{n-1})).$$

If there exists an $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0-1}) = 0$ then $x_{n_0} = f(x_{n_0-1}) = x_{n_0-1}$ and x_{n_0-1} is a fixed point of f and the proof is complete.

On the other hand, if $d(x_{n+1}, x_n) \neq 0$ for any $n \in \mathbb{N}$, then taking into account (2.1) and our assumptions on φ and ψ we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) &\leq \varphi(d(x_n, x_{n-1})) - \psi(d(x_n, x_{n-1})) \\ &< d(x_n, x_{n-1}). \end{aligned}$$

Denoting $d(x_{n+1}, x_n)$ by ρ_n we have

$$\rho_n \leq \varphi(\rho_{n-1}) - \psi(\rho_{n-1}) < \rho_{n-1}. \tag{2.2}$$

Hence $\{\rho_n\}$ is a nonnegative nonincreasing sequence and possesses a limit, say, ρ^* such that $\rho^* \geq 0$. We claim that $\rho^* = 0$.

Now, from (2.2), if $\rho^* > 0$, then by passing over to limit as $n \to \infty$, we get

$$\rho^* \leq \varphi(\rho^*) - \psi(\rho^*) \leq \rho^*.$$

Thus, we have

$$\rho^* = \varphi(\rho^*) - \psi(\rho^*).$$

By (C3), for $\rho^* > 0$ we obtain

$$\rho^* = \varphi(\rho^*) - \psi(\rho^*) < \rho^*,$$

a contradiction. Therefore, $\rho^* = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. Fix $\epsilon > 0$. Since $\rho_n = d(x_{n+1}, x_n) \to 0$ as $n \to \infty$, there exists an $n_0 \in \mathbb{N}$ such that

$$d(x_{n_0+1}, x_{n_0}) \leq \min\left\{\frac{\epsilon}{2},\ \epsilon - \varphi(\epsilon) + \psi(\epsilon)\right\}.$$

We claim that $f\Big(\overline{B(x_{n_0}, \epsilon)} \cap \{y \in X : y \geq x_{n_0}\}\Big) \subset \overline{B(x_{n_0}, \epsilon)}$.

Let $z \in \overline{B(x_{n_0}, \epsilon)} \cap \{y \in X : y \geq x_{n_0}\}$. Then the following cases arise.

Case 1. $0 < d(z, x_{n_0}) \leq \frac{\epsilon}{2}$. In this case, as z and x_{n_0} are comparable, we have

$$\begin{aligned} d(f(z), x_{n_0}) &\leq d(f(z), f(x_{n_0})) + d(f(x_{n_0}), x_{n_0}) \\ &= d(f(z), f(x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\leq \varphi(d(z, x_{n_0})) - \psi(d(z, x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &< d(z, x_{n_0}) + d(x_{n_0+1}, x_{n_0}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon. \end{aligned}$$

Case 2. $\frac{\epsilon}{2} < d(z, x_{n_0}) \leq \epsilon$. In this case, as z and x_{n_0} are comparable; φ is a nondecreasing function; and ψ is a nonincreasing function, $\varphi(d(z, x_{n_0})) \leq \varphi(\epsilon)$ and $\psi(d(z, x_{n_0})) \geq \psi(\epsilon)$, so that

$$\begin{aligned} d(f(z), x_{n_0}) &\leq d(f(z), f(x_{n_0})) + d(f(x_{n_0}), x_{n_0}) \\ &= d(f(z), f(x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\leq \varphi(d(z, x_{n_0})) - \psi(d(z, x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\leq \varphi(\epsilon) - \psi(\epsilon) + d(x_{n_0+1}, x_{n_0}) \\ &\leq \varphi(\epsilon) - \psi(\epsilon) + \epsilon - \varphi(\epsilon) + \psi(\epsilon) \leq \epsilon. \end{aligned}$$

This proves our claim.

Since $x_{n_0+1} \in \overline{B(x_{n_0}, \epsilon)} \cap \{y \in X : y \geq x_{n_0}\}$, our claim gives us that $x_{n_0+2} = f(x_{n_0+1}) \in \overline{B(x_{n_0}, \epsilon)} \cap \{y \in X : y \geq x_{n_0}\}$. Repeating this process it follows that $x_n \in \overline{B(x_{n_0}, \epsilon)}$ for all $n \geq n_0$. Since ϵ is arbitrary, $\{x_n\}$ is a Cauchy sequence.

