DEMONSTRATIO MATHEMATICA
Vol. XLIV No 4 2011

10.1515/dema-2013-0342

Ilinka Dimitrova, Jörg Koppitz

COREGULAR SEMIGROUPS OF FULL TRANSFORMATIONS

Abstract. This paper is mainly dedicated to the description of coregular subsemigroups of the symmetric semigroup T_n of transformations on an n-element set. Namely, we characterize all coregular transformation semigroups S with $|S| \leq 3$. In the subsemigroup E_n of all extensive transformations, the coregular elements coincide with the idempotent ones. We characterize all bands within E_n. Within the subsemigroup OE_n of all order-preserving extensive transformations, we also determine the maximal bands (with respect to the inclusion).

1. Introduction and preliminaries

Regular semigroups play an important role in the semigroup theory and they have been studied from various aspects. We want to investigate a particular class of regular semigroups.

An element α of a semigroup S is called coregular if there is a $\beta \in S$ such that

$$\alpha = \alpha\beta\alpha = \beta\alpha\beta.$$

A semigroup is called coregular if each element of it is coregular ([2]).

Coregular semigroups have been also studied in [4] and [8]. A coregular semigroup can be characterized as a semigroup S with $a = a^3$ for all $a \in S$ or as a union of disjoint groups with elements of order ≤ 2. Since in a group, $a = a^3$ if and only if a has an order ≤ 2, coregular elements in a semigroup generalize elements of a group of order 2. For example, a square matrix with n rows that satisfies $A^3 - A = 0 = (A-1)A(A+1)$, is a coregular element in the semigroup $M_n(\mathbb{R})$ of all square matrices with n rows. The semigroup $M_2(\mathbb{R})$ of all real 2×2 matrices is not regular (also not coregular), but it contains coregular elements. The set

2000 *Mathematics Subject Classification*: 20M20.
Key words and phrases: symmetric semigroup, coregular semigroups, order-preserving transformations, extensive transformations, maximal bands.

$$\left\{ \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \mid 0 \leq \alpha < 2\pi \right\}$$

is a coregular semigroup within $M_2(\mathbb{R})$, which defines the axial symmetries of the figures of a given plane (see also [1]). Let us also note that the pseudoinverse matrix concept introduced by E. H. Moore ([7]) in 1920, is very useful for solving some optimization problems. Every symmetric matrix is equal to its pseudoinverse if and only if it is coregular (see [1]).

We begin by recalling some notation and definitions that will be useful in the paper. For standard terms and concepts in semigroup theory we refer to [6]. Let us start by defining the semigroups that will be objects of study in this paper. For $n \in \mathbb{N}$, let X_n be a finite chain with n elements, say $X_n = \{1 < 2 < \ldots < n\}$. As usual, we denote by T_n the symmetric semigroup of all full transformations on X_n. Every transformation $\alpha \in T_n$ may be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

where $A_1, \ldots, A_r$ are the $\ker\alpha$-classes (the blocks of α) and $a_i\alpha^{-1} = A_i$ for $1 \leq i \leq r \leq n$. Then every idempotent transformation is characterized by the property that $a_i \in A_i$ for $1 \leq i \leq r \leq n$.

We say that a transformation α in T_n is order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and α is extensive if $x \leq x\alpha$ for all $x \in X_n$. Denote by E_n the subsemigroup of all extensive transformations on X_n and by OE_n the semigroup of all order-preserving extensive transformations on X_n. These monoids were studied for example in [5] and [9]. The coregular semigroups within the symmetric semigroup T_3 are characterized in [3].

In Section 2, we characterize all coregular semigroups with ≤ 3 elements within T_n. The coregular elements coincide with the idempotent elements in the semigroup E_n. In this case, the study of the coregular semigroups within E_n means the investigation of the bands (idempotent semigroups) within E_n. For a subsemigroup S of T_n, we denote by $E(S)$ the set of idempotents in S. In an inverse semigroup, the idempotents form a semigroup, but in general, this is not true. The maximal bands within T_n are of particular interest. Their description is still an open problem connected with questions in graph theory. We will give a complete answer for the subsemigroup OE_n of T_n in Section 3. Moreover, we characterize all bands within E_n as well as within OE_n.

For completion, we recall the definition of Green's equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{J}$ on a monoid M: for all $u, v \in M$,

$$\begin{aligned} u\mathcal{R}v &\iff uM = vM, \\ u\mathcal{L}v &\iff Mu = Mv, \\ u\mathcal{H}v &\iff u\mathcal{R}v \text{ and } u\mathcal{L}v, \\ u\mathcal{J}v &\iff MuM = MvM. \end{aligned}$$

For a transformation $\alpha \in T_n$, we denote by $\ker\alpha$ and im α the kernel and the image of α, respectively. The kernel of a transformation is an equivalence on X_n. For an equivalence ρ on X_n, we denote by $a\rho$ the equivalence class (block) containing $a \in X_n$ and we put $X_n/\rho := \{a\rho \mid a \in X_n\}$ as well as $A\rho := \bigcup\{a\rho \mid a \in A\}$ for a set $A \subseteq X_n$. The number rank $\alpha = |X_n/\ker\alpha| = |\text{im } \alpha|$ is called rank of α. The symmetric semigroup T_n is regular, but the submonoids E_n and OE_n are not regular. The next result is well known (see [6, Exercise 16, page 63]): for all $\alpha, \beta \in T_n$,

$$\begin{aligned} \alpha\mathcal{L}\beta &\iff \text{im } \alpha = \text{im } \beta, \\ \alpha\mathcal{R}\beta &\iff \ker\alpha = \ker\beta, \\ \alpha\mathcal{H}\beta &\iff \text{im } \alpha = \text{im } \beta \text{ and } \ker\alpha = \ker\beta, \\ \alpha\mathcal{J}\beta &\iff \text{rank } \alpha = \text{rank } \beta. \end{aligned}$$

For a set $A \subseteq X_n$ and an equivalence ρ on X_n such that $|A| = |X/\rho|$, let $H_{\rho,A}$ be the following $\mathcal{H}$-class:

$$H_{\rho,A} := \{\alpha \in T_n \mid \ker\alpha = \rho, \quad \text{im } \alpha = A\}.$$

Remember that im $\alpha\beta \subseteq$ im β and $\ker\alpha \subseteq \ker\alpha\beta$. Moreover, if $\alpha \in E(T_n)$ then

- im $\beta \subseteq$ im α implies $\beta\alpha = \beta$,
- $\ker\alpha \subseteq \ker\beta$ implies $\alpha\beta = \beta$.

Lemma 1. *Let $\alpha, \beta \in E(T_n)$, $A :=$ im α, $B :=$ im $\alpha\beta$, and $\rho := \ker\beta$. Then $A \subseteq B\rho$ and $B \subseteq A\rho$.*

Proof. We have $A\beta = (X\alpha)\beta = B$. This implies $(A\beta)\rho = B\rho$, where $A \subseteq (A\beta)\rho$ since β is an idempotent. This shows $A \subseteq B\rho$. On the other hand, we have $B = A\beta = (A\rho)\beta \subseteq A\rho$ since β is an idempotent. ∎

We denote by $\Delta_A := \{(a, a) \mid a \in A\}$ for a set $A \subseteq X_n$.

Lemma 2. *Let $\alpha_1, \alpha_2, \alpha_3 \in E(T_n)$, $A_i :=$ im α_i and $\rho_i := \ker\alpha_i$ for $i = 1, 2, 3$. Then $\alpha_1\alpha_2 = \alpha_3$ if and only if $A_3 \subseteq A_2$, $A_3 \subseteq A_1\rho_2$, $A_1 \subseteq A_3\rho_2$, $\rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1 = \rho_3$, and $\Delta_{A_3} \subseteq \rho_1 \circ \Delta_{A_1} \circ \rho_2$.*

Proof. Suppose that $\alpha_1\alpha_2 = \alpha_3$. Then, clearly, $A_3 \subseteq A_2$, $A_1 \subseteq A_3\rho_2$ and $A_3 \subseteq A_1\rho_2$ by Lemma 1.

Let $(x, y) \in \rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1$. Then there are x', $y' \in X_n$ with $(x', y') \in (A_1^2 \cap \rho_2)$ and $(x', x), (y, y') \in \rho_1$. This implies $x\alpha_1\alpha_2 = x'\alpha_1\alpha_2$, $y\alpha_1\alpha_2 = y'\alpha_1\alpha_2$, and $x'\alpha_2 = y'\alpha_2$. Since $x', y' \in A_1$, we have $x' = x'\alpha_1$ and $y' = y'\alpha_1$ and thus $x'\alpha_1\alpha_2 = y'\alpha_1\alpha_2$. So, we have $x\alpha_1\alpha_2 = y\alpha_1\alpha_2$, i.e. $x\alpha_3 = y\alpha_3$ and thus $(x, y) \in \rho_3$. Let $(x, y) \in \rho_3$, i.e. $x\alpha_3 = y\alpha_3$. Thus $x\alpha_1\alpha_2 = y\alpha_1\alpha_2$. This implies the existence of x', $y' \in A_1$, namely $x' = x'\alpha_1 = x\alpha_1$ and $y' = y'\alpha_1 = y\alpha_1$. Hence $x'\alpha_2 = y'\alpha_2$ and $(x', y') \in (A_1^2 \cap \rho_2)$, where $(x', x), (y, y') \in \rho_1$. This shows that $(x, y) \in \rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1$. Consequently, $\rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1 = \rho_3$.

Let $x \in A_3$. Then $x\alpha_1\alpha_2 = x$ since $\alpha_1\alpha_2 = \alpha_3$ is an idempotent, and for $y = x\alpha_1 \in A_1$, $x\alpha_1 = y = y\alpha_1$, i.e. $(x, y) \in \rho_1$. Further, $y\alpha_2 = x\alpha_1\alpha_2 = x = x\alpha_2$ (since $x \in A_3 \subseteq A_2$), i.e. $(y, z) \in \rho_2$. In particular, we have $x = x\alpha_1\alpha_2 = z$. So, we have $(x, x) \in \rho_1 \circ \Delta_{A_3} \circ \rho_2$. This shows $\Delta_{A_3} \subseteq \rho_1 \circ \Delta_{A_1} \circ \rho_2$.

For the converse, we show that $\ker \alpha_1\alpha_2 = \rho_3$, $\mathrm{im}\ \alpha_1\alpha_2 = A_3$, and $\alpha_1\alpha_2$ is an idempotent. If this is the case, then $\alpha_1\alpha_2$ is the idempotent in H_{ρ_3, A_3}, i.e. $\alpha_1\alpha_2 = \alpha_3$.

We have $(x, y) \in \ker \alpha_1\alpha_2$ if and only if $(x\alpha_1, y\alpha_1) \in \rho_2$ if and only if there are x', $y' \in A_1$ with (x, x'), $(y, y') \in \rho_1$, $x' = x'\alpha_1$, $y' = y'\alpha_1$, and $(x', y') \in \rho_2$ if and only if $(x, y) \in \rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1 = \rho_3$.

Let $x \in A_3 \subseteq A_1\rho_2$. Then there is $y \in A_1$ with $x \in y\rho_2$, i.e. $y = y\alpha_1$ and $x\alpha_2 = y\alpha_2$. Since $x \in A_3 \subseteq A_2$, we have $x\alpha_2 = x$. This yields $x = x\alpha_2 = y\alpha_2 \in \mathrm{im}\ \alpha_1\alpha_2$. Let $x \in \mathrm{im}\ \alpha_1\alpha_2$. Then $x \in A_2$ and there is $z \in A_1$ with $z\alpha_2 = x$. Because of $A_1 \subseteq A_3\rho_2$, there is $w \in A_3$ with $(z, w) \in \rho_2$, i.e. $z\alpha_2 = w\alpha_2$. Since $w \in A_3 \subseteq A_2$, we have $w = w\alpha_2$. Hence $x = z\alpha_2 = w\alpha_2 = w \in A_3$. Consequently, $\mathrm{im}\ \alpha_1\alpha_2 \subseteq A_3$ and altogether $\mathrm{im}\ \alpha_1\alpha_2 = A_3$.

Let $x \in \mathrm{im}\ \alpha_1\alpha_2 = A_3$. Then, since $\Delta_{A_3} \subseteq \rho_1 \circ \Delta_{A_1} \circ \rho_2$, there is $y \in A_1$ with $(x, y) \in \rho_1$ and $(y, x) \in \rho_2$. Then $x\alpha_1\alpha_2 = y\alpha_1\alpha_2 = y\alpha_2 = x\alpha_2$. Since $x \in A_3 \subseteq A_2$, we have $x\alpha_2 = x$, i.e. $x\alpha_1\alpha_2 = x$. This shows that $\alpha_1\alpha_2$ is an idempotent. ∎

2. Coregular semigroups within T_n

First we characterize the coregular elements within T_n. Let $\alpha \in T_n$ and $\bar{x}$ be the $\ker \alpha$-class containing x. Then all elements of $\bar{x}$ have the same image under α, denoted by $x\alpha$.

Definition 3. Let $E_2(T_n)$ be the set of all $\alpha \in T_n$ such that for all $x, y \in X_n$,

$$x\alpha \in \bar{y} \Rightarrow y\alpha \in \bar{x}.$$

Proposition 4. *$E_2(T_n)$ is the set of all coregular elements in T_n.*

Proof. Let $\alpha \in E_2(T_n)$ and $x \in X_n$. Then there is a $\ker\alpha$-class $\bar{y}$ with $x\alpha \in \bar{y}$. Then $y\alpha \in \bar{x}$, and so $x\alpha\alpha\alpha = y\alpha\alpha = x\alpha$, i.e. $\alpha^3 = \alpha$.

Conversely, let $\alpha \in T_n$ with $\alpha^3 = \alpha$ and let $x, y \in X_n$ with $x\alpha \in \bar{y}$. Assume that $y\alpha \notin \bar{x}$. Then $x\alpha\alpha\alpha = (y\alpha)\alpha$, where $y\alpha \notin \bar{x}$, i.e. $x\alpha\alpha\alpha \neq x\alpha$. So, $x\alpha = x\alpha^3 \neq x\alpha$, a contradiction. ■

The set $E_2(T_n)$ is not closed under multiplication. The question arises which subsets of $E_2(T_n)$ are closed. Clearly, all trivial semigroups are coregular. We will give a complete answer for subsets with ≤ 3 elements. A characterization for sets with ≥ 4 elements is still an open problem.

Remark 5. For all $\alpha \in E_2(T_n) \setminus E(T_n)$, $\alpha^2 \in E(T_n)$ and both α and α^2 belong to the same $\mathcal{H}$-class. Moreover, $\beta\alpha^2 = \alpha^2\beta = \beta$ for all β in this $\mathcal{H}$-class. Hence a two-element set $S \subseteq E_2(T_n)$ is a semigroup if and only if S is a band or there is an $\mathcal{H}$-class H of T_n such that $S \subseteq H$ and exactly one element of S is an idempotent.

Proposition 6. *Let $S = \{\alpha_1, \alpha_2\} \subseteq E(T_n)$ with $\alpha_i \in H_{\rho_i, A_i}$ $(i \in \{1, 2\})$. Then S is a semigroup if and only if at least one of the following statements is valid:*

(i) $A_1 \subseteq A_2$ *and* $\rho_2 \subseteq \rho_1$.
(ii) $A_2 \subseteq A_1$ *and* $\rho_1 \subseteq \rho_2$.
(iii) $A_1 = A_2$.
(iv) $\rho_2 = \rho_1$.

Proof. Suppose that S is a semigroup. Then $\alpha_1\alpha_2, \alpha_2\alpha_1 \in \{\alpha_1, \alpha_2\}$. First, $\alpha_1\alpha_2 \in \{\alpha_1, \alpha_2\}$ implies $A_1 \subseteq A_2$ or $\rho_1 \subseteq \rho_2$. Moreover, $\alpha_2\alpha_1 \in \{\alpha_1, \alpha_2\}$ gives $\rho_2 \subseteq \rho_1$ or $A_2 \subseteq A_1$. This shows that one of the statements (i)–(iv) is valid.

Suppose that one of the statements (i)–(iv) is valid.

(i) $A_1 \subseteq A_2$, $\rho_2 \subseteq \rho_1$, and $\alpha_1, \alpha_2 \in E(T_n)$ implies $\alpha_1\alpha_2 = \alpha_2\alpha_1 = \alpha_1$, i.e. S is a semilattice.

(ii) $A_2 \subseteq A_1$, $\rho_1 \subseteq \rho_2$, and $\alpha_1, \alpha_2 \in E(T_n)$ implies $\alpha_1\alpha_2 = \alpha_2\alpha_1 = \alpha_2$, i.e. S is a semilattice.

(iii) $A_1 = A_2$ and $\alpha_1, \alpha_2 \in E(T_n)$ implies $\alpha_1\alpha_2 = \alpha_1$ and $\alpha_2\alpha_1 = \alpha_2$, i.e. S is a left-zero-semigroup.

(iv) $\rho_2 = \rho_1$ and $\alpha_1, \alpha_2 \in E(T_n)$ implies $\alpha_1\alpha_2 = \alpha_2$ and $\alpha_2\alpha_1 = \alpha_1$, i.e. S is a right-zero-semigroup. ■

Remark 7. A two-element band within T_n is a semilattice or a left- (right-) zero-semigroup.

Proposition 8. *Let $S \subseteq E(T_n)$ be a three-element set. Then S is a semigroup if and only if there are sets $A_1, A_2, A_3 \subseteq X_n$, equivalences ρ_1, ρ_2, ρ_3*

on X_n, *and pairwise distinct elements* $\alpha_1, \alpha_2, \alpha_3 \in S$ *such that* $\alpha_i \in H_{\rho_i, A_i}$ $(1 \leq i \leq 3)$ *and at least one of the following holds:*

(i) $A_1 \subseteq A_2 \subseteq A_3$ *and* $\rho_3 \subseteq \rho_2 \subseteq \rho_1$.
(ii) $A_1 = A_2 \subseteq A_3$ *and* $\rho_3 \subseteq \rho_2, \rho_1$.
(iii) $A_1 \subseteq A_2 = A_3$ *and* $\rho_3, \rho_2 \subseteq \rho_1$.
(iv) $A_1 \subseteq A_2, A_3$ *and* $\rho_3 = \rho_2 \subseteq \rho_1$.
(v) $A_1, A_2 \subseteq A_3$ *and* $\rho_3 \subseteq \rho_2 = \rho_1$.
(vi) $A_1 = A_2 = A_3$.
(vii) $\rho_3 = \rho_2 = \rho_1$.
(viii) $A_2 \subseteq A_3\rho_1$, $A_3 \subseteq A_2\rho_1$, $A_3 \subseteq A_1$, $\rho_2 \circ (A_2^2 \cap \rho_1) \circ \rho_2 = \rho_3$, $\Delta_{A_3} \subseteq \rho_2 \circ \Delta_{A_2} \circ \rho_1$, *and at least one of the following conditions holds:*
(viii$_1$) $A_3 \subseteq A_2$, $A_3 \subseteq A_1\rho_2$, $A_1 \subseteq A_3\rho_2$, $\rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1 = \rho_3$, *and* $\Delta_{A_3} \subseteq \rho_1 \circ \Delta_{A_1} \circ \rho_2$.
(viii$_2$) $A_1 \subseteq A_2$.
(viii$_3$) $\rho_1 \subseteq \rho_2$.

Proof. Let S be a semigroup. Then any two-element subset of S is a semigroup or at least one two-element subset does not be a semigroup. Suppose that any two-element subset is a semigroup. Then, by Proposition 6, for all $i, j \in \{1, 2, 3\}$ with $i \neq j$, $A_i \subseteq A_j$ and $\rho_j \subseteq \rho_i$, or $A_i = A_j$, or $\rho_i = \rho_j$. Then, there are pairwise distinct $i_1, i_2, i_3 \in \{1, 2, 3\}$ such that:

(i) $A_{i_1} \subseteq A_{i_2} \subseteq A_{i_3}$ and $\rho_{i_3} \subseteq \rho_{i_2} \subseteq \rho_{i_1}$ or
(ii) $A_{i_1} = A_{i_2} \subseteq A_{i_3}$ and $\rho_{i_3} \subseteq \rho_{i_1}, \rho_{i_2}$ or
(iii) $A_{i_1} \subseteq A_{i_2} = A_{i_3}$ and $\rho_{i_3}, \rho_{i_2} \subseteq \rho_{i_1}$ or
(iv) $A_{i_1} \subseteq A_{i_2}, A_{i_3}$ and $\rho_{i_3} = \rho_{i_2} \subseteq \rho_{i_1}$ or
(v) $A_{i_1}, A_{i_2} \subseteq A_{i_3}$ and $\rho_{i_3} \subseteq \rho_{i_2} = \rho_{i_1}$ or
(vi) $A_{i_1} = A_{i_2} = A_{i_3}$ or
(vii) $\rho_{i_3} = \rho_{i_2} = \rho_{i_1}$.

Suppose that there is a two-element subset of S that does not be a semigroup. Without loss of generality, we can assume that $\alpha_2\alpha_1 = \alpha_3$. Then $A_2 \subseteq A_3\rho_1$, $A_3 \subseteq A_2\rho_1$, $A_3 \subseteq A_1$, $\rho_2 \circ (A_2^2 \cap \rho_1) \circ \rho_2 = \rho_3$, and $\Delta_{A_3} \subseteq \rho_2 \circ \Delta_{A_2} \circ \rho_1$ by Lemma 2. Suppose that $\alpha_1\alpha_2 = \alpha_3$. Then $A_3 \subseteq A_2$, $A_3 \subseteq A_1\rho_2$, $A_1 \subseteq A_3\rho_2$, $\rho_1 \circ (A_1^2 \cap \rho_2) \circ \rho_1 = \rho_3$, and $\Delta_{A_3} \subseteq \rho_1 \circ \Delta_{A_1} \circ \rho_2$ by Lemma 2. Suppose that $\alpha_1\alpha_2 = \alpha_1$. This gives $A_1 \subseteq A_2$. If we suppose that $\alpha_1\alpha_2 = \alpha_2$ then we obtain $\rho_1 \subseteq \rho_2$. Thus (viii) holds.

Conversely, suppose that at least one of the properties (i)–(viii) holds. Without loss of generality, we can assume that $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$. We check that in each case ((i)–(viii)), S is a semigroup. In the first seven cases, we use the fact that $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ implies $\alpha\beta = \alpha$ and that $\ker \alpha \subseteq \ker \beta$ implies $\alpha\beta = \beta$ for all $\alpha, \beta \in E(T_n)$.

(i) S is a semilattice isomorphic to the semilattice $A_1 \subseteq A_2 \subseteq A_3$ under set intersection.

(ii) $\{\alpha_1, \alpha_2\}$ is a left-zero semigroup and α_3 serves as the identity.

(iii) $\{\alpha_2, \alpha_3\}$ is a left-zero semigroup and α_1 serves as the zero.

(iv) $\{\alpha_2, \alpha_3\}$ is a right-zero semigroup and α_1 serves as the zero.

(v) $\{\alpha_1, \alpha_2\}$ is a right-zero semigroup and α_3 serves as the identity.

(vi) S is a left-zero semigroup.

(vii) S is a semigroup.

(viii) We have $\alpha_2\alpha_1 = \alpha_3$ by Lemma 2. Then $\alpha_2\alpha_3 = \alpha_2\alpha_2\alpha_1 = \alpha_2\alpha_1 = \alpha_3$ and $\alpha_3\alpha_1 = \alpha_2\alpha_1\alpha_1 = \alpha_2\alpha_1 = \alpha_3$.

(viii$_1$) By Lemma 2, we obtain $\alpha_1\alpha_2 = \alpha_3$. This gives $\alpha_1\alpha_3 = \alpha_1\alpha_1\alpha_2 = \alpha_1\alpha_2 = \alpha_3$ and $\alpha_3\alpha_2 = \alpha_1\alpha_2\alpha_2 = \alpha_1\alpha_2 = \alpha_3$. Thus S is a zero-semigroup.

(viii$_2$) From $A_1 \subseteq A_2$ it follows $\alpha_1\alpha_2 = \alpha_1$. Then $\alpha_1\alpha_3 = \alpha_1\alpha_2\alpha_1 = \alpha_1\alpha_1 = \alpha_1$ and $\alpha_3\alpha_2 = \alpha_2\alpha_1\alpha_2 = \alpha_2\alpha_1 = \alpha_3$. Hence, S is a semigroup.

(viii$_3$) From $\rho_1 \subseteq \rho_2$, it follows $\alpha_1\alpha_2 = \alpha_2$. Then $\alpha_1\alpha_3 = \alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1 = \alpha_3$ and $\alpha_3\alpha_2 = \alpha_2\alpha_1\alpha_2 = \alpha_2\alpha_2 = \alpha_2$, and so S is a semigroup. ∎

Proposition 9. *Let $S \subseteq E_2(T_n)$ be a three-element set. Then S is a semigroup if and only if S is a band or there are sets $A, B \subseteq X_n$ and equivalences ρ, π on X_n such that $S \cap H_{\pi,A} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 \in E(T_n)$ and $\alpha_2 \notin E(T_n)$, $S \cap H_{\rho,B} = \{\alpha_3\}$, where α_3 is an idempotent different from α_1, and at least one of the following holds:*

(i) *$A \subseteq B$ and $\rho \subseteq \pi$.*

(ii) *$B \subseteq A$, $\pi \subseteq \rho$, α_2 is the identity mapping on B, and for all ρ-classes $\bar{x}$, if $y \in \bar{x}$ then $y\alpha_2 \in \bar{x}$.*

Proof. Suppose that there are sets $A, B \subseteq X_n$ and equivalences ρ, π on X_n such that $S \cap H_{\pi,A}$ contains the idempotent α_1 as well as the non-idempotent α_2 and $S \cap H_{\rho,B}$ contains the idempotent α_3. First, we have $\alpha_2\alpha_1 = \alpha_2 = \alpha_1\alpha_2$ since $\alpha_1, \alpha_2 \in H_{\pi,A}$ and $\alpha_1 \in E(T_n)$. Clearly, $\{\alpha_1, \alpha_2\}$ is a two-element group.

Suppose that (i) is satisfied. Then $\alpha_1\alpha_3 = \alpha_1$ and $\alpha_2\alpha_3 = \alpha_2$ since $A \subseteq B$ and α_3 is an idempotent. Moreover, $\alpha_3\alpha_1 = \alpha_1$ and $\alpha_3\alpha_2 = \alpha_2$ since $\rho \subseteq \pi$ and α_3 is an idempotent. This shows that S is a semigroup with α_3 as the identity.

Suppose that (ii) is satisfied. Since $B \subseteq A$ and $\pi \subseteq \rho$, we have $\alpha_1\alpha_3 = \alpha_3\alpha_1 = \alpha_3$. Because of $y\alpha_2 \in \bar{x}$ for all ρ-classes $\bar{x}$ and all $y \in \bar{x}$, we have $\alpha_2\alpha_3 = \alpha_3$. Moreover, from $y\alpha_2 = y$ for all $y \in B$ it follows $\alpha_3\alpha_2 = \alpha_3$. Hence S is a semigroup with α_3 as the zero.

Conversely, let S be a semigroup that is not contained in $E(T_n)$. Then there is a set $A \subseteq X_n$ and an equivalence π on X_n such that $(S \setminus E(T_n)) \cap$

$H_{\pi,A} \neq \emptyset$. Without loss of generality, let $\alpha_2 \in (S \setminus E(T_n)) \cap H_{\pi,A}$ and $\alpha_1 = \alpha_2^2 \in H_{\pi,A} \cap E(T_n) \cap S$. Let $\alpha_3 \in S \setminus \{\alpha_1, \alpha_2\}$.

If $\alpha_3 \in H_{\pi,A}$ then $\alpha_3^2 = \alpha_1$, and so it is easy to verify that S is a 3-element group with $\alpha_2 \in S$ of order 2, which is a contradiction.

Hence $\alpha_3 \notin H_{\pi,A}$ and there is a set $B \subseteq X_n$ and an equivalence ρ on X_n such that $\alpha_3 \in H_{\rho,B} \neq H_{\pi,A}$. In particular, $\alpha_3 \in E(T_n)$ since otherwise $\alpha_3^2 \in E(T_n) \cap H_{\rho,B}$ would be a fourth element in S.

If $\alpha_1\alpha_3 = \alpha_2$ then $\alpha_2 = \alpha_1\alpha_3 = \alpha_1\alpha_3\alpha_3 = \alpha_2\alpha_3 = \alpha_2\alpha_1\alpha_3 = \alpha_2\alpha_2 = \alpha_1$, which is a contradiction. Hence $\alpha_1\alpha_3 \in \{\alpha_1, \alpha_3\}$. By a similar argument, we obtain $\alpha_3\alpha_1 \in \{\alpha_1, \alpha_3\}$.

Assume that $\alpha_1\alpha_3 = \alpha_3$. This implies that $\pi \subseteq \rho$. Suppose to the contrary that $\alpha_3\alpha_1 = \alpha_1$. Then $\rho \subseteq \pi$ and thus $\pi = \rho$. This implies $\alpha_3\alpha_2 = \alpha_2$ since α_3 is an idempotent.

If $\alpha_2\alpha_3 = \alpha_1$ then $\alpha_3 = \alpha_1\alpha_3 = \alpha_2^2\alpha_3 = \alpha_2\alpha_2\alpha_3 = \alpha_2\alpha_1 = \alpha_2$.

If $\alpha_2\alpha_3 = \alpha_2$ then $\alpha_3 = \alpha_1\alpha_3 = \alpha_2\alpha_2\alpha_3 = \alpha_2\alpha_2 = \alpha_1$.

If $\alpha_2\alpha_3 = \alpha_3$ then $\alpha_1 = \alpha_2\alpha_2 = \alpha_2\alpha_3\alpha_2 = \alpha_3\alpha_2 = \alpha_2$.

So, we have a contradiction in each case. Hence $\alpha_3\alpha_1 = \alpha_3$. This implies $B \subseteq A$. Then $\pi \subset \rho$ since otherwise $\pi = \rho$ and $B \subseteq A$ implies $A = B$ because $|A| = |X_n/\ker\pi| = |X_n/\ker\rho| = |B|$, which contradicts $H_{\rho,B} \neq H_{\pi,A}$. From $\pi \subset \rho$, it follows rank $\alpha_2\alpha_3 \leq$ rank $\alpha_3 <$ rank α_2. But $\alpha_2\alpha_3 \in H_{\rho,B} \cup H_{\pi,A}$, which implies $\alpha_2\alpha_3 \in H_{\rho,B}$, i.e. $\alpha_2\alpha_3 = \alpha_3$. Let $\bar{x}$ be a ρ-class. Then $\bar{x}\alpha_2\alpha_3 = \bar{x}\alpha_3$. This is only possible if $y\alpha_2 \in \bar{x}$ for all $y \in \bar{x}$. From $\pi \subset \rho \subseteq \ker\alpha_3\alpha_2$ and $\alpha_3\alpha_2 \in S \subseteq H_{\rho,B} \cup H_{\pi,A}$ it follows $\rho = \ker\alpha_3\alpha_2$ and thus $\alpha_3\alpha_2 \in H_{\rho,B} \cap S$, i.e. $\alpha_3\alpha_2 = \alpha_3$. In particular, $\alpha_3\alpha_2 = \alpha_3$ implies that α_2 is the identity mapping on $B \subseteq X_n$. So, we have (ii).

Suppose that $\alpha_1\alpha_3 = \alpha_1$. This implies that $A \subseteq B$. If $\alpha_3\alpha_2 = \alpha_3$ then $\alpha_2 = \alpha_1\alpha_2 = \alpha_1\alpha_3\alpha_2 = \alpha_1\alpha_3 = \alpha_1$, which is a contradiction. Thus $\alpha_3\alpha_2 \in \{\alpha_1, \alpha_2\}$, which implies $\rho \subseteq \pi$. Hence (i) holds. ∎

3. The monoids E_n and OE_n

In this section, we investigate the bands within the semigroup E_n. The first simple observation is that the bands within E_n coincide with the coregular subsemigroup of E_n.

Lemma 10. *The set of all coregular elements in E_n is $E(E_n)$, i.e. each coregular subsemigroup of E_n lies in $E(E_n)$.*

Proof. Let $\alpha \in E_n$ with $\alpha^3 = \alpha$. Then for every $x \in X_n$, $x\alpha = x\alpha^3 \geq x\alpha^2 \geq x\alpha$, i.e. $x\alpha = x\alpha^2$. This shows that $\alpha^2 = \alpha$, i.e. $\alpha \in E(E_n)$. Conversely, each idempotent transformation α satisfies $\alpha^3 = \alpha$. ∎

Next, we determine the bands within E_n. For a set $\emptyset \neq Y \subseteq X_n$, we denote by Y^* the greatest element in Y.

Remark 11. Let ρ be an equivalence on X_n with the set $\Pi := \{a_1, \dots, a_k\}$ of ρ-classes. Then

$$\varepsilon_\Pi := \begin{pmatrix} a_1 \cdots a_k \\ a_1^* \cdots a_k^* \end{pmatrix}$$

is the only transformation in $E(E_n)$ with the kernel ρ. Thus $X_n/\ker\varepsilon_\Pi = \Pi$ and $\varepsilon_{X_n/\ker\alpha} = \alpha$ for all $\alpha \in E(E_n)$. In particular, $E(E_n)$ is the set of all ε_Π, where Π is a partition of X_n. Moreover, $A\varepsilon_\Pi = A^*$ for all $A \in \Pi$.

Let Π be a partition of X_n and $Y \subseteq X_n$. Then we put

$$Y_\Pi := \bigcup\{A \in \Pi \mid A^* \in Y\}.$$

Definition 12. Let $\mathbf{\Omega}$ be a set of partitions of X_n such that for all $\Pi, \Phi \in \mathbf{\Omega}$:

1. If $B \in \Phi$ with $B_\Pi \neq \emptyset$ then $B_\Pi^* = B^*$.
2. $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} \in \mathbf{\Omega}$.

Denote by $\mathbf{\Omega_d}$ the set of all such sets $\mathbf{\Omega}$.

Lemma 13. *Let Φ and Π be partitions of X_n. Then*

(i) $x\varepsilon_\Pi\varepsilon_\Phi = B^*$ *for all* $B \in \Phi$ *with* $B_\Pi \neq \emptyset$ *and all* $x \in B_\Pi$.
(ii) $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\}$ *is the set of* $\ker\varepsilon_\Pi\varepsilon_\Phi$*-classes.*

Proof. Let $x \in B_\Pi$ for some $B \in \Phi$ with $B_\Pi \neq \emptyset$. Then there is an $A \in \Pi$ with $A^* \in B$ such that $x \in A$. We have $x\varepsilon_\Pi\varepsilon_\Phi = A^*\varepsilon_\Phi = B\varepsilon_\Phi = B^*$. This shows (i). In particular, (i) shows that $B_\Pi\varepsilon_\Pi\varepsilon_\Phi = B^*$ for all $B \in \Phi$ with $B_\Pi \neq \emptyset$. Since $B_1^* \neq B_2^*$ for different elements B_1 and B_2 of Φ, the $\ker\varepsilon_\Pi\varepsilon_\Phi$-classes are given by B_Π, $B \in \Phi$ with $B_\Pi \neq \emptyset$, i.e. (ii) is valid. ∎

Proposition 14. *A set $S \subseteq E(E_n)$ is a subsemigroup of E_n if and only if there is $\mathbf{\Omega} \in \mathbf{\Omega_d}$ with*

$$S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}.$$

Proof. Let $S \subseteq E(E_n)$ be a subsemigroup of E_n. Then we put

$$\mathbf{\Omega} := \{X_n/\ker\gamma \mid \gamma \in S\}$$

and show that $\mathbf{\Omega} \in \mathbf{\Omega_d}$. For this let $\alpha, \beta \in S$. Moreover, let Π and Φ be the set of all $\ker\alpha$-classes and all $\ker\beta$-classes, respectively. Let $B \in \Phi$ with $B_\Pi \neq \emptyset$. Then, by Lemma 13, B_Π is a $\ker\alpha\beta$-class and $B_\Pi^*\alpha\beta = B^*$ (since $\alpha = \varepsilon_\Pi$ and $\beta = \varepsilon_\Phi$ by Remark 11). On the other hand, since $\alpha\beta \in E(E_n)$, $B_\Pi^*\alpha\beta = B_\Pi^*$ (see Remark 11), so $B^* = B_\Pi^*$. Further, since $\alpha\beta \in S$ and since $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\}$ is the set of $\ker\alpha\beta$-classes, i.e. $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} = X_n/\ker\alpha\beta$, we conclude $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} \in \mathbf{\Omega}$. Altogether, we have $\mathbf{\Omega} \in \mathbf{\Omega_d}$, where $\{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\} = \{\varepsilon_{X_n/\ker\gamma} \mid \gamma \in S\} = \{\gamma \mid \gamma \in S\} = S$.

Conversely, let $S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}$ for some $\mathbf{\Omega} \in \mathbf{\Omega_d}$. We have to show that S is a semigroup. For this let $\Pi, \Phi \in \mathbf{\Omega}$. Then $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\}$ is the set of $\ker \varepsilon_\Pi\varepsilon_\Phi$-classes by Lemma 13 (ii). Since $\mathbf{\Omega} \in \mathbf{\Omega_d}$, we have $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} \in \mathbf{\Omega}$, i.e. $X_n/\ker \varepsilon_\Pi\varepsilon_\Phi \in \mathbf{\Omega}$, and $B_\Pi^* = B^*$. Moreover, by Lemma 13 (i), we have $x\varepsilon_\Pi\varepsilon_\Phi = B^* = B_\Pi^* \in B_\Pi$ for all $x \in B_\Pi$ and for all $B \in \Phi$ with $B_\Pi \neq \emptyset$. Hence $\varepsilon_\Pi\varepsilon_\Phi$ is an idempotent and we have $\varepsilon_\Pi\varepsilon_\Phi = \varepsilon_{X_n/\ker \varepsilon_\Pi\varepsilon_\Phi} \in S$. ∎

Now, we want to study the bands within the subsemigroup OE_n of E_n. