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SMOOTH BRUCK LOOPS, SYMMETRIC SPACES, AND NONASSOCIATIVE VECTOR SPACES

Abstract. Our purposes in this work include the following: (1) Extend and expand earlier work on symmetric spaces, particularly that done from a nonassociative algebra point of view, from the finite-dimensional setting to the Banach space setting. (2) Take a careful look at the equivalence of the categories of smooth pointed reflection quasigroups (a special class of symmetric spaces) and uniquely 2-divisible Bruck loops (= K-loops = gyrocommutative gyrogroups). (3) Propose a loop-theoretic analog of topological vector spaces. (4) Derive algebraic consequences and equivalences of smoothness notions, particularly the notion of parallel transport. (5) Illustrate the effective interaction of the algebraic operations of reflection, Bruck addition, and coaddition in the test case of parallelograms in symmetric spaces.

1. Introduction

René Descartes and Pierre de Fermat revolutionized the study of geometry with their introduction of coordinate systems and algebraic methods. Significant outcomes were the rise of analytic geometry and the introduction of Euclidean vector spaces as an appropriate framework for Euclidean geometry. The latter were generalized in time to the notions of Hilbert and Banach spaces.

Attempts to find appropriate algebraic coordinatizations of more general geometric settings have led to the study of more general algebraic structures. A number of these have nonassociative binary operations. In recent years A. A. Ungar has introduced certain gyrocommutative gyrogroups as an appropriate structure for the development of an analytic hyperbolic geometry [16]. M. Kikkawa considered quasigroups as algebraic models of symmetric spaces [4], [5] and several other authors have followed up on this idea over the years. We mention in particular the work of L. V. Sabinin [14]

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and P. Nagy and K. Strambach [11] as having particular relevance for our work.

In this work we consider Bruck loops (or K-loops) as suitable algebraic models for symmetric spaces. While earlier work focused on the finite dimensional setting, we consider the infinite dimensional setting as well. This generalization is made possible through recent work of Neeb [12], who generalized the machinery of connections on finite-dimensional symmetric spaces to sprays on infinite-dimensional Banach symmetric spaces. Our goal is to find an algebraic model that is an enriched Bruck loop and that, as closely as possible, resembles a non-associative Banach space, with the smooth structure and the tangent bundle replaced by algebraic notions and operations internal to the loop. These structures may be viewed as variants of the gyrovector space structures of Ungar [16], gyrocommutative gyrogroups with a type of scalar multiplication, or of the odules of Sabanin [13], [14]. In the latter part of the paper we adopt primarily the terminology and notation of Ungar, since we find this the most suitable for highlighting analogies with the classical theory of vector spaces.

2. Smooth symmetric spaces

One of our principal goals in what follows is to extend to the infinite-dimensional setting (Banach manifolds) significant parts of the well-developed theory of finite-dimensional differentiable loops. We work exclusively with C^∞-differentiability, a property we refer to as smoothness. Smooth manifolds will mean for us manifolds modeled on some Banach space for which the charts are smoothly related.

We begin with the definition of a symmetric space. We employ the axioms of Loos [10], since this approach connects well with nonassociative algebra and has excellent functorial properties as well, such as the fact that the tangent bundle of a symmetric space is again a symmetric space.

Definition 2.1. We say $(M, \bullet)$ is a *Loos symmetric space* if M is a smooth Banach manifold, and $(x, y) \mapsto x \bullet y : M \times M \to M$ is a smooth map with the following properties for all $a, b, c \in M$ (each property is given an alternative equivalent form in terms of the left translations S_a):

(S1) $a \bullet a = a$ ($S_a a = a$);
(S2) $a \bullet (a \bullet b) = b$ ($S_a S_a b = b$);
(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$ ($S_a S_b c = S_{S_a b} S_a c$);
(S4) Every $a \in M$ has a neighborhood U such that $a \bullet x = x$ implies $a = x$ for $x \in U$ (S_a has isolated fixed points).

We say the operation $\bullet$ is *idempotent* (S1), *left involutive* (S2), and *left distributive* (S3). Note that (S2) implies the *right quasigroup* property

($a \bullet x = b$ has a unique solution in x) since

$$a \bullet x = b \Leftrightarrow x = a \bullet (a \bullet x) = a \bullet b. \tag{2.1}$$

We view the left translation $S_x(y) = x \bullet y$ as a *point reflection* or *symmetry* through x. Properties (S1) and (S2) have obvious geometric interpretation.

We recall some basic notions from differential geometry that will be fundamental for our developments. The notion of a(n) (affine) connection on a finite dimensional manifold becomes more subtle in the Banach manifold setting. A useful generalization in this setting is the notion of a *spray* [7].

Definition 2.2. Let M be a Banach manifold and $\pi : TM \to M$ its tangent bundle.

(i) A *second-order vector field on* M is a vector field $F : TM \to TTM$ satisfying: an integral curve of the local flow Φ_F on TM (with infinitesimal generator F) projects under π to a *geodesic* of F in M having the given integral curve as the velocity curve.
(ii) A second order vector field F on TM is called a *spray* if $\pi\Phi_F(s, tv) = \pi\Phi_F(st, v)$ whenever either side is defined.

Definition 2.3. The domain $\mathcal{D}_{\exp} \subseteq TM$ of the exponential function of a spray is the set of all points $v \in T_xM$, $x \in M$, for which the maximal integral (or flow) curve $\gamma_v : J \to TM$ of F with $\gamma_v(0) = v$ satisfies $1 \in J$; in this case we define the *exponential* of v by $\exp(v) = \exp_x(v) := \pi(\gamma_v(1))$.

The *geodesics* have the form $t \mapsto \exp(tv)$ for $v \in \mathcal{D}_x \subseteq TM$. These are the same as the geodesics of Definition 2.2, as one can see from property (ii) of a spray [7, Proposition IV-4.2]. The manifold (M, F) is *geodesically complete* if any two points lie on a geodesic.

A given spray gives rise to a notion of *parallel transport* or *parallel translation* (see [7, Section VIII.3]). For a piecewise smooth curve $\alpha : [s, t] \to M$, we write

$$P_s^t(\alpha) : T_{\alpha(s)}M \to T_{\alpha(t)}M$$

for the corresponding linear map given by *parallel transport* along α.

Remark 2.4. By results of [7, Section IV.4] the domain $\mathcal{D}_{\exp}$ of the exponential function is an open set containing all $0_x \in T_xM$. The exponential function is smooth on $\mathcal{D}_{\exp}$. Each $\exp_x : T_x \cap \mathcal{D}_{\exp} \to M$ has derivative the identity map at 0_x, and hence is diffeomorphism onto some open set containing x for small enough open neighborhoods of 0_x.

The following are important results of Neeb [12, Theorems 3.3, 3.4 and 3.6] that will be crucial in what follows.

Theorem 2.5. *Let $(M, \bullet)$ be a Loos symmetric space.*

(i) *Identifying $T(M \times M)$ with $T(M) \times T(M)$, then*

$$v \bullet w := T(\mu)(v, w) \text{ where } \mu(x, y) := x \bullet y$$

defines a Loos symmetric space on TM. In each tangent space T_xM, $v \bullet w = 2v - w$.

(ii) *The function*

$$F : TM \to TTM, \quad F(v) := -T(S_{v/2} \circ Z)(v)$$

defines a spray on M, where $Z : M \to TM$ is the zero section and $S_{v/2}$ is the point symmetry for $v/2$ from part (i).

(iii) *$Aut(M, \bullet) = Aut(M, F)$, where the former consists of all diffeomorphisms that are automorphisms with respect to $\bullet$ and the latter consists of all diffeomorphisms that preserve the spray F.*

(iv) *F is uniquely defined as the only spray invariant under all symmetries S_x, $x \in M$.*

(v) *Every geodesic of (M, F) extends to a geodesic defined on all of $\mathbb{R}$.*

(vi) *Let $\alpha : \mathbb{R} \to M$ be a geodesic and call the maps $\tau_{\alpha,s} := S_{\alpha(s/2)} \circ S_{\alpha(0)}$, $s \in \mathbb{R}$, translations along α. Then these are automorphisms of $(M, \bullet)$ with*

$$\tau_{\alpha,s}(\alpha(t)) = \alpha(t + s) \quad \text{and} \quad d\tau_{\alpha,s}(\alpha(t)) = P_t^{t+s}(\alpha)$$

for all $s, t \in \mathbb{R}$.

The spray of the preceding theorem will be called the *canonical spray* of the Loos symmetric space $(M, \bullet)$. In light of Theorem 2.5(v), we henceforth assume that geodesics are defined on all of $\mathbb{R}$, unless stated otherwise.

Definition 2.6. A *Loos reflection quasigroup* is a Loos symmetric space $(M, \bullet)$ with Axiom (S4) replaced by

(M4) $x \bullet a = b$ has a unique solution, denoted $x = a\#b$, which is smooth as a function of a and b.

More generally, a *reflection quasigroup* is a pair $(M, \bullet)$ where M is a set and the binary operation $\bullet$ satisfies (S1)–(S3) and (M4).

Remark 2.7.

(i) We note that a Loos reflection quasigroup is indeed a quasigroup since (M4) guarantees the left quasigroup property and equation (2.1) the right.

(ii) By (S2) $x \bullet a = b$ if and only if $a = x \bullet b$. Hence $a\#b = b\#a$.

(iii) The condition that $x \bullet a = b$ has a unique solution is stronger than axiom (S4), since if $x \bullet y = y$, then uniqueness of solution and $y \bullet y = y$

implies $x = y$. Thus S_x has a unique fixed point x, in particular, isolated fixed points.

(iv) It follows from $a \bullet b = c \Leftrightarrow a = b\#c$ that a map between reflection quasigroups preserves the $\bullet$-operation if and only if it preserves the $\#$-operation. In particular from (S3), symmetries S_x preserve the $\#$-operation.

Geometric intuition suggests that reflection through the midpoint m of a and b should carry a to b, i.e., the unique solution of

$$S_x a = x \bullet a = b,$$

if it exists, should be $x = m$, the midpoint. Thus we interpret the left quasigroup property, the existence of a unique solution of $x \bullet a = b$, as asserting the unique existence of a *midpoint* or *mean*, which we denote $x = a\#b$, not the more common notation b/a from quasigroup theory. We note that the midpoint is a midpoint of symmetry, not a midpoint in terms of distance.

Example 2.8. The *core operation* on a group G is defined by $a \bullet b = ab^{-1}a$. A subset P of a group G is a *twisted subgroup* if it contains the identity and is closed under the core operation. In this case $(P, \bullet)$ satisfies the first three axioms of a symmetric space and is further a quasigroup iff P is uniquely 2-divisible (each element of P has a unique square root in P). Hence uniquely divisible twisted subgroups of Lie groups that are submanifolds are Loos reflection quasigroups with respect to the core operation, provided the square root operation is smooth on P.

Proof. The mean $a\#b$ in this case is the unique solution in P of the Ricatti equation: $x \bullet a = xa^{-1}x = b$. To find the solution and establish its uniqueness, we compute

$$\begin{aligned} xa^{-1}x &= b \\ (a^{-1/2}xa^{-1/2})(a^{-1/2}xa^{-1/2}) = a^{-1/2}(xa^{-1}x)a^{-1/2} &= a^{-1/2}ba^{-1/2} \\ a^{-1/2}xa^{-1/2} &= (a^{-1/2}ba^{-1/2})^{1/2} \\ x &= a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}. \end{aligned}$$

The third line follows from the second by uniqueness of square roots. All steps are reversible, so we have found the unique solution

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}.$$

It follows that $a\#b$ is a smooth function of a, b provided the square root function is smooth. ∎

The set of real numbers $\mathbb{R}$ endowed with the core operation $s \bullet t = s + (-t) + s = 2s - t$ defines a Loos reflection quasigroup. A *one-parameter* *reflection quasigroup*, or *one-parameter RQ* for short, into a symmetric space $(M, \bullet)$ is a continuous homomorphism $\alpha : (\mathbb{R}, \bullet) \to (M, \bullet)$. We recall a result from [9, Proposition 4.4].

Theorem 2.9. *The (maximal) geodesics of a Loos symmetric space are precisely the one-parameter RQs. They each have the form $\beta_v(t) = \exp_x(tv)$, where $x \in M$, $v \in T_xM$, and hence are smooth. The correspondence $v \leftrightarrow \beta_v$ with inverse $\beta \leftrightarrow \beta'(0)$ is a bijection between T_xM and all one-parameter RQs taking the value x at* 0.

This is an analog of the basic result in Lie theory that every one-parameter subgroup of a Lie group G has the unique representation $t \mapsto \exp(tX)$, where X is a member of the Lie algebra.

The following lemma provides a weaker set of conditions to verify that a Loos symmetric space is a Loos reflection quasigroup. It also suggests a viewpoint that will be important to our further developments, that of a pointed reflection quasigroup. The category of pointed reflection quasigroups and point-preserving homomorphisms is both vital and suggestive for our later developments.

Lemma 2.10. *Suppose that $(M, \bullet)$ satisfies* (S1), (S2), *and* (S3). *If for some designated $\varepsilon \in M$, M satisfies the requirement that $x \bullet \varepsilon = b$ has unique solution for all b, then M is a quasigroup.*

Proof. By Remark 2.7(i), we need to show that $x \bullet a = b$ has a unique solution for all $a, b \in M$. For each $a \in M$, let us denote the unique solution of $x \bullet \varepsilon = a$ by $a^{1/2}$, i.e., $a^{1/2} = \varepsilon \# a$. Then $a^{1/2} \bullet \varepsilon = a$ and by (S2), $\varepsilon = a^{1/2} \bullet a$. Applying $a^{1/2}$ to $x \bullet a = b$, we obtain (by (S3))

$$a^{1/2} \bullet b = a^{1/2} \bullet (x \bullet a) = (a^{1/2} \bullet x) \bullet (a^{1/2} \bullet a) = (a^{1/2} \bullet x) \bullet \varepsilon.$$

Then by definition $a^{1/2} \bullet x = (a^{1/2} \bullet b)^{1/2}$ or by (S2) we have the equivalent

$$x = a^{1/2} \bullet (a^{1/2} \bullet b)^{1/2}. \tag{2.2}$$

To verify that x is indeed a solution, we note

$$\begin{aligned}(a^{1/2} \bullet (a^{1/2} \bullet b)^{1/2}) \bullet a &= (a^{1/2} \bullet (a^{1/2} \bullet b)^{1/2}) \bullet (a^{1/2} \bullet \varepsilon) \\ &= a^{1/2} \bullet ((a^{1/2} \bullet b)^{1/2} \bullet \varepsilon) \\ &= a^{1/2} \bullet (a^{1/2} \bullet b) = b. \blacksquare\end{aligned}$$

The proof of the preceding lemma motivates the following definition.

Definition 2.11. In a pointed reflection quasigroup $(M, \bullet, \varepsilon)$, we define $a^{1/2} = \varepsilon \# a$, the unique solution of $x \bullet \varepsilon = a$. We also define $a^2 = a \bullet \varepsilon$ and $a^{-1} = \varepsilon \bullet a$. The squaring map x^2, square root map $x^{1/2}$, and inversion x^{-1} are defined by these formulas. Inductively $x^{2^{n+1}} = (x^{2^n})^2$. Note that the squaring and inversion maps are defined in any pointed Loos symmetric space.

Lemma 2.12. *Let $(M, \bullet, \varepsilon)$ be a pointed Loos symmetric space. For all $v \in T_\varepsilon M$, $\exp_\varepsilon(v) = (\exp_\varepsilon(1/2^n)v)^{2^n}$ for each nonnegative n.*

Proof. The proof is by induction on n. Certainly it is true for $n = 0$. Set $\alpha(t) = \exp_\varepsilon(tv)$, which by Theorem 2.9 is a one-parameter RQ. For $n = 1$ we have

$$\left(\exp_\varepsilon \frac{1}{2}v\right)^2 = \alpha\left(\frac{1}{2}\right) \bullet \varepsilon = \alpha\left(\frac{1}{2}\right) \bullet \alpha(0) = \alpha\left(\frac{1}{2} \bullet 0\right) = \alpha(1-0) = \exp_\varepsilon v.$$

If it is true for $n = k$, then

$$\left(\exp_\varepsilon \frac{1}{2^{k+1}}v\right)^{2^{k+1}} = \left(\left(\exp_\varepsilon \frac{1}{2^k}\left(\frac{1}{2}v\right)\right)^{2^k}\right)^2 = \left(\exp_\varepsilon \frac{1}{2}v\right)^2 = \exp_\varepsilon v. \blacksquare$$

We characterize the class of Loos reflection quasigroups important for our study.

Theorem 2.13. *Let $(M, \bullet)$ be a Loos symmetric space endowed with its canonical spray (Theorem 2.5), and let $\varepsilon \in M$. The following are equivalent:*

(1) *M is a geodesically complete Loos reflection quasigroup.*
(2) *The equation $x \bullet \varepsilon = b$ has a unique solution $x = b\#\varepsilon = b^{1/2}$, the square root map is smooth, and every element of M lies on some geodesic containing ε.*
(3) *Given distinct $a, b \in M$, there exists a unique (injective) one-parameter RQ sending 0 to a and 1 to b and the squaring map is a diffeomorphism.*
(4) *The exponential function $\exp_\varepsilon : T_\varepsilon M \to M$ is bijective and the squaring map is a diffeomorphism.*
(5) *The exponential function $\exp_\varepsilon : T_\varepsilon M \to M$ is a diffeomorphism.*
(6) *The exponential function $\exp_x : T_x M \to M$ is a diffeomorphism for all $x \in M$.*
(7) *The square root map exists, is smooth, and for every $a \in M$, $\lim_{n\to\infty} a^{1/2^n} = \varepsilon$.*

Proof. (1)⇒(2): Immediate, since (2) is a weakening of the conditions of (1).

(2)⇒(4): Let $x \in M$ and let α be a geodesic with image containing x and ε. Setting $\beta(t) = \alpha(t - t_0)$, where $\alpha(t_0) = \varepsilon$, and setting $v = \beta'(0)$, we have from Theorem 2.9 that $\beta(t) = \exp_\varepsilon(tv)$, and thus $x \in \exp_\varepsilon(T_\varepsilon M)$.

Suppose that $\exp_\varepsilon v = \exp_\varepsilon w$. By Lemma 2.12 $(\exp_\varepsilon(1/2^n)v)^{2^n} = (\exp_\varepsilon(1/2^n)w)^{2^n}$ for all positive n, and repeated application of the hypothesis of unique square roots yields $\exp_\varepsilon(1/2^n)v = \exp_\varepsilon(1/2^n)w$. Since $\exp_\varepsilon$ is locally injective, $(1/2^n)v = (1/2^n)w$ for some n, so $v = w$.

We note $(a^{1/2})^2 = (\varepsilon \# a) \bullet \varepsilon = a$ and $(a^2)^{1/2} \bullet \varepsilon = (\varepsilon \# (a \bullet \varepsilon)) \bullet \varepsilon = a \bullet \varepsilon$. Thus $x \bullet \varepsilon = a \bullet \varepsilon$ has solutions $x = (a^{1/2})^2$ and $x = (a^2)^{1/2}$. By uniqueness of solution, the two are equal. It follows that the squaring mapping and square root mapping are inverses. The squaring map is smooth by smoothness of the $\bullet$-operation and the square root map is smooth by hypothesis. Thus the squaring map is a diffeomorphism.

(4)⇒(5): There is some δ-ball B_δ around 0_ε such that $\exp_\varepsilon$ restricted to B_δ is a diffeomorphism onto an open subset of M by Remark 2.4. By Lemma 2.12, on any open ball B around 0_ε we can write

$$\exp_\varepsilon v = (\exp_\varepsilon|_{B_\delta}(1/2^n)v)^{2^n};$$

for n large enough the right-hand side is a diffeomorphism equal to $\exp_\varepsilon$.

(5)⇒(6): Suppose that $\exp_\varepsilon$ is a diffeomorphism onto M. For any y, $y = \exp_\varepsilon(v)$ for some unique $v \in T_xM$. Then $\exp(1/2)v$ is a midpoint for $\varepsilon = \exp_\varepsilon(0)$ and $y = \exp_\varepsilon(v)$, since $\beta_v(t) = \exp_\varepsilon(tv)$ is a $\bullet$-homomorphism. Since $S_{\exp_\varepsilon(1/2)v}$ is a diffeomorphic automorphism of $(M, \bullet)$ (by (S2) and (S3)) carrying ε to y, by Theorem 2.5(iii) it is an automorphism of (M, F). We conclude that the exponential function is also a smooth homeomorphism onto M at y.

(6)⇒(1): Let $x, y \in M$, $x \neq y$. By hypothesis $y = \exp_x(v)$ for some $v \in T_xM$. Since the map $\alpha_v(t) = \exp_x(tv)$ is a $\bullet$-homomorphism, it also preserves midpoints, and hence $\alpha_v(1/2) = \exp_x(1/2)v$ is a midpoint of $x = \alpha_v(0)$ and $y = \alpha_v(1) = \exp_x(v)$. If $m = \exp_x(w)$ is another midpoint of x and y, then by the same argument m is a midpoint of x and $\exp_x(2w)$. Hence $\exp_x(2w) = m \bullet x = y = \exp_x(v)$. By hypothesis, $2w = v$, and thus $w = (1/2)v$, $m = \exp_x(w) = \exp_x(1/2)v$. Thus the midpoint is unique; it follows that $(M, \bullet)$ is a reflection quasigroup. In the course of the argument, we have also shown that M is geodesically complete.

We note that the square root function is given by $\exp_\varepsilon(1/2)\log_\varepsilon x$, where $\log_\varepsilon$ is the smooth inverse of $\exp_\varepsilon$, and is thus smooth. It then follows from the formula derived at the end of the proof of Lemma 2.10 that $(x, y) \mapsto x \# y$ is a smooth map from $M \times M$ to M.

(3)⇒(4): Suppose $\exp_\varepsilon(v) = \exp_\varepsilon(w)$. Then $\alpha(t) = \exp_\varepsilon(tv)$ and $\beta(t) = \exp_\varepsilon(tw)$ are both geodesics taking the value ε at 0 and $\exp_\varepsilon(v)$ at 1. By hypothesis they are equal, so $v = \alpha'(0) = \beta'(0) = w$. Hence $\exp_\varepsilon$ is injective.

Let $x \in M$. Then there exists a geodesic γ such that $\gamma(0) = \varepsilon$ and $\gamma(1) = x$. By Theorem 2.9 for $v = \gamma'(0)$, $\gamma(t) = \exp_\varepsilon(tv)$ for all t. Hence $x = \exp_\varepsilon v$. We conclude that $\exp_\varepsilon$ is surjective.

(4),(6)⇒(3): Given distinct a, b, then $b = \exp_a(v)$ for some $v \in T_aM$. Hence $\alpha(t) = \exp_a(tv)$ is a geodesic such that $\alpha(0) = a$ and $\alpha(1) = b$. It is injective since $\exp_a$ is. Since all the other one-parameter RQs taking the value a at 0 are of the form $\exp_a(tw)$, $w \in T_aM$ (Theorem 2.9) and since $\exp_a(w) \neq b$ for $w \neq v$, α is uniquely determined by its values at 0 and 1.

(2),(4)⇒(7): That (2),(4) imply (7) is straightforward from Lemma 2.12.

(7)⇒(5): Similar to (4) implies (5). ∎

Example 2.14. Let $\mathbf{A}$ be a C^*-algebra with identity ε, let $\mathbf{A}^{-1}$ be the set of invertible elements, and let $\operatorname{Sym}\mathbf{A} := \{a \in \mathbf{A} : a = a^*\}$, the closed subspace of self-adjoint elements. Let $M := \{aa^* : a \in \mathbf{A}^{-1}\}$, the set of self-adjoint positive elements. Then M is a uniquely divisible twisted subgroup of the group $\mathbf{A}^{-1}$ of invertible elements, and an open subset of the Banach space $\operatorname{Sym}\mathbf{A}$. It follows readily that M equipped with the restricted core operation $a \bullet b = ab^{-1}a$ is a Loos reflection quasigroup, called the *positive definite core* of $\mathbf{A}$. It is standard that the exponential function from $\operatorname{Sym}\mathbf{A}$ to M, which can be identified with $\exp_1 : T_1M \to M$, is a diffeomorphism and hence by (5) of the preceding theorem all the conditions of the theorem are satisfied. The operator $a\#b$ arising from the quasigroup structure is the standard operator geometric mean of a and b. The unique one-parameter quasigroup determined by a, b yields the weighted means $a\#_tb := a^{1/2}(a^{-1/2}ba^{-1/2})^ta^{1/2}$.

Remark 2.15. The surjectivity of the exponential map, and bijectivity in the simply connected case, can often be deduced from an assumption of some form of nonpositive curvature on the manifold M. Results of this type can be found in [12], along with a review of earlier such results. It is shown in [9] that the Thompson metric on M, the set of self-adjoint positive elements in a C^*-algebra, has nonpositive curvature in the sense of Busemann, that is, for all $x, y, z \in M$, $d(x\#y, x\#z) \leq (1/2)d(y, z)$.

3. Bruck and K-loops, gyrogroups, and B-loops

In this section we recall results that are worked out in detail and often in more generality in [3, Chapter 6] and [15]. The interplay worked out in this and the next section between reflection quasigroups and Bruck loops is often reminiscent of material appearing in [11, Chapter 6], except that in the manifold setting we relax the assumption of finite dimensionality.

We have considered *quasigroups* in the previous section: sets equipped with a binary operation $\odot$ such that the equations $a \odot y = b$ and $x \odot a = b$ have unique solutions in y and x respectively. A *loop* is a quasigroup $(L, \odot)$ with a two-sided identity e. Unique left and right inverses exist and if they agree for an element a, we denote the *inverse* by a^{-1} (recall a' is a *left inverse* of a if $a'a = e$).

Since we do not postulate the associative law, we adopt the following convention to lessen the number of parentheses needed: for $a_n, a_{n-1}, \cdots, a_1$, inductively define

$$a_k \odot a_{k-1} \odot \cdots \odot a_1 = a_k \odot (a_{k-1} \odot \cdots \odot a_1).$$

This means one associates beginning on the right and working to the left. Note in particular that $a \odot b \odot c = a \odot (b \odot c)$. For the special case that $a = a_i$ for every i we write a^n for the n-fold product. In the case inverses exist, we define $a^{-n} = (a^{-1})^n$.

A loop L satisfies the *Bol* property if for all $a, b, c \in L$

$$a \odot b \odot a \odot c = (a \odot b \odot a) \odot c. \tag{3.3}$$

It is a standard fact (see e.g., [3, Theorem 6.4(3)]) that loops L satisfying the Bol property are *left power alternative*, that is, satisfy for all $a, b \in L$ and for all $m, n \in \mathbb{Z}$:

$$a^m \odot a^n \odot b = a^{m+n} \odot b. \tag{3.4}$$

In particular, each $a \in L$ has an inverse a^{-1} and the powers $\{a^m : m \in \mathbb{Z}\}$ form a cyclic subgroup.

The loop L satisfies the *automorphic inverse* property if every element has an inverse and for all $a, b \in L$

$$(a \odot b)^{-1} = a^{-1} \odot b^{-1} \tag{3.5}$$

A loop satisfying both the Bol property and the automorphic inverse property is called a *Bruck loop*.

In a loop the left translations L_a defined by $L_a(x) = a \odot x$ are bijections. One can thus define the *precession map*:

$$\ell(a, b) := L_{a \odot b}^{-1} L_a L_b. \tag{3.6}$$

One sees directly that

$$a \odot b \odot c = (a \odot b) \odot \ell(a, b)(c). \tag{3.7}$$

A loop satisfies the *left inverse* property if inverses exist and for all $a, b \in L$,

$$a^{-1} \odot a \odot b = b. \tag{3.8}$$

A *K-loop* was historically defined to be a loop satisfying (i) the left inverse property, (ii) the automorphic inverse property, (iii) every $l(a, b)$ is an automorphism, and (iv) $\ell(a, b) = \ell(a, b \odot a)$ for all a, b. It is a theorem of Kreuzer's ([6], see also [3, Theorem 6.7]) that a loop is a Bruck loop if and only if it is a K-loop, and hence the terminology is interchangeable.

A. Ungar [16] has introduced and studied gyrocommutative gyrogroups, which have been shown to be equivalent to Bruck loops [15]. The axiom system for a gyrocommutative gyrogroup G is reminiscent of that for a commutative group:

(G1) $a \oplus b \oplus c = (a \oplus b) \oplus \mathrm{gyr}[a,b]c$, where $\mathrm{gyr}[a,b] : G \times G \to G$ (gyroassociativity);
(G2) $0 \oplus a = a \oplus 0 = a$ (existence of identity);
(G3) $a \oplus (\ominus a) = \ominus a \oplus a = 0$ (existence of inverses);
(G4) $a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$ (gyrocommutativity);
(G5) $\mathrm{gyr}[0,a] = \mathrm{id}$;
(G6) $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a,b]$ (loop property).

It turns out in the presence of (G1), (G2), (G3), and (G5) that (G6) is equivalent to the Bol property. These five properties together imply $(G, \oplus, 0)$ is a loop, that $\mathrm{gyr}[a,b] = \ell(a,b)$, the predecession map, is a automorphism of $(G, \oplus)$, and that (G4) is equivalent to the *Bruck identity* $(x \oplus y) \oplus (x \oplus y) = x \oplus (y \oplus y) \oplus x$, which in turn is equivalent to the automorphic inverse property. Conversely, setting $\mathrm{gyr}[a,b] = \ell(a,b)$ in a Bruck loop, one sees that Axioms (G1)–(G6) are satisfied. These observations establish the equivalence between gyrocommutative gyrogroups and Bruck loops. This equivalence yields the following corollary.

Corollary 3.1. *A gyrocommutative gyrogroup satisfies the Bol property, the Bruck identity, and the automorphic inverse property. It is left power alternative and the maps* $\mathrm{gyr}[a,b]$ *are equal to the predecession map* $\ell(a,b)$ *and are automorphisms.*

Let $(G, \oplus)$ be a gyrocommutative gyrogroup and let $\mathrm{Aut}_0(G)$ be a subgroup of its automorphism group containing all automorphisms $\mathrm{gyr}[a,b]$ for $a, b \in G$. Then $G \times \mathrm{Aut}_0(G)$ is a group with respect to the operation

$$(a, A)(b, B) = (a \oplus A(b), \mathrm{gyr}[a, A(b)]AB). \tag{3.9}$$

Using this group structure, one can readily derive several further useful properties of the maps $\mathrm{gyr}[a,b]$.

Lemma 3.2. *Let* $(G, \oplus)$ *be a gyrocommutative gyrogroup, and let* $a, b, c \in G$.

(1) $(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b,a]c)$.
(2) $\mathrm{gyr}[b,a] = \mathrm{gyr}^{-1}[a,b]$, *the inverse of* $\mathrm{gyr}[a,b]$.
(3) $\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a,b]$.
(4) $\mathrm{gyr}[a, b \oplus a] = \mathrm{gyr}[a,b]$.

We call a Bruck loop L a *B-loop* if every element has a unique square root, i.e., for every $y \in L$, there exists a unique $x \in L$ such that $x^2 = y$. The terminology dates back to the work of Glauberman [1], [2], although he only considered the case of finite loops.

There is a close connection between B-loops and pointed reflection quasigroups.

Proposition 3.3. *If $(L, \odot, e)$ is a B-loop with identity e, then $(L, \bullet, e)$ is a pointed reflection quasigroup, where $x \bullet y := x^2 \odot y^{-1}$. Conversely if $(M, \bullet, \varepsilon)$ is a pointed reflection quasigroup, then $(M, \odot, \varepsilon)$ is a B-loop with identity ε, where $x \odot y := x^{1/2} \bullet y^{-1}$. In the given settings the two constructions are inverse.*

Example 3.4. Let P be a uniquely 2-divisible twisted subgroup of a group G with identity e. We have already seen in Example 2.8 that $(P, \bullet, e)$ is a pointed reflection quasigroup with respect to the core operation $x \bullet y = xy^{-1}x$. In addition, $(P, \odot, e)$ is a B-loop with respect to the operation $x \odot y = x^{1/2}yx^{1/2}$ and furthermore the constructions of the previous proposition convert the two operations into each other. It is further the case that integral and hence dyadic powers in P agree in the group and loop multiplications.

The fact that $(P, \odot, e)$ is a B-loop is well-known; see, for example, Theorem 6.14 of [3]. This reference also gives the equivalence of powers.

Remark 3.5. All pointed reflection quasigroups and all B-loops arise, up to isomorphism, via the construction of the previous example. This is the content of Theorem 6.15 of [3] for the case of B-loops, where a B-loop is embedded as a twisted subgroup of the permutation group generated by the left translations. For the case of pointed reflection quasigroups, see, for example [8], where the quadratic embedding into a twisted subgroup of the the displacement group is constructed.

Corollary 3.6. *Let $(L, \odot, e)$ be a B-loop. Then the mean operation of the associated reflection quasigroup $(L, \bullet, e)$ is given by $a\#b = a \odot (a^{-1} \odot b)^{1/2}$. In particular $e\#b = b^{1/2}$, so square roots agree in the loop and quasigroup.*

Proof. By the previous remark we may assume, up to isomorphism, that we are working in a uniquely 2-divisible twisted subgroup of a group L with $a \odot b = a^{1/2}ba^{1/2}$ and $a \bullet b = ab^{-1}a$. By Example 2.8

$$a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2} = a \odot (a^{-1} \odot b)^{1/2}. \blacksquare$$

Remark 3.7. There are obvious categories *BLoop* consisting of B-loops and homomorphisms and *RQgp* consisting of pointed reflection quasigroups and distinguished point preserving homomorphisms. These categories are straightforwardly isomorphic via the construction of Proposition 3.3 at the object level and the set-theoretic identity at the morphism level, since via the interrelationship of the operations $\odot$ and $\bullet$, one sees that a map is a morphism in one category if and only if it is in the other. One obtains alternatively an equivalence of categories between the category of gyrocommutative gyrogroups and homomorphisms and *RQgp*.

4. Smooth gyrocommutative gyrogroups

In the previous section we have observed that $(X, \oplus, \mathbf{0})$ is a gyrocommutative gyrogroup if and only if it is a Bruck loop (resp. K-loop). We now assume that X is a smooth (equal C^∞) manifold over a Banach space E and that the operation $\oplus$ is smooth as a map from $X \times X$ to X. Our goal is to exhibit properties of X that are analogues to vector space properties, and this is made more transparent when we use the additive notation and axiom system of gyrogroups. In addition Ungar [16] has developed a theory of gyrovector spaces, and we wish to compare and contrast his theory with ours. The whole theory of course easily translates over to Bruck loops. The theory that we develop in this section is a variant of the theory of smooth left odules of Sabinen [13].

We begin with some remarks about the interplay of uniquely 2-divisible gyrocommutative gyrogroups $(X, \oplus, \mathbf{0})$ and pointed reflection quasigroups. In Proposition 3.3 we have seen that $(X, \oplus, \mathbf{0})$ is equivalent to a pointed reflection quasigroup $(X, \bullet, \mathbf{0})$ via the following formulas :

$$\begin{aligned} a \bullet b &= a \ominus (\ominus a \oplus b) = a \oplus (a \ominus b) = 2a \ominus b, \\ a \oplus b &= S_{a^{1/2}}(S_{\mathbf{0}} b) = (a \# \mathbf{0}) \bullet (\mathbf{0} \bullet b), \end{aligned} \tag{4.10}$$

where S_x represents the point reflection at x.

Remark 4.1. From Corollary 3.6 the mean operation is given by

$$a \# b = a^{1/2} \bullet (a^{1/2} \bullet b)^{1/2}$$

for any a and b in a reflection quasigroup $(X, \bullet)$. Using the above equivalence of operations, we obtain

$$a \# b = a \oplus (1/2) \cdot (\ominus a \oplus b)$$

in a uniquely 2-divisible gyrocommutative gyrogroup $(X, \oplus)$.

Lemma 4.2. *Let $(X, \oplus, \mathbf{0})$ be a uniquely 2-divisible gyrocommutative gyrogroup, and let $a \bullet b = 2a \ominus b$ be the corresponding reflection quasigroup operation (Equation 4.10). If we choose some point $a \in X$ besides $\mathbf{0}$ as distinguished point for $(X, \bullet)$, then the gyroaddition corresponding to $(X, \bullet, a)$ is given in terms of the original gyroaddition by*

$$x \oplus_a y = a \oplus ((\ominus a \oplus x) \oplus (\ominus a \oplus y)).$$

Proof. By Equation 4.10, $x \oplus_a y = (x \# a) \bullet (a \bullet y)$. We compute:

$$\begin{aligned} a \oplus ((\ominus a \oplus x) \oplus (\ominus a \oplus y)) &= a \oplus ((\ominus a \oplus x) \# \mathbf{0}) \bullet (\mathbf{0} \bullet (\ominus a \oplus y)) \\ &= a \oplus ((1/2) \cdot (\ominus a \oplus x) \bullet (a \ominus y)) \end{aligned}$$

$$= (a \oplus (1/2) \cdot (\ominus a \oplus x)) \bullet (a \oplus (a \ominus y)$$
$$= (a \# x) \bullet (2a \ominus y)$$
$$= (x \# a) \bullet (a \bullet y) = x \oplus_a y,$$

where first equality is an application of $u \oplus v = (u\#\mathbf{0}) \bullet (\mathbf{0} \bullet v)$, the second the facts that $u\#\mathbf{0} = (1/2) \cdot u$, $0 \bullet u = u^{-1}$, and the automorphic inverse property, the third the fact that $L_a = S_{(1/2)\cdot a} S_{\mathbf{0}}$, a composition of maps left-distributive over $\bullet$, the fourth Remark 4.1 and the power associative property, the fifth equation (4.10), and the sixth the beginning observation of the proof. ∎

In the remainder of this section we assume $(X, \oplus, \mathbf{0})$ *is a smooth gyrocommutative gyrogroup with unique square roots, where the gyroaddition* $\oplus : X \times X \to X$ *is smooth.* (In loop-theoretic terms, we are working with a smooth B-loop.) *We further make the blanket assumptions that the square root map* $x \mapsto (1/2)x$ *is smooth and that* $\lim_{n\to\infty} 1/2^n x = \mathbf{0}$ *for all* $x \in X$.

Remark 4.3. The blanket assumptions correspond to condition (7) of Theorem 2.13 converted to the gyrogroup setting. We could alternatively assume gyrogroup versions of any of the other conditions as our standing assumption. For example, condition (2) translates to the condition that the square root map is smooth and every element lies on a continuous one-parameter group, i.e., a continuous homomorphism from $(\mathbb{R}, +)$ into the gyrogroup.

In order to show that the binary operation $\bullet$ is smooth, we first apply the Implicit Mapping Theorem (Theorem 5.9 of Chapter I, [7]) with the function

$$f(a, b) = a \oplus b : X \times X \to X.$$

Then L_a, the left translation by a, is a diffeomorphism, whose inverse map is the left translation by $\ominus a$. It follows that $dL_a(b) = D_2 f(a, b)$ is an isomorphism for all b. Thus, the function

$$g(x) = \ominus x : U_0 \to V$$

satisfying $f(x, g(x)) = \mathbf{0}$ is also smooth on a sufficiently small open neighborhood U_0 of a. So the binary operation $\bullet$ obtained from the following compositions is smooth :

$$(a, b) \mapsto (a, \ominus b) \mapsto (a, a \ominus b) \mapsto a \oplus (a \ominus b).$$

From Axiom (M4) of a reflection quasigroup, we know that each S_x has only one fixed point. Thus, $(X, \bullet)$ becomes a Loos symmetric space. (Note that $(x, y) \mapsto x\#y$ is smooth by formula (2.2) in the proof of Lemma 2.10 and our standing assumption that the square root function is smooth.) Therefore, we can apply Theorems 2.5 and 2.9 to the smooth uniquely 2-divisible gyrocommutative gyrogroup $(X, \oplus)$ equipped with unique square roots. In particular X has a distinguished spray and a corresponding exponential function $\exp : TX \to X$.

In the following we abbreviate the exponential functions $\exp_{\mathbf{0}}$ at the identity $\mathbf{0} \in X$ by exp, and the corresponding log function by log. We note by Theorem 2.13 that these are diffeomorphisms between $T_{\mathbf{0}}X$ and X.

Lemma 4.4. *The continuous homomorphisms from* $(\mathbb{R}, +)$ *to* $(X, \oplus)$ *are the maps* $\alpha_x(t) = \exp(tv)$*, where* $x = \exp v$.

Proof. We know (Theorem 2.9) that the continuous $\bullet$-homomorphisms from $(\mathbb{R}, \bullet, 0)$ to $(X, \bullet, \mathbf{0})$ are precisely those of the form $\beta_v(t) = \exp(tv)$ for $v \in T_{\mathbf{0}}X$. By Remark 3.7, these must be precisely the continuous homomorphisms. The result now follows from Theorem 2.13(5). ∎

In the next proposition we define a smooth scalar multiplication on X.