As X is complete, there exists an $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Again, since $\rho_n \to 0$ and f is continuous it follows that x^* is a fixed point of f. ∎

THEOREM 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous, nondecreasing and weakly $(\varphi - \psi)$-contractive map of type* (II) *satisfying the condition*

$$d(fx, fy) \leq \varphi(d(x, y)) - \psi(d(x, y)) \quad \text{for all } x \geq y \quad (x, y \in X)\,. \tag{2.3}$$

If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Proof. Following the proof of Theorem 2.1 we only need to check that $\{x_n\}$ is a Cauchy sequence. Fix $\epsilon > 0$. Since $\rho_n = d(x_{n+1}, x_n) \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_{n_0+1}, x_{n_0}) \leq \min\bigl\{q\epsilon, (1-q)\epsilon\bigr\}.$$

We claim that $f\bigl(\overline{B(x_{n_0}, \epsilon)} \cap \{y \in X : y \geq x_{n_0}\}\bigr) \subset \overline{B(x_{n_0}, \epsilon)}$.

Let $z \in \overline{B(x_{n_0}, \epsilon)} \cap \{y \in X : y \geq x_{n_0}\}$. Then the following cases arise.

Case 1. $0 < d(z, x_{n_0}) \le q\,\epsilon$. In this case, as z and x_{n_0} are comparable, we have

$$\begin{aligned} d(f(z), x_{n_0}) &\le d(f(z), f(x_{n_0})) + d(f(x_{n_0}), x_{n_0}) \\ &= d(f(z), f(x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\le \varphi(d(z, x_{n_0})) - \psi(d(z, x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\le \varphi(q\,\epsilon) - \psi(0) + d(x_{n_0+1}, x_{n_0}) \\ &\le q\,\epsilon + (1-q)\,\epsilon \le \epsilon. \end{aligned}$$

Case 2. $q\epsilon < d(z, x_{n_0}) \le \epsilon$. In this case, as z and x_{n_0} are comparable and φ is a nondecreasing function and ψ is a nonincreasing function, $\varphi((d(z, x_{n_0})) \le \varphi(\epsilon)$ and $\psi(d(z, x_{n_0})) \ge \psi(\epsilon)$, we have

$$\begin{aligned} d(f(z), x_{n_0}) &\le d(f(z), f(x_{n_0})) + d(f(x_{n_0}), x_{n_0}) \\ &= d(f(z), f(x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\le \varphi(d(z, x_{n_0})) - \psi(d(z, x_{n_0})) + d(x_{n_0+1}, x_{n_0}) \\ &\le \varphi(\epsilon) - \psi(q\epsilon) + d(x_{n_0+1}, x_{n_0}) \\ &\le (1-q)\epsilon + q\epsilon \le \epsilon. \end{aligned}$$

This proves our claim. ■

In what follows we prove that Theorem 2.1 is still valid for f not necessarily continuous, assuming the following hypothesis on X (see [10, Theorem 3] and [13, Theorem 2.2]): if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then

$$x_n \le x \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Theorem 2.3. *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Assume that the condition (2.4) holds. Let $f : X \to X$ be a nondecreasing and weakly $(\varphi - \psi)$-contractive map of type (I) satisfying the condition*

$$d(fx, fy) \le \varphi(d(x, y)) - \psi(d(x, y)) \quad \textit{for all } x \ge y \quad (x, y \in X)\,.$$

If there exists $x_0 \in X$ with $x_0 \le f(x_0)$, then f has a fixed point.

Proof. Following the proof of Theorem 2.1 we only need to check that $f(z) = z$. In fact,

$$\begin{aligned} d(f(z), z) &\le d(f(z), f(x_n)) + d(f(x_n), z) \\ &\le \varphi(d(z, x_n)) - \psi(d(z, x_n)) + d(x_{n+1}, z) \end{aligned}$$

and taking limit as $n \to \infty$, $d(f(z), z) \le 0$ and this proves that $d(f(z), z) = 0$ and, consequently $f(z) = z$. ■

