Agata Pilitowska, Anna B. Romanowska

EMBEDDING MODES INTO SEMIMODULES, PART III

Abstract. In the first part of this paper, we considered the problem of constructing a (commutative unital) semiring defining the variety of semimodules whose idempotent subreducts lie in a given variety of modes. We provided a general construction of such semirings, along with basic examples and some general properties. In the second part of the paper we discussed some selected varieties of modes, in particular, varieties of affine spaces, varieties of barycentric algebras and varieties of semilattice modes, and described the semirings determining their semi-linearizations, the varieties of semimodules having these algebras as idempotent subreducts. The third part is devoted to varieties of differential groupoids and more general differential modes, and provides the semirings of the semi-linearizations of these varieties.

This paper is a direct continuation of the first and second parts appearing with the same title [4] and [5]. In the first part, we considered the problem of constructing a (commutative unital) semiring defining the variety of semimodules whose idempotent subreducts lie in a given variety of modes, and such that each semimodule-embeddable member of this mode variety embeds into a semimodule over the constructed semiring. We described a general construction of such semirings, with basic examples and some general properties. In the second part, we investigated selected varieties of modes, and described the semirings determining varieties of semimodules having algebras of these classes as subreducts, and discussed properties of the corresponding semi-affine spaces. In particular, we investigated varieties of affine spaces, varieties of barycentric algebras and varieties of semilattice modes. The third part is devoted to varieties of differential groupoids and more general differ-

ential modes, and provides the semirings of the semi-linearizations of these varieties. Apart from having interesting properties of their own, differential groupoids and differential modes play an important role in the problem of embedding modes into semimodules, and also in the theory of finitely generated modes. (See [2], [3], [6], [9], [10].)

We use the terminology, notation and results of the first and second parts of the paper, and continue the section numbering from the second paper.

5. The semiring of the variety of differential groupoids

Differential groupoids are binary modes $(G, \cdot)$ defined by the identity $x \cdot (x \cdot y) = x$. (See [9].) As shown in [10, Section 7.1], the (affinization) ring of the variety $\mathcal{D}_2$ of differential groupoids is the ring $R(\mathcal{D}_2) = \mathbb{Z}[X]/\langle X^2 \rangle \cong \mathbb{Z}[d]$, where $d^2 = 0$, of *integral dual numbers*. (See also [9].) Its elements are represented as $m + dn$ for $m, n \in \mathbb{Z}$. The differential groupoid operation on an $R(\mathcal{D}_2)$-space is defined as $x \cdot y = xy\underline{X}$ or equivalently as $x \cdot y = xy\underline{d}$. As the quotient of $\mathbb{Z}[X, Y]$, the ring $R(\mathcal{D}_2)$ can be written as

$$\mathbb{Z}[X, Y]/\langle 1 - X - Y, X^2 \rangle \cong \mathbb{Z}[X, Y]/\alpha,$$

where

$$\alpha = cg((X + Y, 1), (X^2, 0)).$$

It is isomorphic to the ring

$$\mathbb{Z}[d, e] \text{ with } d + e = 1 \text{ and } d^2 = 0,$$

and with elements represented as $em + dn$ for $m, n \in \mathbb{Z}$. Denote this ring by R. The differential groupoid operation on an affine R-space is defined as $x \cdot y = xe + yd$.

Note that differential groupoids form an irregular variety. Hence the semiring $S(\mathcal{D}_2)$ of the variety $\mathcal{D}_2$ can be calculated similarly as its ring and is determined by the same relations. (Cp. [4].) In particular, $S(\mathcal{D}_2)$ is isomorphic to the semiring $\mathbb{N}[X, Y]/\alpha$ and to the semiring $\mathbb{N}[d, e]$ with $d + e = 1$, and $d^2 = 0$. Denote this semiring by S. As before, the differential groupoid operation on a semi-affine S-space is defined as $x \cdot y = xe + yd$. In what follows we identify the semiring $S(\mathcal{D}_2)$ with S (or with the corresponding quotient semiring), and the ring $R(\mathcal{D}_2)$ with R (or with the corresponding quotient ring).

