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THE VARIETY OF ALL COMMUTATIVE BCK-ALGEBRAS IS GENERATED BY ITS FINITE MEMBERS AS A QUASIVARIETY

Abstract. We prove the result announced by the title as well as some of its consequences.

It is well known that the variety of Łukasiewicz algebras is generated by its finite members (see *e.g.* [20]). W. Blok and I. Ferreirim [3] proved that the variety of all Łukasiewicz algebras is generated by its finite members as a quasi-variety. In this paper we show that this is also the case for the variety of all commutative BCK-algebras as well as some of its subvarieties. It is worth to note that unlike the case of subvarieties of the variety of all Łukasiewicz algebras, there are subvarieties of the variety of all commutative BCK-algebras which are not generated by their finite members [29, 38].

The main result of the paper was obtained in early '90s. Since then there were many papers on BCK-algebras, BCK-algebras with condition (S) (pocrims) and bounded commutative BCK-algebras (Wajsberg algebras, MV-algebras). Let me list some important papers suggested by an anonymous referee.

- A good study of the varieties of BCK-algebras (including some aspects of commutative BCK-algebras) was done by W. Blok and J. Raftery in [5]. The references given in this paper provide a fairly comprehensive vision of the status of BCK-algebras until 1995.
  The implicative presentation of BCK-algebras tends to be used today, because they are the implicational subreduct of commutative integral residuated lattices.
- BCK-*algebras with condition S* are currently known as *pocrims*: partially ordered, commutative, residuated, integral monoids. The varieties of

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pocrims were also studied by W. Blok and J. Raftery in [6]. This paper contains a complete bibliography on pocrims and BCK-algebras.

- *Partially naturally ordered commutative monoids with residuation* are a special case of pocrims known as *hoops*, and they are studied in [4, 2, 3].
- *Bounded Commutative* BCK-*algebras*, in their implicational presentation, are also known as CN-algebras [20] and Wajsberg algebras [15]. In fact, they are term-wise equivalent to Chang's MV-algebras introduced in [9] (see also [20, 15, 23] and [11]). MV-algebras can be seen as the unit segment of commutative lattice ordered groups with a strong unit (see [10] and [20] for MV-chains, and [24] for the general case).
- The literature on MV-*algebras* is very large. Since Mundici's work [23], in which he studies the relationship between the many valued Łukasiewicz logic and AF C*-algebras, using the categorical equivalence between MV-algebras and commutative ℓ-groups with strong unit, the theory of the MV-algebras has developed a great deal in different directions. A general approach to the theory of MV-algebras can be found in the book [11] and the paper [16]. The lattice of all subvarieties of MV-algebras was studied in [20], and a complete equational presentation of these subvarieties can be found in [13].

Preliminaries

The notation and terminology used in this paper are rather standard. For background on universal algebra we refer the reader to G. Grätzer [17] and for BCK-algebras to K. Iséki and S. Tanaka [18].

We will use capital letters A, B, C (possibly with indices) to denote algebras as well as their base sets. By $\boldsymbol{N}$ we will denote the set of natural numbers.

For the reader's convenience we list below some fundamental definitions and facts which we are going to use.

Definition. A BCK-algebra is an algebra $A = < A, *_A, 0_A >$ of type (2,0) such that for all $x, y, z \in A$ the following conditions are satisfied:

(1) $((x *_A y) *_A (x *_A z)) *_A (z *_A y) = 0_A$;
(2) $(x *_A (x *_A y)) *_A y = 0_A$;
(3) $x *_A x = 0_A$;
(4) $0_A *_A x = 0_A$;
(5) $x *_A y = 0_A$ and $y *_A x = 0_A$, then $x = y$.

In a BCK-algebra A the symbol $\leq_A$ denotes the partial order (so called BCK-order) relation $(x \leq_A y) \Leftrightarrow x *_A y = 0_A$. By (4) 0_A is the smallest element of A with respect to $\leq_A$.

Definition. A BCK-algebra A satisfies the condition (S) if for any x, y in A there is the largest element in the set

$$\{z \in A \colon z *_A x \leq_A y\}.$$

We will denote this element by $x \circ_A y$.

The definition of $x \circ_A y$ allows us to look at a BCK-algebra satisfying the condition (S) as a BCK-algebra with an additional binary operation $\circ_A$. Thus we have the following definition (see [28]).

Definition. A BCK-algebra with the operation (S) is an algebra $A = \langle A, *_A, \circ_A, 0_A\rangle$ of type (2,2,0) such that $\langle A, *_A, 0_A\rangle$ is a BCK-algebra and for all $x, y, z \in A$

(6) $(x \circ_A y) *_A x \leq_A y$;
(7) $z *_A x \leq_A y$ implies $z \leq_A x \circ_A y$.

It was shown by H. Yutani [36] that conditions (6) and (7) can be replaced with the single condition

(8) $(x *_A y) *_A z = x *_A (y \circ_A z)$.

In the sequel we shall often omit the subscript $_A$ and write $*, \circ, 0, \leq$ instead of $*_A, \circ_A, 0_A, \leq_A$. The symbol $*$ will be also omitted, so we shall write xy for $x * y$.

Definition. A BCK-algebra A is called commutative iff the identity

$$(C) \qquad x(xy) = y(yx)$$

holds in A.

Proposition 1.

(i) (see [36]) *The class of all commutative* BCK-*algebras is a variety.*
(ii) (see [35]) *If A is a commutative* BCK-*algebra then $(A, \leq)$ is a lower semilattice, and moreover $x \wedge y = x(xy)$.*

Definition. A commutative BCK-algebra satisfying the identity

$$(L) \qquad xy = (xy)(yx)$$

is called a Łukasiewicz algebra.

Proposition 2. (see [20]) *The variety of Łukasiewicz algebras is generated by its finite members.*

Proposition 3. (see [26]) *If A is a commutative* BCK-*algebra and for any $a, b \in A$ there is c in A such that $a \leq c$ and $b \leq c$ then A is a Łukasiewicz algebra.*

Remark. The term "Łukasiewicz algebra" has been used in the literature (see *e.g.* [1]) with a different meaning. The term "Łukasiewicz algebra of order n" was introduced by G. C. Moisil and his collaborators for algebras which were thought to form an algebraic counterpart of many-valued Łukasiewicz logic but it appeared that they were not. Namely it was shown by W. Suchon [34] that Łukasiewicz implication could not be defined in terms of operations of Łukasiewicz algebras in Moisil sense. Łukasiewicz algebras in our sense form an algebraic counterpart of the purely implicational fragment of Łukasiewicz logic and, if bounded (see Definition below), form an algebraic counterpart for Łukasiewicz logic (see [21]).

Algebras which are polynomially equivalent to Łukasiewicz algebras were intensively studied under different names by different authors, *e.g.* unbounded Wajsberg algebras considered by W. Blok and D. Pigozzi in [4].

Definition. An algebra $A =< A, *, 0, 1 >$ of type $< 2, 0, 0 >$ is called bounded (commutative) BCK-algebra if and only if $< A, *, 0 >$ is a (commutative) BCK-algebra and $x1 = 0$ is an identity of A.

Definition. A bounded BCK-algebra A is said to have involution if and only if $1(1x) = x$ is an identity of A.

Proposition 4. *If A is a bounded commutative* BCK-*algebra then it has involution.*

Proof. The fact easily follows from Proposition 1 (ii). ∎

Amalgamation property. (see [22]) The variety of all bounded Łukasiewicz algebras has the amalgamation property.

Let us recall the theorem proved recently by W. Blok and I. Ferreirim [3].

Theorem 1. (see [3]) *Let V be the variety of bounded commutative* BCK-*algebras (=bounded Łukasiewicz algebras) or Łukasiewicz algebras, K the class of all finite subdirectly irreducible algebras in V then $V = SPP_U(K)$.*

In case of a bounded BCK-algebra with the involution, the operation (S) can be defined in terms of $*$, namely

(9) $x \circ y = 1 * ((1 * x) * y)$.

Notation. We define for any natural number n:

$$xy^n \text{ as } x \text{ for } n = 0 \text{ and } xy^{n+1} = (xy^n)y,$$
$$nz \text{ as } 0 \text{ for } n = 0 \text{ and } (n+1)z = (nz) \circ z.$$

We will also use $x < y$ for $x \leq y$ and $x \neq y$.

The results below, Proposition 5, 6, 7, are well known and can be found in literature.

Proposition 5. (see [18]) *Let A be a* BCK-*algebra. Then the following hold for all $x, y, z \in A$*

(10) $(xy)z = (xz)y$;
(11) $x0 = x$;
(12) $x \leq y$ *implies* $xz \leq yz$ *and* $zy \leq zx$.

Proposition 6. (see [19]) *If A is a* BCK-*algebra with the operation (S) then for all $x, y, z \in A$ and any natural numbers m, n*

(13) $x \circ y = y \circ x$;
(14) $x \circ 0 = x$;
(15) *if* $y \leq z$ *then* $x \circ y \leq x \circ z$;
(16) $xz^{m+n} = (xz^m)z^n = (xz^m)(nz) = (x(mz))(nz)$.

Proposition 7. (see [18]) *If A is a commutative* BCK-*algebra then for all $x, y \in A$*

(17) $xy = x(x(xy)) = x(x \wedge y)$;
(18) *if moreover A is bounded then $x \leq y$ if and only if $1y \leq 1x$.*

Let us consider the following identity

(H) $(xy)(zy) = (xz)(yz)$.

BCK-algebras satisfying the identity (H) correspond to naturally ordered commutative integral monoids with residuation (see [39]).

We have the following

Proposition 8. (see [12]) *If A is bounded commutative* BCK-*algebra then A satisfies (H).*

Lemma 1. *If a* BCK-*algebra A with the operation (S) satisfies identity (H) then for all $x, y, z \in A$*

(i) $(xy) \circ y = (yx) \circ x$;
(ii) *if* $x \leq y$ *then* $x \circ (yx) = y$;
(iii) *if* $x \leq y$ *then* $zy = (zx)(yx)$.

Proof. For (i) we have

$$\begin{aligned} ((xy) \circ y)((yx) \circ x) &= (((xy) \circ y)x)(yx) && \text{by (8) and (13)} \\ &= (((xy) \circ y)y)(xy) && \text{by } (H) \\ &= ((xy) \circ y)(y \circ (xy)) && \text{by (8)} \\ &= 0 && \text{by (3) and (13).} \end{aligned}$$

As for (ii) let $x \leq y$. By definition, $(x \circ (yx))x \leq yx$ and, since $yx \leq yx$ one has

(19) $y \leq x \circ (yx)$.

Using (12) we have $yx \leq (x \circ (yx))x$ and according to (5) $yx = (x \circ (yx))x$.

Thus we have

$$\begin{aligned} 0 &= ((x \circ (yx))x)(yx) = ((x \circ (yx))y)(xy) && \text{by } (H) \\ &= (x \circ (yx))y && \text{as } xy = 0. \end{aligned}$$

So $x \circ (yx) \leq y$ which together with (19) and (5) gives $x \circ (yx) = y$.

Now let x, y be as above. Then

$$\begin{aligned} zy &= z(x \circ (yx)) && \text{by (ii)} \\ &= (zx)(yx) && \text{by (8).} \ \blacksquare \end{aligned}$$

Lemma 2. *Let A be any bounded* BCK-*algebra with the involution satisfying identity (H). Then for any $y, z \in A$*

$$(y \circ z)z = y(y(1z)).$$

Proof.

$$\begin{aligned} (y \circ z)z &= (1((1y)z))z && \text{by (9)} \\ &= (1z)((1y)z) && \text{by (10)} \\ &= (1(1y))(z(1y)) && \text{by } (H) \\ &= y(z(1y)) && \text{by ``involution''} \\ &= y((1(1z))(1y)) && \text{by ``involution''} \\ &= y((1(1y))(1z)) && \text{by (10)} \\ &= y(y(1z)) && \text{by ``involution''.} \end{aligned}$$

BCK-**union**

A notion of BCK-union was defined incorrectly in [27], as it was pointed out by T. Traczyk. We start with the correct definition and next we list all theorems of [27] which are true and proofs of which remain correct under the new definition.

Remark. A similar construction (in the finite case) was used by A. Romanowska and T. Traczyk in [30].

Definition. A subalgebra B of a BCK-algebra A is called convex iff for any $b \in B$ and $a \in A$ if $a \leq b$ then $a \in B$.

Remark. The designation "convex" was accepted by BCK-researchers. It was pointed out by the referee that in ordered sets, this is called a downset or order ideal.

Definition. A non-empty family $\{A_i\}_{i \in I}$ of BCK-algebras is called connected iff for every $i, j \in I$ the following conditions are satisfied:

1. The operations of A_i and A_j coincide within the set $A_i \cap A_j$ which is a convex subalgebra of both A_i and A_j.

2. If $x \in A_i - A_j$, $y \in A_j - A_i$ then the set of all lower bounds of $\{x, y\}$ in $A_i \cap A_j$ has a largest element.

Definition. Given a connected family $\{A_i\}_{i\in I}$ of BCK-algebras, we say that a BCK-algebra $< A, *, 0 >$ is a BCK-union of the family $\{A_i\}_{i\in I}$ if and only if

1. $A = \bigcup_{i\in I} A_i$;
2. the operation $*$ restricted to the set A_i coincides with the respective operation of the algebra A_i for every $i \in I$;
3. for every $i \in I$ A_i is a convex subalgebra of $< A, *, 0 >$;
4. for $x \in A_i - A_j$, $y \in A_j - A_i$, $xy := x(x \wedge y)$.