We now present an example where it can be appreciated that hypotheses in Theorems 2.1 and 2.2 do not guarantee uniqueness of the fixed point (see [10]). Let $X = \{(1,0),(0,1)\} \subset \mathbb{R}^2$ and consider the usual order

$$(x,y) \le (z,t) \Leftrightarrow x \le z \text{ and } y \le t.$$

Thus, $(X, \le)$ is a partially ordered set, whose different elements are not comparable. Besides, (X,d) is a complete metric space considering d as the Euclidean distance. Put $\varphi(t) = \frac{2}{3}(t+1)$, $\psi(t) = \frac{2}{3}$ for all $t \in [0,\infty)$. The identity map $f(x,y) = (x,y)$ is trivially continuous and nondecreasing since elements in X are only comparable to themselves. Observe that conditions (2.1) and (C1)–(C3) of Theorem 2.1 are satisfied i.e., f is a weakly $(\varphi - \psi)$-contractive mapping of type (I).

On the other hand, if we consider $\varphi(t) = \frac{2}{3}t$, $\psi(t) = \frac{1}{3}t$ for all $t \in [0,\infty)$, then conditions (2.3) and (D1)–(D3) of Theorem 2.2 are satisfied i.e, f is a weakly $(\varphi - \psi)$-contractive map of type (II). To see this, let us take $q = \frac{1}{2}$. Moreover, $(1,0) \le f(1,0) = (1,0)$ and f has two fixed points in X.

Now there arises a natural question whether there are any sufficient conditions that ensure the uniqueness of the fixed point in Theorems 2.1 and 2.2. The answer is in affirmative. These conditions are:

(SC1) for $x, y \in X$ there exists a lower bound or an upper bound.

(SC2) X is such that if $\{x_n\}$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that every term is comparable to the limit x.

(SC3) f maps comparable elements to comparable elements, that is:

$$\text{for } \; x, y \in X, x \le y \Rightarrow f(x) \le f(y) \; \text{ or } \; f(x) \ge f(y).$$

In [13] it is proved that condition (SC1) is equivalent to: for $x, y \in X$ there exists a $z \in X$ which is comparable to x and y.

It may be remarked that corresponding results for Theorems 2.1 and 2.2 pertaining to uniqueness of the fixed point under conditions (SC1)–(SC3) can be obtained by applying similar arguments as in [10], so we omit the details.

Now we state the following theorems without proof which ensure the uniqueness of fixed points in Theorems 2.1 and 2.2 respectively. An appropriate blend of the proofs of Theorem 2.1 and [10, Theorem 4] works.

Theorem 2.4. *Let $(X, \le)$ be a partially ordered set, $\mathcal{C}$ a chain in X and suppose that there exists a metric d on $\mathcal{C}$ such that $(\mathcal{C}, d)$ is a complete metric space. Let $f : \mathcal{C} \to \mathcal{C}$ be a nondecreasing mapping such that*

$$d(fx, fy) \le \varphi(d(x,y)) - \psi(d(x,y)) \text{ for all } x, y \in \mathcal{C} \tag{2.5}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous and nondecreasing function and $\psi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous and nonincreas-

ing function satisfying conditions (C1)–(C3). *If there exists* $x_0 \in \mathcal{C}$ *with* $x_0 \le f(x_0)$, *then* f *has a unique fixed point in* $\mathcal{C}$.

Theorem 2.5. *Let* $(X, \le)$ *be a partially ordered set,* $\mathcal{C}$ *a chain in* X *and suppose that there exists a metric* d *on* $\mathcal{C}$ *such that* $(\mathcal{C}, d)$ *is a complete metric space. Let* $f : \mathcal{C} \to \mathcal{C}$ *be a nondecreasing mapping such that*

$$d(fx, fy) \le \varphi(d(x, y)) - \psi(d(x, y)) \text{ for all } x, y \in \mathcal{C} \tag{2.6}$$

where $\varphi : [0, \infty) \to [0, \infty)$ *is an upper semicontinuous and nondecreasing function and* $\psi : [0, \infty) \to [0, \infty)$ *is a lower semicontinuous and nonincreasing function satisfying conditions* (D1)–(D3). *If there exists* $x_0 \in \mathcal{C}$ *with* $x_0 \le f(x_0)$, *then* f *has a unique fixed point in* $\mathcal{C}$.