It is clear that $\mathbb{N}[X, Y]$ is a subsemiring of the semiring $\mathbb{Z}[X, Y]$. By the Second Isomorphism Theorem (see e.g. [10, Theorem 1.2.4]), it follows that the quotient $\mathbb{N}[X, Y]/\alpha$ is isomorphic with a subsemiring of $\mathbb{Z}[X, Y]/\alpha$. To describe the semiring S and its relation to the ring R in a more direct way, we will first need several technical lemmas.

LEMMA 5.1. *The following hold for elements of both R and S for all natural numbers k, l, m and n:*

(a) $de^k = d$;
(b) $e^k = e^{k+1} + d$;
(c) $e^k + dk = 1$;
(d) $e^k m + dn = e^{k+l} m + d(ml + n)$;
(e) $m + dn = e^l m + d(ml + n)$;
(f) $e^k m + dn = e^{k-l} m + d(n - ml)$ *in the case $k \geq l$ and $n - ml \geq 0$;*
(g) $e^k m + dn = m + d(n - mk)$, *in the case $n - mk \geq 0$.*

Proof. First note that $d + e = 1$ and $d^2 = 0$ imply that $de = d$, whence (a) holds.

Now $e^2 + d = e^2 + de = (e + d)e = e$. If $e^{k+1} + d = e^k$, then $e^{k+2} + d = e^{k+2} + de = (e^{k+1} + d)e = e^{k+1}$. This simple induction proves (b).

The third equality (c) follows from the previous ones:

$$1 = (d + e)^k$$
$$= d^k + d^{k-1}ek + d^{k-2}e^2(k(k-1)/2) + \cdots + de^{k-1}k + e^k$$
$$= dk + e^k.$$

The equality (d) follows from the first and the third ones. Indeed,

$$e^k m + dn = e^k m(e^l + dl) + dn = e^{k+l} m + d(ml + n).$$

The equality (e) follows from (d) for $k = 0$. And (f) follows directly from (d). Then (g) is a special case of (f) obtained for $k = l$. ∎

The last three equalities of Lemma 5.1 easily generalize to elements of the ring R with integer coefficients.

LEMMA 5.2. *The following hold for elements of the ring R for all integers m, n and s and natural numbers k and l:*

(a) $m + ds = e^k m + d(mk + s)$;
(b) $e^k m + dn = e^l m + d(m(l - k) + n)$;
(c) $e^k m + dn = m + d(n - mk)$.

Proof. The first equality follows by Lemma 5.1(c), the second from the first one, and the third from the second by taking $l = 0$. ∎

LEMMA 5.3. *For each element $m + ds \in R$ with $m > 0$ and $s < 0$, there is a natural number k such that*

$$m + ds = e^k m + d(mk + s)$$

and $mk + s \geq 0$.

Proof. By Lemma 5.2(a), for any natural number k

$$m + ds = e^k m + d(mk + s).$$

In particular, $k := \lceil -s/m \rceil$ is the least natural number such that $mk + s \geq 0$. ∎

LEMMA 5.4. *Each element of the semiring S can be written as*

$$e^k m + dn$$

for some natural numbers k, m, n.

Proof. First note that each element a in S has the form

$$a = n_0 + e n_1 + \cdots + e^k n_k + d n_d,$$

where $n_0, n_1, \ldots, n_k, n_d \in \mathbb{N}$. Note that if $k = 0$, then a has the required form. Suppose that $k > 0$. We will show that there are natural numbers m and n such that

$$a = e^k m + dn.$$

By Lemma 5.1, (a) and (c), one has:

$$a = n_0(e^k + dk) + n_1 e(e^{k-1} + d(k-1)) + \cdots + e^k n_k + d n_d$$
$$= e^k(n_0 + n_1 + \cdots + n_k) + d(kn_0 + (k-1)n_1 + \cdots + n_{k-1} + n_d).$$