Lemma 3. *For every connected family $\{A_i\}_{i\in I}$ of* BCK-*algebras the following conditions hold:*

1. *the set theoretical union of the partial orderings of the algebras A_i, $i \in I$ is a partial ordering of the set $\bigcup_{i\in I} A_i$;*
2. *if $x \in A_k - A_j$, $y \in A_j - A_k$, for $j, k \in I$, then inf(x, y) in the poset $\bigcup_{i\in I} A_i$ exists and it belongs to $A_k \cap A_j$.*

Theorem 2. *The* BCK-*union exists for every connected family of* BCK-*algebras and it is unique.*

Theorem 3. *If $\{A_i\}_{i\in I}$ is a connected family of commutative* BCK-*algebras then the* BCK-*union of the family $\{A_i\}_{i\in I}$ is a commutative* BCK-*algebra.*

Theorem 4. *Every commutative* BCK-*algebra is the* BCK-*union of a connected family of Łukasiewicz algebras.*

The following theorem characterizes subdirectly irreducible commutative BCK-algebras in terms of the BCK-union.

Theorem 5. *A commutative* BCK-*algebra is subdirectly irreducible if and only if it is the* BCK-*union of a connected family $\{A_i\}_{i\in I}$ of subdirectly irreducible Łukasiewicz algebras such that for any $i, j \in I$ $A_i \cap A_j \neq \{0\}$.*

Remark. A version of Theorem 5 was proved earlier, using a different method, by A. Romanowska and T. Traczyk (see [32], Th. 3.1.). A complete and detailed characterization of subdirectly irreducible Łukasiewicz algebras was given by the same authors in [30, 31, 32, 33].

Proof. Let $\{A_i\}_{i\in I}$ be a connected family of subdirectly irreducible Łukasiewicz algebras such that for any $i, j \in I$ $A_i \cap A_j \neq \{0\}$. Let $i \in I$ and $a \in A_i$, $a \neq 0$ be any element generating the smallest non-trivial ideal of A_i. It is easy to see that a generates the smallest non-trivial ideal of the BCK-union of the family $\{A_i\}_{i\in I}$.

Now let A be a subdirectly irreducible commutative BCK-algebra. Then according to Theorem 4 A is the BCK-union of a connected family $\{A_i\}_{i\in I}$ of Łukasiewicz algebras. If for some i, j in I, $i \neq j$, $A_i \cap A_j = \{0\}$ then 0 is not meet irreducible in A and, as it follows from Proposition 3 of [25], A is not subdirectly irreducible. Let J be the smallest non-trivial ideal of A. It is easy to see that $J \cap A_i$ is the smallest non-trivial ideal of A_i for each $i \in I$.

Corollary. *Any subdirectly irreducible commutative* BCK-*algebra is a tree in which the least element 0 is meet-irreducible.*

Definition. By a branching element of a subdirectly irreducible commutative BCK-algebra A we will mean each element x of A such that for any element y of A such that $x < y$ there is an element z in A such that $x < z$ and y and z are incomparable with respect to the BCK-order of A.

The set BR_A of branching elements of a given subdirectly irreducible commutative BCK-algebra A is partially ordered by the BCK-order relation of A. Moreover BR_A is a lower semilattice with respect to that relation.

The last easily follows from the fact stating that any subdirectly irreducible commutative BCK-algebra is a tree.

Definition. An element x in A is called a maximal branching element if $x \in BR_A$ and moreover x is maximal in BR_A with respect to BCK-order of A.

Bounded commutative BCK-algebras

Let A be any linearly ordered commutative BCK-algebra, x and z two elements of A.

Definition. By A_z we denote the subalgebra of the algebra A with base set $\{y \in A : y \leq z\}$. By A_z^x we denote a subalgebra of the algebra A generated by $A_z \cup \{x\}$, thus $A_z^x = Sg^A(A_z \cup \{x\})$.

Theorem 6. *For the algebra A the following is true:*

(i) *A_z^x is a bounded commutative* BCK-*algebra. In particular it is with the condition (S).*

(ii) *If there is an $n \in \mathbf{N}$, $n > 0$ such that $nz < x$ and $x \leq (n+1)z$ then all elements of A_z^x are of the form $(kz)y$ for $k = 1, ..., n$, $y \in A_z$ and $(nz) \circ y$ for $y \leq b \in A_z$ where b is the least element of all elements y of A_z for which $x = (nz) \circ y$.*

(iii) *If for all natural numbers n $nz < x$ then the algebra A_z^x contains an infinite chain $0 < z < 2z < ... < x(2z) < xz < x$ and all elements of the form $(nz)y$ and $(x(nz))y$, where n is a natural number and y is an element of A_z.*

Proof. Part (i) is trivial.

Parts (ii) and (iii) follow from Lemma 4 – Lemma 13 below. ∎

Definition. We will say that the algebra A^x_z is of type $(1, n, b)$ if Theorem 6 (ii) holds and is of type 2 if Theorem 6 (iii) holds.

Corollary 1.

(i) *If algebra A^x_z is of type $(1, n, b)$ and $z \leq z'$ then algebra $A^x_{z'}$ is of type $(1, n', b')$ for some natural n' and some $b' \leq z'$.*
(ii) *If algebra A^x_z is of type 2 and $z' \leq z$ then algebra $A^x_{z'}$ is of type 2 as well.*

We start with the following

Lemma 4. *Let A be a bounded linearly ordered commutative* BCK-*algebra, x, z elements of A, m, n natural numbers and $mz \neq 1$. Then*

(i) *if $m \geq n$ then $(mz)(nz) = (m-n)z$;*
(ii) *if $x \leq z$ then $(m-1)z \leq (mz)x$;*
(iii) *if $(m-1)z \leq x \leq mz$ then there is exactly one $y \in A$ such that $0 \leq y \leq z$ and $x = (mz)y$;*
(iv) *if $0 \leq x \leq z$ then $(mz)x = ((m-1)z) \circ (zx)$.*

Proof. (i) It can be easily shown that for any natural number k, $kz = 1(1z^k)$. Using this we have

$$\begin{aligned} (mz)(nz) &= (1(1z^m))(1(1z^n)) \\ &= (1z^n)(1z^m) && \text{by (10) and Proposition 4} \\ &= (1z^n)((1z^n)((m-n)z) && \text{by (16).} \end{aligned}$$

To end the proof of (i) it is sufficient to show that $(m-n)z \leq 1z^n$. Suppose it is not the case so $1z^n \leq (m-n)z$. Then

$$\begin{aligned} 0 &= (1z^n)((m-n)z) \\ &= 1z^m && \text{by (16)} \\ &= 1(mz) && \text{by (16).} \end{aligned}$$

So $1 \leq mz$ and as 1 is the largest element $mz = 1$, a contradiction.

(ii) By (12) we have $((m+1)z)z \leq ((m+1)z)x$ and by (i) $((m+1)z)z = mz$.

(iii) Suppose $(m-1)z \leq x \leq mz$. Then

$$x = (mz)((mz)x)$$

and

$$(mz)x \leq (mz)((m-1)z) = z \quad \text{by (12) and (i).}$$

As for uniqueness, let us suppose that for some t, u such that $0 \leq t, u \leq z$ $x = (mz)t = (mz)u$. Then

$$t = (mz)((mz)t) = (mz)((mz)u) = u.$$

(iv) We have

$$\begin{aligned} ((mz)x)(((m-1)z) \circ (zx)) &= (((mz)x)((m-1)z))(zx) && \text{by (8)} \\ &= (((mz)((m-1)z))x)(zx) && \text{by (10)} \\ &= (zx)(zx) && \text{by (i)} \\ &= 0, \end{aligned}$$

so

(20) $((mz)x) \leq ((m-1)z) \circ (zx)$.

Now

$$\begin{aligned} &((m-1)z) \circ (zx))((mz)x) \\ &\quad = [((m-1)z) \circ (((mz)((m-1)z))x)]((mz)x) && \text{by (i)} \\ &\quad = [((m-1)z) \circ (((mz)x)((m-1)z))]((mz)x) && \text{by (10)} \\ &\quad = [(((m-1)z)((mz)x)) \circ ((mz)x)]((mz)x) && \text{by Lemma 1 (i)} \\ &\quad = (0 \circ ((mz)x))((mz)x) && \text{by (ii)} \\ &\quad = ((mz)x)((mz)x) && \text{by (14)} \\ &\quad = 0. \end{aligned}$$

Thus $((m-1)z) \circ (zx) \leq (mz)x$, which together with (20) and (5) gives

$$((m-1)z) \circ (zx) = (mz)x.$$

This finishes the proof of Lemma 4. ■

To prove Theorem 6 let us take a close look at the structure of any bounded subalgebra A_z^x of a linearly ordered commutative BCK-algebra A. We have the following situation $A_z^x = Sg^A(\{d \in A\colon 0 \leq d \leq z\} \cup \{x\})$ where $z \in A$, $z \neq 0$, $x \in A$, $z < x$ (the case $x \leq z$ is trivial). We have to consider two cases:

CASE 1. There is an $n \in \boldsymbol{N}$, $n > 0$ such that $nz < x$ and $x \leq (n+1)z$.

It is easy to see that A_z^x contains the following chain $0 < z < 2z < \ldots < nz < x$ and because of Lemma 4 (iii) for any natural number m such that $0 \leq m \leq n$ each element u in A_z^x such that $(m-1)z \leq u \leq mz$ is of the form $(mz)a$ for some a such that $0 \leq a \leq z$ and a is uniquely determined. Now we prove that the operation $*$ is uniquely determined in A_z^x.

Let $v = ((mz)a)((kz)b)$ and $a \leq z$, $b \leq z$, $1 \leq k \leq m$. We will show that v is equal to exactly one element of A of the form $(m'z)a'$ where $a' \leq z$.

It is easy to see that $v = 0$ whenever a, b are such that $a, b \leq z$ $m < k$ or $m = k$ and $b \leq a \leq z$.

Lemma 5. *Let* $1 \leq m \leq n$, $k = 1$. *Then*

(i) $v = (mz)(a \circ (zb))$;
(ii) *if* $z \leq a \circ (zb)$ *then* $v = ((m-1)z)((a \circ (zb))z)$ *and* $(a \circ (zb))z \leq z$.

Proof (i). We have by (8)

(21) $v = ((mz)a)(zb) = (mz)(a \circ (zb))$.

Let $c := zb$. If $a \circ c \leq z$ we are done.

(ii) If $z \leq a \circ c$ then as $a, c \leq z$ $a \circ c \leq 2z$ and $(a \circ c)z \leq (2z)z = z$. According to Lemma 1 (ii) $a \circ c = z \circ ((a \circ c)z)$ and

$$\begin{aligned}(mz)(a \circ c) &= (mz)(z \circ ((a \circ c)z)) = ((mz)z)((a \circ c)z) \text{ by } (8)\\ &= ((m-1)z)((a \circ c)z).\end{aligned}$$

So we have

(22) $(mz)(a \circ c) = ((m-1)z)((a \circ c)z)$. ∎

Lemma 6. *Let* $1 < k < m \leq n$ *and* $a = b$. *Then* $v = (m-k)z$.

Proof. We have

$$\begin{aligned}((mz)a)((kz)a) &= ((mz)(kz))(a(kz)) \text{ by } (H)\\ &= (m-k)z \text{ by Lemma 4 (i) and fact that } a \leq z \leq kz.\end{aligned}$$ ∎

Lemma 7. *Let* $1 < k < m \leq n$ *and* $a < b$. *Then*

$$v = ((m-k+1)z)(z(ba)).$$

Proof. By Lemma 1 (iii) $(kz)b = ((kz)a)(ba)$ and

$$\begin{aligned}((mz)a)((kz)b) &= ((mz)a)(((kz)a)(ba))\\ &= ((mz)a)(((kz)(ba))a) && \text{by } (10)\\ &= ((mz)((kz)(ba)))(a((kz)(ba))) && \text{by } (H)\\ &= (mz)((kz)(ba)) && \text{as } a((kz)(ba)) = 0 \text{ for } k > 1\\ &= (mz)(((k-1)z) \circ (z(ba))) && \text{by Lemma 4 (iv)}\\ &= ((mz)((k-1)z))(z(ba)) && \text{by } (8)\\ &= ((m-k+1)z(z(ba)) && \text{by Lemma 4 (i),}\end{aligned}$$

note that $z(ba) \leq z$. ∎

Lemma 8. *Let* $1 < k < m \leq n$ *and* $b < a$. *Then* $v = ((m-k)z)(ab)$.

Proof.

$$\begin{aligned}((mz)a)((kz)b) &= (((mz)b)(ab))((kz)b) && \text{by Lemma 1 (iii)}\\ &= (((mz)b)((kz)b))(ab) && \text{by (10)}\\ &= (((mz)(kz))(b(kz)))(ab) && \text{by } (H)\\ &= ((m-k)z)(ab) && \text{and } ab \le z. \ \blacksquare\end{aligned}$$

Till now we have described the $*$-operation on a subalgebra of A_z^x with the carrier set $\{c\colon 0 \le c \le nz\}$. Now, let $a := x(nz)$. We will show the following

Lemma 9. *If $a := x(nz)$ then:*

(i) $a \le z$;
(ii) $x = (nz) \circ a$;
(iii) *for any* $b < a$ $(nz) \circ b < x$.

Proof. As to (i) we have

$$\begin{aligned}(x(nz))z &= x((nz) \circ z) && \text{by (8)}\\ &= xx = 0 && \text{as } x = (n+1)z.\end{aligned}$$

So (i) is proved.

As to (ii) $(nz) \circ a = (nz) \circ (x(nz)) = x$ by Lemma 1 (ii), since $xz < x$.

As to (iii) let us suppose that $(nz) \circ b = (nz) \circ a$. Then

$$\begin{aligned}0 &= ((nz) \circ a)((nz) \circ b)\\ &= (x((xa)(nz)))(x((xb)(nz))) && \text{by (9)}\\ &= (x(x((xb)(nz))))((xa)(nz)) && \text{by (10)}\\ &= ((xb)(nz))((xa)(nz)) && \text{by Proposition 1 (ii)}\\ &= ((xb)(xa))((nz)(xa)) && \text{by } (H)\\ &= ((x(xa))b)((nz)(xa)) && \text{by (10)}\\ &= (ab)((nz)(x(x(nz)))) && \text{by Proposition 1 (ii)}\\ & && \text{and the definition of } a\\ &= (ab)((nz)(nz)) && \text{by Proposition 1 (ii)}\\ &= (ab)0 = ab.\end{aligned}$$

Thus $a \le b$, which is a contradiction. This finishes the proof of (iii). ∎

Note that $x = (nz) \circ b$ for any b such that $a \le b$. It can be easily shown that each y such that $nz \le y < x$ is of the form $(nz) \circ b$ for some unique $b < a$. Now let $u := ((nz) \circ c)((kz) \circ b)$ where $k \ge 0$, $c \le a$ and $b \le z$. If $k = n$ and $c \le b \le a$ then $u = 0$.

Lemma 11. *Let $k = n$ and $b < c \le a$. Then $u = cb$.*

Proof. Note first that, by Lemma 9 (ii) and (8)

(23) $(nz)(xc) = 0$,

as $c \leq a$ and the definition of a imply $nz = xa \leq xc$, and next we have

$$\begin{aligned} u &= ((nz) \circ c)((nz) \circ b) = (((nz) \circ c)(nz))b && \text{by (8)} \\ &= ((x((xc)(nz)))(nz))b && \text{by (9)} \\ &= ((x(nz))((xc)(nz)))b && \text{by (10)} \\ &= ((x(xc))((nz)(xc)))b && \text{by } (H) \\ &= (c((nz)(xc)))b && \text{by Proposition 1 (ii)} \\ &= cb && \text{by (23).} \ \blacksquare \end{aligned}$$

Lemma 11. *Let* $1 \leq k < n$. *Then*

(i) *if* $(zc) \circ b < z$ *then* $u = ((n-k+1)z)\ ((zc) \circ b)$;
(ii) *if* $z \leq (zc) \circ b$ *then* $u = ((n-k)z)(((zc) \circ b)z)$ *and* $((zc) \circ b)z \leq z$.

Proof. We first prove that

(24) $((n-k)z) \circ a \leq x(kz)$

which is equivalent to $(((n-k)z) \circ a)(x(kz)) = 0$. We have

$$\begin{aligned} (((n-k)z) \circ a)(x(kz)) &= (((n-k)z) \circ (x(nz)))(x(kz)) && \text{by the definition of } a \\ &= (x((x(x(nz)))((n-k)z)))(x(kz)) && \text{by (9) and (13)} \\ &= (x((nz)((n-k)z)))(x(kz)) && \text{by Proposition 1 (ii)} \\ &= (x(kz))(x(kz)) && \text{by Lemma 4 (i)} \\ &= 0. \end{aligned}$$

If $c \leq a$ then $(n-k)z \circ c \leq ((n-k)z) \circ a$.
Thus

$$\begin{aligned} u &= ((nz) \circ c)((kz) \circ b) = (((nz) \circ c)(kz))b && \text{by (8)} \\ &= ((x((x(nz))c))(kz))b && \text{by (9)} \\ &= ((x(kz))((x(nz))c))b && \text{by (10)} \\ &= ((x(kz))(((x(kz))((n-k)z))c))b && \text{by (16)} \\ &= ((x(kz))((x(kz))(((n-k)z) \circ c)))b && \text{by (8)} \\ &= (((n-k)z) \circ c)b && \text{by (24) and Proposition 1 (ii)} \\ &= (((n-k+1)z)(zc))b && \text{by Lemma 4 (iv)} \\ &= ((n-k+1)z)((zc) \circ b) && \text{by (8).} \end{aligned}$$

So we have

(25) $((nz \circ c)((kz) \circ b) = (((n-k)z) \circ c)b.$

The last element is of the form $(mz)d$ for some natural m and $d \le z$ if $(zc) \circ b \le z$. If not then by (22)

$$((n-k+1)z)((zc) \circ b) = ((n-k)z)(((zc) \circ b)z),$$

where $((zc) \circ b)z \le z$. ∎

Lemma 12. *Let* $0 = k < n$, $b \le c \le a$. *Then* $u = (nz) \circ (cb)$ *and* $cb \le a$.

Proof. First we show that

(26) $(nz) \circ (cb) \le xb.$

Let us note that (26) is equivalent to

(26′) $b \le x((nz) \circ (cb))$ by (12), linearity and (13).

We have

$$\begin{aligned} x((nz) \circ (cb)) &= (x(nz))(cb) && \text{by (8)} \\ &= a(cb) && \text{by definition of } a. \end{aligned}$$

To finish the proof of (26) we have to show that $b \le a(cb)$.
As $b \le a$ and $cb \le ab$ we have

$$\begin{aligned} b &= a(ab) && \text{by Proposition 1 (ii)} \\ &\le a(cb) && \text{by (12).} \end{aligned}$$

Now

$$\begin{aligned} ((nz) \circ c)b &= ((nz) \circ ((cb) \circ b))b && \text{by Lemma 1 (ii)} \\ &= ((nz) \circ (cb))(((nz) \circ (cb))(xb)) && \text{by Lemma 2} \\ &= (nz) \circ (cb) && \text{by (26) and Proposition 1 (ii).} \end{aligned}$$

Note that $cb \le a$.

The fact that $u = (nz) \circ (cb)$ under assumptions of Lemma 12 easily follows from the definition of u. ∎

Lemma 13. *Let* $k < n$, $c < b$. *Then* $u = ((n-k)z)(bc)$.

Proof. We have

$$\begin{aligned} u = ((nz) \circ c)((kz) \circ b) &= (((nz) \circ c)(kz))b && \text{by (8)} \\ &= (((n-k)z) \circ c)b && \text{by (25)} \\ &= (((n-k)z) \circ c)(c \circ (bc)) && \text{by Lemma 1 (ii)} \\ &= ((((n-k)z) \circ c)c)(bc) && \text{by (8)} \\ &= (((n-k)z)(((n-k)z)(xc)))(bc) && \text{by Lemma 2.} \end{aligned}$$

Now as $c \leq a$ $(n-k)z \leq nz = xa \leq xc$, so

$$((n-k)z)(((n-k)z)(xc)) = (n-k)z$$

and

$$((nz) \circ c)((kz) \circ b) = ((n-k)z)(bc). \ \blacksquare$$

CASE 2. For all natural numbers n $nz < x$. In this case the algebra A_z^x contains an infinite chain $0 < z < 2z < \ldots < x(2z) < xz < x$.