Example 2.6. Let $X = \mathbb{R}$ and let $\le$ denote the usual ordering in $\mathbb{R}$. Then $(X, \le)$ is a partially ordered set. Put

$$\mathcal{C} = \{0\} \cup \{\pm 2^{-n} : n \in \mathbb{N}\}$$

and let d be the usual metric on $\mathcal{C}$. Define the mapping $f : \mathcal{C} \to \mathcal{C}$ by

$$fx = \frac{1}{2}x \text{ for all } x \in \mathcal{C}.$$

It is obvious that $\mathcal{C}$ is a chain in X and it is complete. The map f is continuous and nondecreasing. Put

$$\varphi(t) = \frac{2}{3}t + \frac{1}{1+t}, \quad \psi(t) = \frac{1}{1+t} \quad \text{for all } t \in [0, \infty).$$

Then conditions (C1)–(C3) are satisfied. Notice that (2.5) obviously holds. Moreover, $-\frac{1}{2} \le f(-\frac{1}{2}) = -\frac{1}{4}$ and f has a unique fixed point 0 in $\mathcal{C}$. Besides, we notice that the iterative sequence $\{x_n\}$ given by $x_0 = -\frac{1}{2}$, $x_1 = f(-\frac{1}{2}) = -\frac{1}{4}$, $x_2 = f(-\frac{1}{4}) = -\frac{1}{8}, \cdots$ is nondecreasing and converges to 0. Furthermore, we observe that each $x_n \le 0$.

3. Application to ordinary differential equations

In this section we apply our main results of Section 2 to obtain a solution of first order periodic problem.

Consider the space $C(I)$, the class of real-valued continuous functions defined on $I = [0, T]$, endowed with the metric d given by

$$d(x, y) = \sup\{|x(t) - y(t)| : t \in I\} \text{ for all } x, y \in C(I).$$

Clearly, $(C(I), d)$ is a complete metric space. Further, note that $C(I)$ can also be equipped with a partial order given by

$$x, y \in C(I), x \le y \Leftrightarrow x(t) \le y(t) \text{ for } t \in I.$$

We now prove the existence of a solution for the following first-order periodic problem:

$$(3.1)\qquad \begin{cases} u'(t) = f(t, u(t)),\ t \in I = [0, T], \\ u(0) = u(T) \end{cases}$$

where $T > 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 3.1. A lower solution for (3.1) is a function $\alpha \in C^1(I)$ such that

$$\begin{aligned} &\alpha'(t) \le f(t, \alpha(t)) \text{ for } t \in I = [0, T], \\ &\alpha(0) \le \alpha(T). \end{aligned}$$

Theorem 3.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and suppose that there exist $\lambda > 0$, $0 < a < 1$ such that for all $x, y \in \mathbb{R}$ with $y \ge x$*

$$(3.2)\qquad \begin{aligned} &0 \le f(t, y) + \lambda y - [f(t, x) + \lambda x] \\ &\le \lambda\left[\ln(y - x + 1) + \frac{(y - x)^2}{2(y - x + 1)}\right]. \end{aligned}$$

If a lower solution for first order periodic problem (3.1) *exists then there exists a unique solution of* (3.1).

Proof. The first order periodic problem (3.1) can be written as

$$\begin{aligned} u'(t) + \lambda u(t) &= f(t, u(t)) + \lambda u(t),\ t \in I = [0, T], \\ u(0) &= u(T). \end{aligned}$$

The above problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]\, ds$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} & \text{if } 0 \le s < t \le T, \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} & \text{if } 0 \le t < s \le T. \end{cases}$$

Define $F : C(I) \to C(I)$ by

$$(Fu)(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)] ds.$$

Clearly, $u \in C^1(I)$ is a solution of (3.1) if $u \in C(I)$ is a fixed point of F.

From hypothesis (3.2) on f, the mapping F is nondecreasing and so, for $u \ge v$ we have

$$f(t, u) + \lambda u \ge f(t, v) + \lambda v$$

which implies, using the fact that $G(t,s) > 0$ for $(t,s) \in I \times I$, that

$$\begin{aligned}(Fu)(t) &= \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s)]ds \\ &\geq \int_0^T G(t,s)[f(s,v(s)) + \lambda v(s)]ds = (Fv)(t)\end{aligned}$$

for $t \in I$. For $u \geq v$, we have

$$\begin{aligned}d(Fu,Fv) &= \sup_{t\in I} |(Fu)(t) - (Fv)(t)| \\ &\leq \sup_{t\in I} \int_0^T G(t,s)[f(s,u(s)) + \lambda u(s) - f(s,v(s)) - \lambda v(s)]ds \\ &\leq \sup_{t\in I} \int_0^T G(t,s)\cdot\lambda\Big[ln\big(u(s)-v(s)+1\big) + \frac{\big(u(s)-v(s)\big)^2}{2\big(u(s)-v(s)+1\big)}\Big]ds.\end{aligned}$$