Hence $a = e^k m + dn$ for $m = n_0 + n_1 + \cdots + n_k$ and $n = kn_0 + (k-1)n_1 + \cdots + n_{k-1} + n_d$. ∎

Let $R^+ := \{m + ds \mid m \in \mathbb{Z}^+, s \in \mathbb{Z}\} \cup \{dn \mid n \in \mathbb{N}\}$. Note that R^+ is a subsemiring of the semiring R. And observe that one can identify the elements of the ring $\mathbb{Z}[X, Y]/\alpha$ with the elements of R, and the elements of the semiring S with the elements of R^+. As a corollary to Lemmas 5.1–5.4, one obtains the following proposition.

PROPOSITION 5.5. *The semiring S of the variety of differential groupoids embeds into its ring R, and is isomorphic to the semiring R^+.*

The following example illustrates an application to semilattice modes with a differential groupoid reduct.

EXAMPLE 5.6. Let $\mathcal{V}$ be the variety of semilattice modes $(G, \cdot, +)$ with a differential groupoid reduct $(G, \cdot)$. By [5, Section 6] and Proposition 5.5, each element of the semiring $S(\mathcal{V})$ can be represented by $m + dn$ for a positive integer m and an integer n or by dn for a natural number n. The addition of the semiring $S(\mathcal{V})$ is idempotent and satisfies the identity $s + 1 = 1$ for each $s \in S(\mathcal{V})$. In particular, $m + dn = (m - 1 + dn) + 1 = 1$ for each positive integer m. And since $n = 1$ for all positive integers n, we also have that $dn = d$. It follows that the semiring $S(\mathcal{V})$ consists precisely of three elements $0, d, 1$ forming the chain $0 < d < 1$.

6. The semiring of the variety of differential modes

Ternary counterparts of differential groupoids, so-called (ternary) differential modes, are discussed in [3] and [6]. However, as noted in [3], it would be very easy to extend all the notions and results of these papers to modes with one n-ary operation for all $n \geq 4$. The ternary case was chosen only to avoid technical complications. In this section we consider differential modes of arbitrarily fixed arity, and call such algebras simply differential modes.

Differential modes are modes $(G, (x_1 \ldots x_{n+1}))$ with one $(n+1)$-ary operation, where n is a fixed positive integer, defined by the left reduction law

$$(6.1) \qquad (x(y_1 y_{11} \ldots y_{1n}) \ldots (y_n y_{n1} \ldots y_{nn})) = (xy_1 \ldots y_n).$$

The affinization of the variety $\mathcal{D}_{n+1}$ of $((n+1)$-ary) differential modes may be found in a similar way as in the binary case.

The ring $R(\mathcal{D}_{n+1})$ is a quotient of the polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ (with commuting indeterminates). The indeterminates $X_1, \ldots, X_n$ furnish the differential mode operation on affine $R(\mathcal{D}_{n+1})$-spaces as

$$(xx_1 \ldots x_n) = x(1 - \sum_{i=1}^{n} X_i) + \sum_{i=1}^{n} x_i X_i.$$

Now the operation $(xx_1 \ldots x_n)$ satisfies the left reduction law (6.1), whence

$$(x(y_1 y_{11} \ldots y_{1n}) \ldots (y_n y_{n1} \ldots y_{nn}))$$
$$= x(1 - \sum_{i=1}^{n} X_i) + \sum_{j=1}^{n} y_j X_j (1 - \sum_{i=1}^{n} X_i) + \sum_{i,j=1}^{n} y_{ij} X_i X_j$$
$$= x(1 - \sum_{i=1}^{n} X_i) + \sum_{i=1}^{n} y_i X_i = (xy_1 \ldots y_n).$$

Equating coefficients of y_{ij}, where $i, j = 1, \ldots, n$, shows that all $X_i X_j$ annihilate affine space elements, so that $R(\mathcal{D}_{n+1})$ is a quotient of the ring $\mathbb{Z}[X_1, \ldots, X_n]/\langle X_i X_j \mid i, j = 1, \ldots, n \rangle \cong \mathbb{Z}[d_1, \ldots, d_n]$, where $d_i d_j = 0$ for $i, j = 1, \ldots, n$. Its elements are represented as $m + \sum_{i=1}^{n} d_i s_i$ for $m, s_1, \ldots, s_n \in \mathbb{Z}$.