Proposition 4.5. *Let* $(X, \oplus)$ *be the smooth gyrocommutative gyrogroup satisfying the standing hypotheses of this section. Then* X *admits a scalar multiplication defined by*

$$t \cdot x := \exp(t \log(x))$$

for any $t \in \mathbb{R}$ *and* $x \in X$*. The scalar multiplication* $(t, x) \mapsto t \cdot x$ *from* $\mathbb{R} \times X$ *to* X *is smooth and satisfies:*

(1) $1 \cdot x = x$, $0 \cdot x = \mathbf{0}$;
(2) $s \cdot (t \cdot x) = (st) \cdot x$;
(3) $s \cdot x \oplus t \cdot x = (s + t) \cdot x$;
(4) $\operatorname{gyr}[a, b](t \cdot x) = t \cdot \operatorname{gyr}[a, b]x$.

Proof. By Theorems 2.9 and 2.13 there is a geodesic, or $\bullet$-homomorphism,

$$\alpha_x : \mathbb{R} \to X, \ \alpha_x(t) = \exp(tv)$$

arising from the canonical spray, where $v = \log(x) \in T_{\mathbf{0}}(X)$ for any $x \in X$. We define the scalar multiplication by

$$t \cdot x := \alpha_x(t) = \exp(t \log(x)).$$

The scalar multiplication is the composition of the following smooth maps

$$(t, x) \mapsto (t, \log(x)) \mapsto t \log(x) \mapsto \exp(t \log(x)),$$

hence smooth.

(1) $1 \cdot x = \exp(\log(x)) = x$, and $0 \cdot x = \exp(0) = \mathbf{0}$.
(2) For $s, t \in \mathbb{R}$, $s \cdot (t \cdot x) = \exp(s \log(t \cdot x)) = \exp(st \log(x)) = (st) \cdot x$.

(3) Since $\alpha_x(t) = \exp(tv)$ is a $\bullet$-homomorphism preserving identities, it is also an $\oplus$-homomorphism by Remark 3.7. Thus

$$s \cdot x \oplus t \cdot x = \alpha_x(s) \oplus \alpha_x(t) = \alpha_x(s+t) = (s+t) \cdot x.$$

(4) We consider for $a, b \in X$ a map

$$\beta : \mathbb{R} \to X,\ \beta(t) = \mathrm{gyr}[a,b](\alpha_x(t)) = \mathrm{gyr}[a,b](t \cdot x).$$

Since $\mathrm{gyr}[a,b]$ is a $\oplus$-homomorphism, the map β is a $\bullet$-homomorphism with $\beta(0) = \mathbf{0}$. Thus by Theorem 2.9 we obtain $\beta(t) = \exp(tw)$ for some $w \in T_{\mathbf{0}}(X)$. Therefore,

$$\mathrm{gyr}[a,b](t \cdot x) = \exp(tw) = t \cdot \exp(w) = t \cdot \beta(1) = t \cdot \mathrm{gyr}[a,b]x. \ \blacksquare$$

We use the preceding to define the notion of a Banach gyrovector space.

Definition 4.6. A *smooth gyrovector space* consists of a smooth gyrocommutative gyrogroup on a Banach manifold equipped with a smooth scalar multiplication satisfying

(1) $1 \cdot x = x$, $0 \cdot x = \mathbf{0}$;
(2) $s \cdot (t \cdot x) = (st) \cdot x$;
(3) $s \cdot x \oplus t \cdot x = (s+t) \cdot x$;
(4) $\mathrm{gyr}[a,b](t \cdot x) = t \cdot \mathrm{gyr}[a,b]x$.

Proposition 4.7. *In a smooth gyrovector space G the map*

$$(a,b,c) \mapsto \mathrm{gyr}[a,b]c : G^3 \to G,\ \mathrm{gyr}[a,b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)),$$

is smooth, and the following additional properties hold:

(i) *For all integers m, $mx = m \cdot x$. In particular, $(-1) \cdot x = \ominus x$.*
(ii) *For all $s, t \in \mathbb{R}$, $s \cdot a \oplus t \cdot a \oplus b = (s+t) \cdot a \oplus b = (s \cdot a \oplus t \cdot a) \oplus b$.*
(iii) *For all $s, t \in \mathbb{R}$ and $a \in G$, $\mathrm{gyr}[s \cdot a, t \cdot a] =$id.*

Proof. The alternative characterization of $\mathrm{gyr}[a,b]c$ follows from the fact that $\mathrm{gyr}[a,b]$ is the precession map. The asserted smoothness then follows from the smoothness of the gyroaddition and the inversion map.

(i) The proof follows by induction for $m > 0$ since $1 \cdot x = x = 1x$ and

$$(m+1)x = x \oplus mx = 1 \cdot x \oplus m \cdot x = (1+m) \cdot x$$

by (3) if $mx = m \cdot x$. For $m = -1$, $(-1) \cdot x \oplus x = (-1) \cdot x \oplus 1 \cdot x = 0 \cdot x = \mathbf{0}$, so $(-1) \cdot x = \ominus x$ by uniqueness of inverses. For $m > 1$,

$$(-m)x = m(\ominus x) = m \cdot (\ominus x) = m \cdot ((-1) \cdot x) = (-m) \cdot x.$$

(ii) Using (i), we see that (ii) holds for integers since a gyrocommutative gyrogroup is left power alternative by Corollary 3.1. Then for integers i, j and $n > 0$, we have

$$\left(\frac{i}{2^n}\right)\cdot a\oplus\left(\frac{j}{2^n}\right)\cdot a\oplus b = i\cdot\left(\frac{1}{2^n}\cdot a\right)\oplus j\cdot\left(\frac{1}{2^n}\cdot a\right)\oplus b$$
$$= (i+j)\cdot\left(\frac{1}{2^n}\cdot a\right)\oplus b$$
$$= \left(\frac{i+j}{2^n}\right)\cdot a\oplus b.$$

Thus the first equality in (ii) holds for all dyadic rationals and then by continuity for all s, t. A similar argument establishes the second equality.

(iii) Item (iii) follows directly from item (ii) and the gyroassociativity law. ∎

Remark 4.8. The scalar multiplication of a smooth gyrovector space G is the one induced in the manner of this section from the canonical spray on the corresponding reflection quasigroup.

Proof. We note that $x\oplus x = 2\cdot x = b$ has the unique solution $x = (1/2)\cdot b$, so that G has unique square roots. It follows from the definition of a smooth gyrovector space that the map $x\mapsto(1/2)\cdot x$ is smooth. Furthermore, $\lim_{n\to\infty}(1/2^n)\cdot a = 0\cdot a = \mathbf{0}$, so the standing hypothesis of this section is satisfied, and a scalar multiplication is induced, as in Proposition 4.5, which satisfies the axioms of a gyrovector space. In the given and induced scalar multiplications, it must be the case, as we have just seen, that $(1/2)\cdot x$ is the unique square root of x, so scalar multiplication by $1/2$ is the same map for both scalar multiplications. It follows from Proposition 4.7(i) that the scalar multiplications by any integer m agree. It follows from the scalar multiplication laws that the scalar multiplications agree for all dyadic rationals and hence by continuity for all real numbers. ∎

Definition 4.9. For a smooth gyrovector space, its *canonical spray* and *exponential map* are by definition those of the corresponding reflection quasigroup.

5. Parallel transport

We have already encountered the notion of parallel transport along geodesics arising from a given spray in Theorem 2.5(vi). We develop the parallel transport along geodesics in a smooth gyrovector space X.

Proposition 5.1. *Let X be a smooth gyrovector space. Let*

$$\alpha:\mathbb{R}\to X,\ \alpha_(t) = \exp(tv),$$

where $a = \alpha(1)$. Then parallel transport from $T_{\mathbf{0}}X$ along α to T_aX is given by

$$P_0^1(\alpha) = dL_a(\mathbf{0}).$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} T_{\mathbf{0}}(X) & \xrightarrow{dL_a(\mathbf{0})=P_0^1(\alpha)} & T_a(X) \\ \Big\downarrow \exp & & \Big\downarrow \exp_a \\ X & \xrightarrow{L_a} & X \end{array}$$

Proof. Via Theorem 2.5(vi) we know that the following diagram commutes for a geodesic α such that $\alpha(t) = x$ and $\alpha(s+t) = y$:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{d\tau_{\alpha,s}(\alpha(t))=P_t^{t+s}(\alpha)} & T_y(X) \\ \Big\downarrow \exp_x & & \Big\downarrow \exp_y \\ X & \xrightarrow{\tau_{\alpha,s}=S_{\alpha(s/2)}S_{\alpha(0)}} & X \end{array}$$

For the given geodesic $\alpha_a(t) = \exp(tv)$, we have $\alpha(0) = \mathbf{0}$ and $\alpha(1) = a$. By equation (4.10)

$$\tau_{\alpha,1} = S_{\alpha(1/2)}S_{\alpha(0)} = S_{\alpha(1/2)}S_{\mathbf{0}} = L_a.$$

Thus, we obtain

$$dL_a(\mathbf{0}) = d\tau_{\alpha,1}(\alpha_a(0)) = P_0^1(\alpha)$$

and the desired commutative diagram from the one just given. ■

A *geodesic loop*, as introduced by Kikkawa [4, 5], is one satisfying equation (5.11) in the following corollary, which corollary follows immediately from the preceding proposition.

Corollary 5.2. *In a smooth gyrovector space X, the addition is given by*

$$x \oplus y = \exp_x \circ P_0^1(\alpha) \circ \log y, \tag{5.11}$$

where $\alpha(t) = \exp(tv)$ is a geodesic with value $\mathbf{0}$ at 0 and x at 1 and $P_0^1 : T_{\mathbf{0}}X \to T_xX$ is parallel transport along α. Thus X is a geodesic loop.

We now introduce rooted and free gyrovectors along the lines of A. Ungar [16, Chapter 5].

Definition 5.3. Let a and b be elements, or points, in a smooth gyrovector space. A *rooted gyrovector* (alternatively, *bound gyrovector*) ab is an ordered pair of points $a, b \in X$. The points a and b of the rooted gyrovector ab are called, respectively, the *tail* and the *head* of the rooted gyrovector.

The value in X of the rooted gyrovector ab is $\ominus a \oplus b$. Accordingly, we write

$$v = \ominus a \oplus b.$$

Furthermore, any point $a \in X$ is identified with the rooted gyrovector $\mathbf{0}a$ with head a, rooted at the origin $\mathbf{0}$. Such gyrovectors are sometimes called *positional gyrovectors.*

Given $v \in T_aX$, the rooted gyrovector $a \exp_a(v)$ is called the *gyrorepresentation* of v.

Two rooted gyrovectors ab and $a'b'$ are said to be equivalent,

$$ab \sim a'b',$$

if they have the same value in X, that is, if

$$\ominus a \oplus b = \ominus a' \oplus b'.$$

Then the relation $\sim$ is given in terms of an equality so that, being reflexive, symmetric, and transitive, it is an equivalence relation. The resulting equivalence classes are called *free gyrovectors.*

We have another version of Proposition 5.1 in terms of rooted gyrovectors.

Corollary 5.4. *Let X be a smooth gyrovector space. Then the parallel transport of a vector v in $T_{\mathbf{0}}X$ along a geodesic from $\mathbf{0}$ to a has gyrorepresentation ab, where*

$$b = a \oplus \exp(v).$$

We next define a notion of parallel transport for rooted gyrovectors that is equivalent to that at the tangent vector level when tangent vectors and rooted gyrovectors are identified via internal representation.

Definition 5.5. A rooted gyrovector $a_1b_1 = a_1 \exp_{a_1}(v_1)$ is said to be a *parallel transport* of a rooted gyrovector $a_0b_0 = a_0 \exp_{a_0}(v_0)$ if the tangent vector $v_1 \in T_{a_1}X$ is the parallel transport of the tangent vector $v_0 \in T_{a_0}X$ along the geodesic

$$\alpha(t) = \exp_{a_0}(tw)$$

with $a_1 = \alpha(1)$ for some $w \in T_{a_0}X$.

Theorem 5.6. *Let X be a smooth gyrovector space. A rooted gyrovector a_1b_1 is a parallel transport of a rooted gyrovector a_0b_0 if and only if their values satisfy*

$$\ominus a_1 \oplus b_1 = \mathrm{gyr}[a_1, \ominus a_0](\ominus a_0 \oplus b_0). \tag{5.12}$$

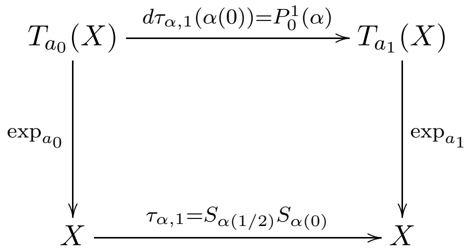
Proof. Assume that a_1b_1 is a parallel transport of a_0b_0. By definition of parallel transport of rooted gyrovectors, we have that

$$b_0 = \exp_{a_0}(v_0),\ b_1 = \exp_{a_1}(v_1),$$

where

$$v_1 = P_0^1(\alpha)(v_0) \tag{5.13}$$

for the geodesic α with $\alpha(0) = a_0$ and $\alpha(1) = a_1$. From Theorem 2.5(vi) we have the commutivity of the following diagram:



Taking $\exp_{a_1}$ on both sides in the equation (5.13) and using the above diagram, we have

$$\begin{aligned}\exp_{a_1}(v_1) &= \exp_{a_1}(d\tau_{\alpha,1}(\alpha(0))v_0)\\ &= \tau_{\alpha,1}(\exp_{a_0}(v_0))\\ &= S_{\alpha(1/2)}S_{\alpha(0)}(\exp_{a_0}(v_0)).\end{aligned}$$

Equivalently,

$$\begin{aligned}b_1 &= S_{\alpha(1/2)}S_{\alpha(0)}\ (b_0)\\ &= \alpha(1/2) \bullet (\alpha(0) \bullet b_0)\\ &= (a_0\#a_1) \bullet (a_0 \bullet b_0).\end{aligned}$$

We fix the point a_0 as the distinguished point of $(X, \bullet)$ and consider the corresponding gyrogroup $(X, \oplus_{a_0}, a_0)$. By Corollary 5.4 applied to this gyrogroup, we have that the parallel transport of v_0 along α has gyrorepresentation a_1b_1, where $b_1 = a_1 \oplus_{a_0} \exp_{a_0}(v_0) = a_1 \oplus_{a_0} b_0$. We then have

$$\begin{aligned}b_1 &= a_1 \oplus_{a_0} b_0\\ &= a_0 \oplus ((\ominus a_0 \oplus a_1) \oplus (\ominus a_0 \oplus b_0))\\ &= (a_0 \oplus (\ominus a_0 \oplus a_1)) \oplus \mathrm{gyr}[a_0, \ominus a_0 \oplus a_1](\ominus a_0 \oplus b_0)\\ &= a_1 \oplus \mathrm{gyr}[a_1, \ominus a_0](\ominus a_0 \oplus b_0),\end{aligned}$$

where the second equality follows from Lemma 4.2, the third is the left inverse property and the fourth follows from

$$\begin{aligned}\operatorname{gyr}[a_0, \ominus a_0 \oplus a_1] &= \operatorname{gyr}[a_0 \oplus (\ominus a_0 \oplus a_1), \ominus a_0 \oplus a_1] \\ &= \operatorname{gyr}[a_1, \ominus a_0 \oplus a_1] = \operatorname{gyr}[a_1, \ominus a_0].\end{aligned}$$

Therefore, by the left inverse property, we have

$$\begin{aligned}\ominus a_1 \oplus b_1 &= \ominus a_1 \oplus (a_1 \oplus \operatorname{gyr}[a_1, \ominus a_0](\ominus a_0 \oplus b_0)) \\ &= \operatorname{gyr}[a_1, \ominus a_0](\ominus a_0 \oplus b_0). \ \blacksquare\end{aligned}$$

Remark 5.7. In the special case with $a_0 = \mathbf{0}$, the equation (5.12) becomes

$$\ominus a_1 \oplus b_1 = \operatorname{gyr}[a_1, \mathbf{0}]b_0 = b_0.$$

Equivalently, we have $b_1 = a_1 \oplus b_0$, the result of Corollary 5.4.

Remark 5.8. Ungar [16] defines $a_1 b_1$ to be a parallel transport of $a_0 b_0$ if equation (5.12) is satisfied. However, we have derived this fact simply from the assumption that we are working in a smooth gyrovector space. This provides some justification for Ungar's definition on the one hand, and on the other demonstrates how the smoothness assumptions lead naturally to appropriate generalizations of classical vector analysis to the context of gyrovector spaces.

6. Parallelogram vector addition

The parallelogram vector addition law in Euclidean geometry is an alternative statement of the triangle vector addition law. The parallelogram vector addition law for a parallelogram *abdc* in the Euclidean vector space gives us two equivalent conditions

$$\begin{aligned}d &= c + (b - a), \\ d &= b + (c - a).\end{aligned}$$

Furthermore, the two diagonal segments in a parallelogram *abdc* in a Euclidean vector space have a common midpoint. In particular,

$$\frac{a+d}{2} = \frac{b+c}{2}.$$

To generalize these ideas, and a number of other ideas, to gyrovector spaces, we introduce another addition into a gyrogroup, an operation that Ungar [16] has called coaddition. Both addition and coaddition in a gyrovector space collapse to the usual vector addition in the associative case, but several standard properties of vector addition tend to split between these two operations in gyrovector spaces. Thus both turn out to be useful. As an important "test case", we observe the interplay of the two operations for the study of parallelograms and midpoints in gyrovector spaces.

Definition 6.1. In a gyrogroup $(X, \oplus, \mathbf{0})$ we define the coaddition $\boxplus$ by

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b.$$

We set $a \boxminus b := a \boxplus (\ominus b) = a \oplus \mathrm{gyr}[a, b](\ominus b) = a \ominus \mathrm{gyr}[a, b]b$.

Remark 6.2. We have remarked that a gyrogroup X is a loop. Indeed, the equation $a \oplus x = b$ in unknown x has a unique solution $x = \ominus a \oplus b$ in X. Furthermore, the equation $x \oplus a = b$ in unknown x has a unique solution $x = b \boxminus a$ in X.

Proof. The first assertion follows from the left inverse property. For the second,

$$\begin{aligned}
(b \boxminus a) \oplus a &= (b \ominus \mathrm{gyr}[b, a]a) \oplus a \\
&= (b \oplus l(a, b)(\ominus a)) \oplus a \\
&= (b \oplus (\ominus(b \oplus a) \oplus b)) \oplus a \\
&= b \oplus (\ominus(b \oplus a) \oplus (b \oplus a)) \\
&= b,
\end{aligned}$$

where the next-to-last equality is an application of the Bol property. The uniqueness follows from the fact that X is a loop. ∎

We consider an alternative method of computing the solution of $x \oplus a = b$ in a gyrocommutative gyrogroup with unique square roots. We recall that $x = a \# b$ is the unique solution of $x \bullet a = b$, so

$$b = x \bullet a = 2x \oplus (\ominus a),$$

where the second equality is the first equation in (4.10). From Remark 6.2 we conclude that

$$2 \cdot (a \# b) = 2x = b \boxminus (\ominus a) = b \boxplus a.$$

Remark 6.3. From the preceding and Remark 4.1, we have the following mixed formula for the midpoint of a and b:

$$a \# b = (1/2) \cdot (b \boxplus a) = a \oplus \frac{1}{2} \cdot (\ominus a \oplus b).$$

Since $a \# b = b \# a$, we conclude that $a \boxplus b = b \boxplus a$ and have the alternative version of the preceding equations:

$$\begin{aligned}
a \# b &= (1/2) \cdot (a \boxplus b) = b \oplus \frac{1}{2} \cdot (\ominus b \oplus a); \\
2(a \# b) &= a \boxplus b.
\end{aligned}$$

In the smooth gyrovector space X where there is no parallel postulate, we still have an analogous definition.

Definition 6.4. Let a, b, and c be distinct noncollinear points in a smooth gyrovector space X. Then the points a, b, c, and d are the vertices of parallelogram $abdc$, ordered either clockwise or counterclockwise, if

$$d = (b \boxplus c) \ominus a.$$

We call $abdc$ the vertices of the parallelogram.

We have equivalent conditions for the vertices of a parallelogram.

Proposition 6.5. *Let a, b, c, and d be distinct points in a smooth gyrovector space X. Then the following are equivalent :*

(1) *$abdc$ are the vertices of a parallelogram.*
(2) *The midpoint of the two points a and d coincides with the midpoint of the two points b and c.*
(3) *The following conditions hold:*

$$\begin{aligned} a &= (b \boxplus c) \ominus d, \\ b &= (a \boxplus d) \ominus c, \\ c &= (a \boxplus d) \ominus b, \\ d &= (b \boxplus c) \ominus a. \end{aligned}$$

(4) *The rooted gyrovectors satisfy the following conditions:*

$$\begin{aligned} \ominus b \oplus d &= \mathrm{gyr}[b, \ominus c]\,\mathrm{gyr}[c, \ominus a](\ominus a \oplus c), \\ \ominus c \oplus d &= \mathrm{gyr}[c, \ominus b]\,\mathrm{gyr}[b, \ominus a](\ominus a \oplus b). \end{aligned}$$

Proof. By Remark 6.2 (letting $x = b \boxplus c$) the condition for the vertices $abdc$ to be a parallelogram

$$d = (b \boxplus c) \ominus a$$

is equivalent to

$$a \boxplus d = d \boxplus a = b \boxplus c.$$

Multiplying $1/2$ on both sides we have equivalently

$$\frac{1}{2} \cdot (a \boxplus d) = \frac{1}{2} \cdot (b \boxplus c).$$

This gives us the equivalence among (1), (2), and (3). We now prove the equivalence between (1) and (4).

(1) $\Rightarrow$ (4): Assume that $abdc$ are the vertices of a parallelogram. Then we have

$$d = (b \boxplus c) \ominus a.$$

We note by Corollary 3.1, several applications of property [G6] of a gyrocommutative gyrogroup and Lemma 3.2(4), and Lemma 3.2(3) that

$$\begin{aligned}\mathrm{gyr}[\mathrm{gyr}[b,\ominus c]c,b] &= \mathrm{gyr}[l(b,\ominus c)c,b] = \mathrm{gyr}[(\ominus b\oplus c)\oplus b,b]\\ &= \mathrm{gyr}[\ominus b\oplus c,b] = \mathrm{gyr}[\ominus b\oplus c,c] = \mathrm{gyr}[\ominus b,c]\\ &= \mathrm{gyr}[b,\ominus c].\end{aligned}$$

By Lemma 3.2(1) and the gyrocommutativity, we obtain

$$\begin{aligned}d &= (b\boxplus c)\ominus a\\ &= (b\oplus \mathrm{gyr}[b,\ominus c]c)\ominus a\\ &= b\oplus(\mathrm{gyr}[b,\ominus c]c\ominus \mathrm{gyr}[\mathrm{gyr}[b,\ominus c]c,b]a)\\ &= b\oplus(\mathrm{gyr}[b,\ominus c]c\ominus \mathrm{gyr}[b,\ominus c]a)\\ &= b\oplus \mathrm{gyr}[b,\ominus c](c\ominus a)\\ &= b\oplus \mathrm{gyr}[b,\ominus c]\,\mathrm{gyr}[c,\ominus a](\ominus a\oplus c).\end{aligned}$$

Thus, we have equivalently

$$\ominus b\oplus d = \mathrm{gyr}[b,\ominus c]\,\mathrm{gyr}[c,\ominus a](\ominus a\oplus c).$$

Using the commutativity of coaddition and following the above steps, we can obtain the other condition.

(4)⇒(1): Reversing the preceding steps yields (4)⇒(1). ■

Remark 6.6. We have seen that a uniquely 2-divisible gyrocommutative gyrogroup X gives rise to a symmetric mean by setting

$$S_x y = x\bullet y = x\oplus(x\ominus y) = 2x\ominus y.$$

By Proposition 6.5(2) we have that distinct points $abdc$ are the vertices of a parallelogram if and only if

$$a\#d = b\#c.$$

It follows from Remark 2.7(iv) that parallelograms are preserved under all symmetries S_a, and hence under all translations $L_a = S_{a^{1/2}}S_{\mathbf{0}}$.

In Euclidean space the parallelogram vector addition law is satisfied; for a parallelogram $abdc$,

$$(-a+b)+(-a+c) = -a+d.$$

We have the analogous parallelogram addition law of rooted gyrovectors.

Corollary 6.7. *Let $abdc$ be the vertices of a parallelogram in a smooth gyrovector space. Then*

$$(\ominus a\oplus b)\boxplus(\ominus a\oplus c) = (\ominus a\oplus d).$$

Proof. By Remark 6.6 we have that $\mathbf{0}(\ominus a\oplus b)(\ominus a\oplus d)(\ominus a\oplus c)$ are the vertices of a parallelogram, and hence by Proposition 6.5(2)

$$(1/2)\cdot(\ominus a\oplus d) = (\ominus a\oplus b)\#(\ominus a\oplus c).$$

Multiplying both sides by 2 yields, in light of Remark 6.3

$$\ominus a \oplus d = (\ominus a \oplus b) \boxplus (\ominus a \oplus c). \ \blacksquare$$

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