Put

$$\Phi(x) = \left[\ln(x+1) + \frac{x^2}{2(x+1)}\right].$$

Obviously, $\Phi : [0,\infty) \to [0,\infty)$ is nondecreasing, positive in $(0,\infty)$ $\Big(\Phi'(x) = \frac{2x+1}{(x+1)^2}\Big)$ and if $u \geq v$, then

$$\left[\ln(u(s)-v(s)+1) + \frac{(u(s)-v(s))^2}{2(u(s)-v(s)+1)}\right] \leq \left[\ln(\|u-v\|+1) + \frac{\|u-v\|^2}{2(\|u-v\|+1)}\right].$$

Now considering the above inequality, we obtain

$$\begin{aligned}(3.3)\quad & d(Fu,Fv) \\ &\leq \sup_{t\in I} \int_0^T G(t,s)\cdot\lambda\Big[ln\big(u(s)-v(s)+1\big) + \frac{\big(u(s)-v(s)\big)^2}{2\big(u(s)-v(s)+1\big)}\Big]ds \\ &\leq \left[\ln(\|u-v\|+1) + \frac{\|u-v\|^2}{2(\|u-v\|+1)}\right]\cdot\lambda\sup_{t\in I}\int_0^T G(t,s)ds.\end{aligned}$$

Note that

$$\lambda\sup_{t\in I}\int_0^T G(t,s)ds = \lambda\sup_{t\in I}\frac{1}{(e^{\lambda T}-1)}\left(\frac{1}{\lambda}e^{\lambda(T+s-t)}\Big]_0^t + \frac{1}{\lambda}e^{\lambda(s-t)}\Big]_t^T\right) = 1.$$

Therefore from (3.3) we obtain

$$d(Fu,Fv) \le \left[\ln(\|u-v\|+1) + \frac{\|u-v\|^2}{2(\|u-v\|+1)}\right] = \left[\ln(\|u-v\|+1) + \frac{\|u-v\|^2+1}{2(\|u-v\|+1)} - \frac{1}{2(\|u-v\|+1)}\right]. \tag{3.4}$$

Put $\varphi(x) = ln(x+1) + \frac{x^2+1}{2(x+1)}$, $\psi(x) = \frac{1}{2(x+1)}$. Clearly, $\varphi : [0,\infty) \to [0,\infty)$ is upper semicontinuous, $\psi : [0,\infty) \to [0,\infty)$ is lower semicontinuous, φ is nondecreasing and ψ is nonincreasing for all $x \in (0,\infty)$ and satisfy conditions (C1)–(C3).

Thus, from (3.4), for $u \ge v$ we obtain

$$d(Fu,Fv) \le \varphi(d(u,v)) - \psi(d(u,v)).$$

Let $\alpha(t)$ be a lower solution of (3.1). Then we will show that $\alpha \le F\alpha$. Now

$$\alpha'(t) + \lambda\alpha(t) \le f(t,\alpha(t)) + \lambda\alpha(t) \text{ for } t \in I.$$

Multiplying by $e^{\lambda t}$, we obtain that

$$(\alpha(t)e^{\lambda t})' \le [f(t,\alpha(t)) + \lambda\alpha(t)]e^{\lambda t} \text{ for } t \in I.$$

Integrating the above expression, we get

$$\alpha(t)e^{\lambda t} \le \alpha(0) + \int_0^t [f(s,\alpha(s)) + \lambda\alpha(s)]e^{\lambda s} ds \text{ for } t \in I. \tag{3.5}$$

Therefore

$$\alpha(0)e^{\lambda T} \le \alpha(T)e^{\lambda T} \le \alpha(0) + \int_0^T [f(s,\alpha(s)) + \lambda\alpha(s)]e^{\lambda s} ds,$$

so that

$$\alpha(0)e^{\lambda t} \le \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s)]\, ds.$$

From the above ineqality and (3.5), we obtain

$$\alpha(t) \le \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s)]\, ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s)]\, ds.$$

Consequently, we have

$$\alpha(t) \le \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s)]\, ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}[f(s,\alpha(s)) + \lambda\alpha(s)]\, ds.$$

Hence

$$\alpha(t) \leq \int_0^T G(t,s)[f(s,\alpha(s)) + \lambda\alpha(s)]\,ds = (F\alpha)(t) \text{ for } t \in I.$$

Finally, Theorems 2.1 and 2.2 give that F has a unique fixed point. ∎

Remark 3.3. We remark that:

(i) In the proof of Theorem 3.2, the unique solution of (3.1) can be obtained as $\lim_{n\to\infty} F^n(x)$, for every $x \in C(I)$. If we choose $x(t) = \alpha(t)$, then $F^n(\alpha)$ is a monotone nondecreasing sequence uniformly convergent to the unique solution of (3.1).

(ii) Condition (3.2) of Theorem 3.2 can be replaced by

$$0 \leq f(t,y) + \lambda y - [f(t,x) + \lambda x] \leq \lambda\Phi(y-x) \text{ for } y \geq x.$$

In this case, one may assume that $\Phi : [0,\infty) \to [0,\infty)$ is a function given by

$$\Phi(x) = \varphi(x) - \psi(x) \;\forall\; x \in [0,\infty),$$

where $\varphi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous and nondecreasing function and $\psi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous and nonincreasing function satisfying the conditions (C1)–(C3). Examples of such functions are:

(a) $\varphi(x) = \ln(x+1) + \frac{x^2+1}{2(x+1)}$ and $\psi(x) = \frac{1}{2(x+1)}$ (which appear in Theorem 3.2).

(b) $\varphi(x) = \frac{x}{x^2+1} + \arctan x$ and $\psi(x) = \frac{x}{x^2+1}$.

4. Application to control theory

In a recent paper, Pathak and Shahzad [18] studied the possibility of optimally controlling the solution of ordinary differential equations via dynamic programming. As an application of our main results in Section 2, we continue our discussion to solve certain problems in control theory in an ordered space. In what follows we use the terminology of Pathak and Shahzad [18] and Evans [8].

Let A be a compact subset of $\mathbb{R}^m$, and let for each given $a \in A$, $F_a : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and nondecreasing weakly $(\varphi-\psi)$-contractive map of type (I) such that

$$F_a(x) = \mathbf{f}(x,a) \quad \forall x \in \mathbb{R}^n,$$

where $\mathbf{f} : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a given bounded, continuous and nondecreasing weakly $(\varphi-\psi)$-contractive map of type (I). Consider the usual order

$$(x_1, x_2, \cdots, x_n) \leq (y_1, y_2, \cdots, y_n) \Leftrightarrow x_i \leq y_i \; i = 1, 2, \cdots, n.$$

Then, $(\mathbb{R}^n, \leq)$ is a partially ordered set. Besides, $(\mathbb{R}^n, d)$ is a complete metric space, where $d(x, y) = |x - y|$, the Euclidean distance between x and y.

We will now study the possibility of optimally controlling the solution $\mathbf{x}(\cdot)$ of the ordinary differential equation

$$\begin{cases} \dot{x}(s) = f\left(x(s),\ \alpha(s)\right), & (t < s < T), \\ x(t) = x. \end{cases} \tag{4.1}$$

Here $\cdot\ = \frac{d}{ds}$, $T > 0$ is a fixed terminal time, and $x \in \mathbb{R}^n$ is a given initial point, taken on by our solution $\mathbf{x}(\cdot)$ at the starting time $t \geq 0$. At later times $t < s < T$, $\mathbf{x}(\cdot)$ evolves according to the ODE (4.1). The function $\alpha(\cdot)$ appearing in (4.1) is a control; that is, some appropriate scheme for adjusting parameters from the set A as time evolves, thereby affecting the dynamics of the system modeled by (4.1). Let us write

$$\mathcal{A} = \{\alpha : [0, T] \to A : \alpha(\cdot) \text{ is measurable}\} \tag{4.2}$$

to denote the set of admissible controls. Then since

$$\begin{aligned} |\mathbf{f}(x, a)| &\leq C, \\ |\mathbf{f}(x, a) - \mathbf{f}(y, a)| &\leq \varphi(|x - y|) - \psi(|x - y|), \end{aligned} \tag{4.3}$$

for some constant $C > 0$ and $x, y \in \mathbb{R}^n$, $a \in A$ with $x \geq y$, we have

$$|F_a(x) - F_a(y))| \leq \varphi(|x - y|) - \psi(|x - y|) \tag{4.4}$$

for all $x, y \in \mathbb{R}^n$ with $x \geq y$, where $\varphi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous and nondecreasing function and $\psi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous and nonincreasing function satisfying the conditions (C1)–(C3). Suppose that there exists $x_0 \in X$ with $x_0 \leq f(x_0)$.

We see that for each control $\alpha(\cdot) \in \mathcal{A}$, the ODE (4.1) has a unique, generalized Lipschitzian continuous solution $\mathbf{x}(\cdot) = \mathbf{x}^{\alpha(\cdot)}(\cdot)$, existing on the time interval $[t, T]$ and solving the ODE for a.e. time $t < s < T$. We call $\mathbf{x}(\cdot)$ the response of the system to the control $\alpha(\cdot)$, and $\mathbf{x}(s)$ the state of the system at time s.

Our goal is to find a control $\alpha^*(\cdot)$ which optimally steers the system. In order to define what "optimal" means however, we must first introduce a cost criterion. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, let us define for each admissible control $\alpha(\cdot) \in \mathcal{A}$ the corresponding cost

$$C_{x,t}[\alpha(\cdot)] := \int_t^T h(\mathbf{x}(s), \alpha(s))ds + g(\mathbf{x}(T)), \tag{4.5}$$

where $\mathbf{x}(\cdot) = \mathbf{x}^{\alpha(\cdot)}(\cdot)$ solves the ODE (4.1) and

$$h : \mathbb{R}^n \times A \to \mathbb{R}, \quad g : \mathbb{R}^n \to \mathbb{R}$$

are given functions. We call h the running cost per unit time and g the terminal cost, and will henceforth assume that

$$(4.6)\qquad \begin{cases} |H_a(x) - \ g(x)| \le C, \\ |H_a(x) - H_a(y)| \le \varphi(|x-y|) - \psi(|x-y|), \\ |g(x) - g(y)| \le \varphi(|x-y|) - \psi(|x-y|), \ (x, y \in \mathbb{R}^n, \ a \in A) \end{cases}$$

for some constant $C > 0$, and for each given $a \in A$, $H_a : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded, continuous and nondecreasing weakly $(\varphi - \psi)$-contractive map of type (I) defined by

$$H_a(x) = h(x, a) \ \ \forall x \in \mathbb{R}^n.$$

Given now $x \in \mathbb{R}^n$ and $0 \le t \le T$, we would like to find if possible a control $\alpha^*(\cdot)$ which minimizes the cost functional (4.5) among all other admissible controls.

To investigate the above problem we shall apply the method of dynamic programming. We now turn our attention to the value function $u(x, t)$ defined by

$$(4.7)\qquad u(x,t) := \mathop{inf}_{\alpha(\cdot)\in\mathcal{A}} \ C_{x,t}[\alpha(\cdot)] \quad (x \in \mathbb{R}^n, 0 \le t \le T.)$$

The plan is this: having defined $u(x, t)$ as the least cost given we start at the position x at time t, we want to study u as a function of x and t. We are therefore embedding our given control problem (4.1), (4.5) into the larger class of all such problems, as x and t vary. This idea can then be used to show that u solves a certain Hamilton–Jacobi type PDE, and finally to show conversely that a solution of this PDE helps us to synthesize an optimal feedback control.

Let us fix $x \in \mathbb{R}^n, 0 \le t \le T$. Following the technique of Evans [8, p. 553], we can obtain the optimality conditions in the form given below.

For each $\xi > 0$ so small that $t + \xi \le T$

$$u(x,t) := \inf_{\alpha(\cdot)\in\mathcal{A}} \left\{ \int_t^{t+\xi} h(\mathbf{x}(s), \alpha(s))ds + u(\mathbf{x}(t+\xi), t+\xi) \right\},$$

where $\mathbf{x}(\cdot) = \mathbf{x}^{\alpha(\cdot)}$ solves the ODE (4.1) for the control $\alpha(\cdot)$. ■

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