Conversely, affine spaces over $\mathbb{Z}[d_1, \ldots, d_n]$ are differential modes under the operation

$$(xx_1 \ldots x_n) := x(1 - \sum_{i=1}^{n} d_i) + \sum_{i=1}^{n} x_i d_i.$$

It follows that $R(\mathcal{D}_{n+1}) \cong \mathbb{Z}[X_1, \ldots, X_n]/\langle X_i X_j \mid i, j = 1, \ldots, n \rangle$.

Similarly, as in the binary case, the ring $R(\mathcal{D}_{n+1})$ may also be written as

$$\mathbb{Z}[X_1, \ldots, X_n, X_{n+1}] / \left\langle 1 - \sum_{i=1}^{n+1} X_i, \ X_i X_j \mid i, j = 1, \ldots, n \right\rangle$$

$$\cong \mathbb{Z}[X_1, \ldots, X_n, X_{n+1}] / \beta,$$

where

$$\beta = cg\left(\left(\sum_{i=1}^{n+1} X_i, 1\right), (X_i X_j, 0) \mid i, j = 1, \ldots, n\right),$$

and is isomorphic to the ring

$$\mathbb{Z}[d_1, \ldots, d_n, f] \text{ with } \sum_{i=1}^{n} d_i + f = 1 \text{ and } d_i d_j = 0, \text{ where } i, j = 1, \ldots, n.$$

As in the case of differential groupoids, differential modes form an irregular variety. The semiring of the variety $\mathcal{D}_{n+1}$ is calculated similarly as its ring, and is determined by the same relations. In particular, the semiring $S(\mathcal{D}_{n+1})$ is isomorphic to the semiring $\mathbb{N}[d_1, \ldots, d_n, f]$ with $\sum_{i=1}^{n} d_i + f = 1$ and $d_i d_j = 0$ for $i, j = 1, \ldots, n$.

As in the case of differential groupoids, it is clear that the semiring $\mathbb{N}[X_1, \ldots, X_n, X_{n+1}] / \beta$ embeds into the semiring $\mathbb{Z}[X_1, \ldots, X_n, X_{n+1}] / \beta$. The following calculations will provide a simple description of the elements of the semiring $S(\mathcal{D}_{n+1})$, and their relation to the elements of the ring $R(\mathcal{D}_{n+1})$.

LEMMA 6.1. *The following hold for elements of both $R(\mathcal{D}_{n+1})$ and $S(\mathcal{D}_{n+1})$ for all natural numbers k, m and $s_1, \ldots, s_n$:*

(a) $d_i f^k = d_i$ for $i = 1, \ldots, n$;

(b) $(\sum_{i=1}^{n} d_i)^k = 0$;

(c) $f^k = f^{k+1} + \sum_{i=1}^{n} d_i$;

(d) $f^k + k \sum_{i=1}^{n} d_i = 1$;

(e) $m + \sum_{i=1}^{n} d_i s_i = f^k m + \sum_{i=1}^{n} d_i (km + s_i)$.

Proof. First note that $\sum_{i=1}^{n} d_i + f = 1$ and $d_i d_j = 0$ for all $i, j = 1, \ldots, n$, imply that $d_i f = d_i$ for all $i = 1, \ldots, n$. Moreover,

$$\left(\sum_{i=1}^{n} d_i\right)^2 = \sum_{i=1}^{n} d_i d_j = 0,$$

whence (a) and (b) hold.

Now

$$f^2 + \sum_{i=1}^{n} d_i = f^2 + \sum_{i=1}^{n} d_i f = \left(f + \sum_{i=1}^{n} d_i\right) f = f.$$

If

$$f^{k+1} + \sum_{i=1}^{n} d_i = f^k,$$

then

$$f^{k+2} + \sum_{i=1}^{n} d_i = f^{k+2} + \sum_{i=1}^{n} d_i f = \left(f^{k+1} + \sum_{i=1}^{n} d_i \right) f = f^{k+1}.$$

This simple induction proves (c).

The equality (d) follows from the previous ones:

$$1 = \left(\sum_{i=1}^{n} d_i + f \right)^k$$

$$= \left(\sum_{i=1}^{n} d_i \right)^k + \left(\sum_{i=1}^{n} d_i \right)^{k-1} fk + \cdots + \left(\sum_{i=1}^{n} d_i \right) f^{k-1} k + f^k$$

$$= \left(\sum_{i=1}^{n} d_i \right) k + f^k.$$

The last equality (e) follows from (d):

$$m + \sum_{i=1}^{n} d_i s_i = f^k m + \left(\sum_{i=1}^{n} d_i \right) km + \sum_{i=1}^{n} d_i s_i = f^k m + \sum_{i=1}^{n} d_i (km + s_i). \ \blacksquare$$

The last equality easily generalizes to elements of the ring $R(\mathcal{D}_{n+1})$ with integer coefficients.

LEMMA 6.2. *The following holds for elements of the ring $R(\mathcal{D}_{n+1})$ for all integers $s_1, \ldots, s_n$ and natural numbers m and k:*

$$m + \sum_{i=1}^{n} d_i s_i = f^k m + \sum_{i=1}^{n} d_i (km + s_i).$$

Proof. The equality follows by Lemma 6.1(d). $\blacksquare$

LEMMA 6.3. *For each element $m + \sum_{i=1}^{n} d_i s_i \in R(\mathcal{D}_{n+1})$ with $m > 0$ and at least one of $s_1, \ldots, s_n$ negative, there is a natural number k such that*

$$m + \sum_{i=1}^{n} d_i s_i = f^k m + \sum_{i=1}^{n} d_i (km + s_i)$$

with all $km + s_1, \ldots, km + s_n$ non-negative.

Proof. By Lemma 6.2, for any natural number k, one has

$$m + \sum_{i=1}^{n} d_i s_i = f^k m + \sum_{i=1}^{n} d_i (km + s_i).$$

In particular, for $k := \max(\lceil |s_1|/m \rceil, \ldots, \lceil |s_n|/m \rceil)$, one has $km + s_1 \geq 0$, $\ldots, km + s_n \geq 0$. $\blacksquare$

LEMMA 6.4. *Each element of the semiring $S(\mathcal{D}_{n+1})$ can be written as*

$$f^k m + \sum_{i=1}^{n} d_i p_i$$

for some natural numbers $k, m, p_1, \ldots, p_n$.

Proof. First note that each element a in $S(\mathcal{D}_{n+1})$ has the form

$$a = n_0 + f n_1 + \cdots + f^k n_k + \sum_{i=1}^{n} d_i m_i,$$

where $n_0, n_1, \ldots, n_k, m_1, \ldots, m_n \in \mathbb{N}$. Note that if $k = 0$, then a has the required form. Suppose that $k > 0$. We will show that there are natural numbers $m, p_1, \ldots, p_n$ such that

$$a = f^k m + \sum_{i=1}^{n} d_i m_i.$$

By Lemma 6.1(d), one has:

$$a = n_0 \left(f^k + \left(\sum_{i=1}^{n} d_i \right) k \right) + n_1 f \left(f^{k-1} + \left(\sum_{i=1}^{n} d_i \right)(k-1) \right) + \ldots$$

$$\cdots + f^k n_k + \sum_{i=1}^{n} d_i m_i$$

$$= f^k (n_0 + n_1 + \cdots + n_k) + \sum_{i=1}^{n} d_i (k n_0 + (k-1) n_1 + \ldots$$

$$\cdots + n_{k-1} + m_i).$$

Hence $a = f^k m + \sum_{i=1}^{n} d_i p_i$ for $m = n_0 + n_1 + \cdots + n_k$ and $p_i = k n_0 + (k-1) n_1 + \cdots + n_{k-1} + m_i$ for $i = 1, \ldots, n$. $\blacksquare$

Let T be the set:

$$\left\{ m + \sum_{i=1}^{n} d_i s_i \mid m \in \mathbb{Z}^+, s_1, \ldots, s_n \in \mathbb{Z} \right\} \cup \left\{ \sum_{i=1}^{n} d_i m_i \mid m_1, \ldots, m_n \in \mathbb{N} \right\}.$$
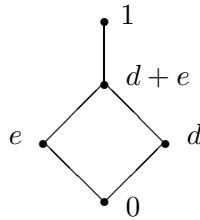
It is clear that T is a subsemiring of the semiring $R(\mathcal{D}_{n+1})$.

PROPOSITION 6.5. *The semiring $S(\mathcal{D}_{n+1})$ of the variety of differential modes embeds into its ring $R(\mathcal{D}_{n+1})$, and is isomorphic to the semiring T.*

Proof. The proof follows by Lemmas 6.1–6.4. ∎

The next example generalizes Example 5.6 for semilattice modes with a differential groupoid reduct.

EXAMPLE 6.6. Let $\mathcal{V}$ be the variety of semilattice modes $(G, (xyz), +)$ with ternary differential mode reduct $(G, (xyz))$. As in Example 5.6, the addition of the semiring $S(\mathcal{V})$ is idempotent and satisfies the identity $s + 1 = 1$. Similar calculations as in Example 5.6 (but based on lemmas of this section) show that $S(\mathcal{V}) = \{0, d, e, d + e, 1\}$ with elements ordered as shown on the following picture.

Finally let us note that a similar method for finding the semiring of the semi-linearization can be applied to any irregular variety of modes, in particular to varieties of reductive binary mode varieties (see [10, section 8.4]).

References

[1] U. Hebisch, H. J. Weinert, *Semirings - Algebraic Theory and Application in Computer Science,* World Scientific, Singapore, 1998.

[2] K. A. Kearnes, *Semilattice modes I: the associated semiring,* Algebra Universalis 34 (1995), 220–272.

[3] A. Kravchenko, A. Pilitowska, A. Romanowska, D. Stanovský, *Differential modes,* Internat. J. Algebra Comput. 18(3) (2008), 567–588.

[4] A. Pilitowska, A. Romanowska, *Embedding modes into semimodules, Part I,* Demonstratio Math. 44 (2011), 523–534.

[5] A. Pilitowska, A. Romanowska, *Embedding modes into semimodules, Part II,* Demonstratio Math., this volume, 781–790.

[6] A. Pilitowska, A. Romanowska, D. Stanovský, *Varieties of differential modes embeddable into semimodules,* Internat. J. Algebra Comput. 19 (2009), 669–680.

[7] A. B. Romanowska, *Semi-affine modes and modals,* Sci. Math. Jpn. 61 (2005), 159–194.

[8] A. B. Romanowska, J. D. H. Smith, *Modal Theory,* Heldermann Verlag, Berlin, 1985.

[9] A. B. Romanowska, J. D. H. Smith, *Differential groupoids,* Contributions to General Algebra 7 (1991), 283–290.

[10] A. B. Romanowska, J. D. H. Smith, *Modes,* World Scientific, Singapore, 2002.

A. Pilitowska, A. B. Romanowska
FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 WARSAW, POLAND
E-mail: apili@alpha.mini.pw.edu.pl
        aroman@alpha.mini.pw.edu.pl
http://www.mini.pw.edu.pl/~apili
http://www.mini.pw.edu.pl/~aroman