First we will show that for any natural number n and any u such that $x((n+1)z) \leq u \leq x(nz)$ there is exactly one a such that $a \leq z$ and $u = (x(nz))a$. Suppose $u = (x(nz))a = (x(nz))b$ and $b \leq z$. Then

$$\begin{aligned} a &= ((x(nz))((x(nz))a)) && \text{by Proposition 1 (ii)} \\ &= ((x(nz))((x(nz))b)) \\ &= b && \text{by Proposition 1 (ii).} \end{aligned}$$

Now we have:

If $n \leq m$ then

$$\begin{aligned} (x(mz))(x(nz)) &= (x(x(nz)))(mz) && \text{by (10)} \\ &= (nz)(mz) && \text{by Proposition 1 (ii)} \\ &= 0. \end{aligned}$$

If $m < n$ then $(x(mz))(x(nz)) = (nz)(mz) = (n-m)z$ by Lemma 4 (i).

If $m < n$ and $a, b \leq z$ then

$$\begin{aligned} &((x(mz))a)((x(nz))b) \\ &\qquad = (x((mz) \circ a))(x((nz) \circ b)) && \text{by (8)} \\ &\qquad = (x(x((nz) \circ b)))((mz) \circ a) && \text{by (10)} \\ &\qquad = ((nz) \circ b)((mz) \circ a) && \text{by Proposition 1 (ii)} \\ &\qquad = (((n+1)z)(zb))(((m+1)z)(za)) && \text{by Lemma 4 (iv),} \end{aligned}$$

and now we can use Lemmas 5–13.

If $n \leq m$, $a, b \leq z$ then

$$\begin{aligned} &((x(mz))a)((x(nz))b) \\ &\quad = (((x(nz))((m-n)z))a)((x(nz))b) && \text{by (16) and (8)} \\ &\quad = ((x(nz))(((m-n)z) \circ a))((x(nz))b) && \text{by (8)} \\ &\quad = ((x(nz))((x(nz))b))(((m-n)z) \circ a) && \text{by (10)} \\ &\quad = b(((m-n)z) \circ a) && \text{by Proposition 1 (ii) and } b \leq z. \end{aligned}$$

The last element is equal to 0 in case $m - n > 0$ or $m - n = 0$ and $b \leq a$, or ba in case $m - n = 0$ and $a < b$.

If $a, b \leq z$ then

$$((x(nz))a)b = (x(nz))(a \circ b) \text{ by (8).}$$

If $a \circ b \leq z$ then we are done. If $z < a \circ b$ then using Lemma 1 (ii) we have

$$\begin{aligned}(x(nz))(a \circ b) &= (x(nz))(z \circ ((a \circ b)z) \\ &= ((x(nz))z)((a \circ b)z) \quad \text{by (8),}\end{aligned}$$

note that $(a \circ b)z \leq z$.

Now let us consider $((x(nz))a)((mz)b)$ where $m > 1$, $a, b \leq z$ (if $m = 1$ then we have the previous case as $zb \leq z$). We have

$$\begin{aligned}((x(nz))a)((mz)b) &= ((x(nz))a)(((m-1)z) \circ (zb)) \text{ by Lemma 4 (iv)} \\ &= (((x(nz))a)((m-1)z))(zb) \quad \text{by (8)} \\ &= (((x(nz))((m-1)z))a)(zb) \quad \text{by (10)} \\ &= ((x((n+m-1)z))a)(zb) \quad \text{by (16)}\end{aligned}$$

and again we have the same situation as in the previous case.

The proof of Theorem 6 is now complete. ■

Corollary 2. *The set of all elements a of A_z^x such that $a \leq nz \circ b$, where n is a natural number and $b \leq z$ is a carrier set of a convex subalgebra of the algebra A.*

Corollary 3. *Let A, B be linearly ordered commutative* BCK-*algebras, $z \in A \cap B$, $z \neq 0$, $x \in A$, $y \in B$, $z < x, y$ $A_z = B_z$ then*

- (i) *if A_z^x is of type $(1, k, b)$, B_z^y is of type $(1, n, c)$ where $k < n$ or $n = k$ and $b \leq c$ or B_z^y is of type 2 then there is an embedding $f\colon A_z^x \longrightarrow B_z^y$ such that $f|_{A_z} = id_{A_z}$.*
- (ii) *if A_z^x, B_z^y are both of type 2 then they are isomorphic and a mapping establishing the isomorphism restricted to A_z is an identity. Moreover A_z^x, B_z^y are isomorphic as subalgebras of bounded linearly ordered commutative* BCK-*algebras $(A_x, *, 0, x)$ and $(B, *, 0, y)$.*

L-terms and valuations

Let t be any term in the BCK-language. By $\mathrm{var}(t)$ we will denote the set of all variables occurring in the term t. Let A be a subdirectly irreducible commutative BCK-algebra and $\mathrm{val}\colon \mathrm{var}(t) \longrightarrow A$. The set of all elements of A which are below some element of the form $\mathrm{val}(v)$, where v belongs to $\mathrm{var}(t)$ is a base set of a subalgebra A' of A. Without loss of generality we can assume that $A = A'$. As $\mathrm{var}(t)$ is finite it follows that A is the BCK-union of a finite connected family $A_1, \ldots, A_n$ of subdirectly irreducible bounded Łukasiewicz algebras such that for all $1 \leq i, j \leq n$ $A_i \cap A_j \neq \{0\}$.

Observation. It will be shown later that $A \in W_n$, where for any natural number $n \geq 1$ W_n denotes the variety of all commutative BCK-algebras with at most n pairwise incomparable elements.

Now, using induction on the complexity of t we define a term t' and a valuation $\mathrm{val}_1 \colon \mathrm{var}(t') \longrightarrow A$ as follows:
if t is a variable, say $t = v$, then we put $t' := t$ and $\mathrm{val}_1(v) := \mathrm{val}(v)$;
if $t = su$, the leftmost variable of s is v_1, the leftmost variable of u is v_2 then, if $\mathrm{val}(v_1)$ and $\mathrm{val}(v_2)$ are comparable with respect to the BCK-order of A, we put $t' := s'u'$ and $t' := s'(w \wedge u')$ otherwise, where w is a new variable not in $\mathrm{var}(t) \cup$ $\mathrm{var}(s') \cup \mathrm{var}(u')$. We put $\mathrm{val}_1(w) := \mathrm{val}(v_1) \wedge \mathrm{val}(v_2)$.

Definition. The term t' described above will be called later an L-version of the term t the valuation val_1 described above will be called the first valuation associated with the valuation val.

Remark. Please note that val (as well as the other valuations mentioned) extends to uniquely determined valuation of BCK-terms to A (corresponding algebra).

Lemma 14. *For any* BCK-*term* t

$$\mathrm{val}(t) = \mathrm{val}_1(t').$$

Proof. Induction on complexity of t. The proof is easy and is left to the reader. ∎

Now we describe a process of "trimming" of A.

Lemma 15. *Given a commutative subdirectly irreducible* BCK-*algebra A and valuations val and val$_1$ as above, there is a commutative subdirectly irreducible* BCK-*algebra B such that each branch of B has the largest element and one of the following two conditions holds*

1) *B is linearly ordered (i.e. B is a Łukasiewicz algebra);*
2) *if z is a maximal branching element of A, x maximal element of A such that $z \leq x$ then the algebra A_z^x is of type* 2;

and there is a valuation $\mathrm{val}_2 \colon \mathrm{var}(t') \longrightarrow B$ such that

$$\mathrm{val}_1(t') = 0 \text{ iff } \mathrm{val}_2(t') = 0.$$

Proof. We describe a construction of B using induction on the number n of branches of A.

Step 1. $n = 1$. Then A satisfies condition 1) and we put $B := A$ and $\mathrm{val}_2 := \mathrm{val}_1$.

Step 2. $n > 1$. If A satisfies the condition 2) we put $B := A$ and $\mathrm{val}_2 := \mathrm{val}_1$. Otherwise, there is a maximal branching element z and maximal elements x, x' in A such that $z \leq x, x'$ such that A_z^x is of type $(1, k, b)$ for some natural number k and element $b \leq z$. Without loss of generality we can assume that there is an embedding $f : A_z^x \longrightarrow A_z^{x'}$, by Corollary 3 (i). We define B to be the subalgebra of the algebra A with base set $A - \{y \in A : z < y\}$ and define valuation val_1' as $\mathrm{val}_1'(v) := f(\mathrm{val}_1(v))$ if $z < \mathrm{val}_1(v) \leq y$ and $\mathrm{val}_1(v)$ otherwise. It is easy to see that $\mathrm{val}_1(t') = 0$ iff $\mathrm{val}_1'(t') = 0$. Now the induction hypothesis can be applied to the algebra B and the valuation val_1'. ∎

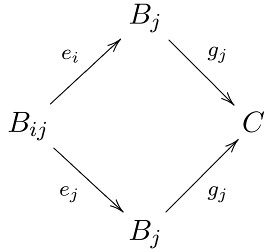
Lemma 16. *Let the algebra B (from Lemma 15) be a BCK-union of a connected family of subdirectly irreducible bounded Łukasiewicz algebras $\{B_1, \ldots, B_m\}$. There are*

(i) *a bounded Łukasiewicz algebra L and embeddings $f_i : B_i \longrightarrow L$ for $i = 1, \ldots, n$ such that for all $i, j = 1, \ldots, n$, $i \neq j$ and all $x \in B_i \cap B_j$ $f_i(x) = f_j(x)$,*

(ii) *a valuation $\mathrm{val}_3 : \mathrm{var}(t') \longrightarrow L$ such that*

$$\mathrm{val}_2(t') = 0 \text{ iff } \mathrm{val}_3(t') = 0.$$

Proof. As each of the algebras $B_i, i = 1, \ldots, n$ is bounded we will consider all of them as algebras of type (2,0,0). By Corollary 3 (ii) any two algebras B_i, B_j have isomorphic subalgebras, namely algebras $(B_i)_z^x$ and $(B_j)_z^y$, where z is the largest element in $B_i \cap B_j$, x, y largest elements of B_i and B_j respectively. Let B_{ij} denote a bounded Łukasiewicz algebra of type (2,0,0) isomorphic to the algebra $(B_i)_z^x$. Let e_i, e_j be embeddings of B_{ij} into B_i and B_j respectively. Using the Amalgamation Property for the variety of bounded Łukasiewicz algebras, we can find a bounded Łukasiewicz algebra C and embeddings g_i, g_j , for $i, j = 1, \ldots, n, i \neq j$, such that the following diagram commutes



Put $L := C$, $f_1 := g_1$, $f_2 := g_2$ and define val_3 as follows:
if $v \in \mathrm{var}(t')$, $\mathrm{val}_2(v) \in B_i$ then $\mathrm{val}_3(v) := g_i(\mathrm{val}_2(v))$.

It is easy to see that condition (i) is satisfied. To show that (ii) holds one can use induction on complexity of the term t. ∎

Corollary. *For any term t and its L-version t'*

$$\mathrm{val}(t) = 0 \text{ iff } \mathrm{val}_3(t') = 0.$$

Main result

Lemma 17. *Let $Q := t_1 = 0, \ldots, t_p = 0 \longrightarrow t = 0$ be a quasiidentity refutable in a commutative* BCK-*algebra A. Then there is a finite subdirectly irreducible commutative* BCK-*algebra D in which the quasiidentity Q can be refuted.*

Proof. Let $\mathrm{var}(Q) := \mathrm{var}(t_1) \cup \mathrm{var}(t_p) \cup \mathrm{var}(t)$ and $\mathrm{val}\colon \mathrm{var}(Q) \longrightarrow A$ be a refuting valuation for Q. We can assume that the algebra A is subdirectly irreducible and that each branch of A has the largest element. Let Q' be a quasiidentity we get from Q by replacing terms $t_1, \ldots, t_p, t$ with their L-versions $t'_1, \ldots, t'_p, t'$ respectively and by adding to its antecedent all equalities $v_i v_j = 0$ for all $v_i, v_j \in \mathrm{var}(Q')$ such that $\mathrm{val}_1(v_i)\mathrm{val}_1(v_j) = 0$ in A where val_1 is the first valuation associated with the valuation val. We can also assume that for each branching point z of A there is a variable v in $\mathrm{var}(Q')$ such that $z = \mathrm{val}_1(v)$. (If it is not the case we can add new variables and extend the valuation val_1 in such a way that the last condition is satisfied and add proper equalities to the predecessor of Q').

Let us note that the valuation val_1 refutes quasiidentity Q' in A. Now using Lemma 16 and its Corollary we can find a bounded Łukasiewicz algebra L and a valuation val_3 such that $\mathrm{val}_3(t'_i) = 0$, $i = 1, \ldots, p$, $\mathrm{val}_3(v_i v_j) = 0$ for all equalities $v_i v_j = 0$ added to the antecedent and $\mathrm{val}_3(t') \neq 0$ in L, *i.e.* val_3 refutes Q' in L.

By Theorem 1 we can find a finite subdirectly irreducible Łukasiewicz algebra K in which the quasiidentity Q' can be refuted. Let val_4 denote the refuting valuation. We will construct an algebra D as a BCK-sum of some subalgebras of isomorphic copies of the algebra K.

Let us recall that the algebra A is the BCK-sum of bounded subdirectly irreducible Łukasiewicz algebras $A_1, \ldots, A_n$. Let x_i be the largest element of A_i, $i = 1, \ldots, n$, z_j, $j = 1, \ldots, m$, be all branching elements of A. Let K_i, $i = 1, \ldots, n$ be an isomorphic copy of the algebra K, and $f_i\colon K \longrightarrow K_i$, be a mapping establishing the isomorphism (there is exactly one such f_i). We define partial valuations val_{5i} of variables from $\mathrm{var}(Q')$ as follows

$$\mathrm{val}_{5i}(v) = \begin{cases} f_i(\mathrm{val}_4(v)) \text{ if } \mathrm{val}_1(v) \leq x_i \\ \text{undefined otherwise.} \end{cases} \tag{1}$$

By y_i we denote any element of $K_1 \cup \ldots \cup K_n$ of the form $\mathrm{var}_{5i}(v)$, where v is a variable in $\mathrm{var}(Q')$ for which $\mathrm{val}_1(v) = z_i$, for some $i = 1, \ldots, n$. We require that if z_i is the largest element in $A_j \cap A_\ell$ then y_i is the largest element in $K_j \cap K_\ell$ and $K_j \cap K_\ell = \{x \in K_j : x \leq y_i\} = \{x \in K_\ell : x \leq y_i\}$, $1 \leq j,\ell \leq n$. Algebras $K_1,...,K_n$ form a connected family of finite subdirectly irreducible bounded Łukasiewicz algebras. Let D be a sum of the family $K_1,\ldots,K_n$ and $\mathrm{val}_5 \colon \mathrm{var}(Q') \longrightarrow D$ defined as follows $\mathrm{val}_5(v) = \mathrm{val}_{5i}(v)$ where i, $1 \leq i \leq n$, is such that $\mathrm{val}_{5i}(v)$ is defined. It is easy to see that

(i) val_5 is well defined (see definition of K_j, $j = 1,\ldots,n$);
(ii) for any term t $\mathrm{val}_5(t') = 0$ iff $\mathrm{val}_4(t') = 0$, where t' is the L-version of t;
(iii) val_5 refutes Q' in D;
(iv) val_5 refutes Q in D. ∎

Corollary 1. *If $A \in W_n$ then $D \in W_n$.*

Theorem 7. *The variety of all commutative* BCK-*algebras is generated by its finite subdirectly irreducible members as a quasivariety.*

Corollary 1 . *The set of quasiidentities of commutative* BCK-*algebras is recursive.*

Corollary 2. *The set of identities of commutative* BCK-*algebras is recursive.*

Generic algebra for the variety of commutative BCK-algebras

It was shown in [20] that the algebra $N = <\boldsymbol{N}, *, 0>$, where for natural numbers m,n $m * n = \max(0, m - n)$ is a generic algebra for the variety of all Łukasiewicz algebras. It follows now from Theorem 7 that the variety of all commutative BCK-algebras also has a nice generic algebra. Let us start with the following easy

Lemma 18. *Let $< A, \leq >$ be a tree in which each element except the least one has countably many successors and the least element of A, 0, has exactly one successor. We define a binary operation $*$ on A in such a way that*

(i) *each branch in A is a carrier set of a subalgebra of A which is isomorphic to N,*
(ii) *if x,y are incomparable in $< A, \leq >$ then $x * y = x * (x \wedge y)$.*

Then $< A, *, 0 >$ is a subdirectly irreducible commutative BCK-*algebra and the* BCK-*order coincides with the tree order.*

Theorem 8. *The algebra A defined in Lemma 18 is a generic algebra for the variety of all commutative* BCK-*algebras.*

Proof. It is easy to see that each finite subdirectly irreducible commutative BCK-algebra is a subalgebra of the algebra A. ∎

Varieties of commutative BCK-algebras of finite width

It was shown that in the variety of Łukasiewicz algebras all subdirectly irreducible algebras are linearly ordered (see *e.g.* [26]). To explain this fact informally let us look at the identity

(L) $xy = (xy)(yx)$

defining the variety of Łukasiewicz algebras in the variety of all commutative BCK-algebras. The identity (L) can be reformulated equivalently as

(L') $(xy) \wedge (yx) = 0.$

It is an easy observation that in any subdirectly irreducible BCK-algebra A, for any two non-zero elements x and y of A there is an element $z \neq 0$ in A such that $z \leq x$ and $z \leq y$ with respect to the BCK-order. So if (L') holds in a nontrivial subdirectly irreducible Łukasiewicz algebra A then for all a, b in A $ab = 0$ or $ba = 0$ which means A is linearly ordered.

We can now generalize (L') (or (L) which is the same) to (L_n), where n is any natural number, $n \geq 1$, in such a way that a subdirectly irreducible commutative BCK-algebra satisfies (L_n) if and only if it has at most n pairwise incomparable elements. It is easy to see that we can take

$$(L_n) \qquad \bigwedge_{0 \leq i,j \leq n} x_i x_j = 0.$$

Let W_n, where n is any natural number, $n \geq 1$, denote the variety of all commutative BCK-algebras satisfying (L_n). Let us note that the variety W_1 is the variety of all Łukasiewicz algebras.

We have the following version of Theorem 5 for varieties W_n.

Theorem 9. *Let $A \in W_n$. The algebra A is subdirectly irreducible if and only if it is the* BCK-*union of a connected family $\{A_i\}_{i \in I}$ of subdirectly irreducible Łukasiewicz algebras such that for any $i, j \in I$ $A_i \cap A_j \neq \{0\}$ and I has at most n elements.*

By Lemma 17 and its Corollary we get

Theorem 10. *For each natural number n, $n \geq 1$, the variety W_n is generated by its finite subdirectly irreducible elements.*

Corollary. *For each natural number n, $n \geq 1$, the set of quasiidentities as well as the set of identities of the variety W_n is recursive.*

We have the following analog of Lemma 18.

Lemma 19. *Let $T_n = <T_n, \leq>$, where $n \in N$, $n \geq 1$, be a tree in which each element except the least one has n successors and the least element 0 of T_n has exactly one successor. We define a binary operation $*$ on T_n in such a way that*

(i) *each branch in T_n is a carrier set of a subalgebra of T_n which is isomorphic to N,*
(ii) *if x, y are incomparable in $<T_n, \leq>$ then $x * y = x * (x \wedge y)$.*

Then $<T_n, *>$ is a subdirectly irreducible commutative BCK-*algebra in W_n and the* BCK-*order coincides with the tree order. Moreover the algebra T_n is a generic algebra for W_n.*

Moreover we have

Theorem 11.

(i) $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_n \ldots.$
(ii) $\cup_{1 \leq n} W_n$ *is the variety of all commutative* BCK-*algebras.*

Proof. (i) is trivial.

(ii) Each finite commutative BCK-algebra belongs to some variety W_n for some natural number n. ■

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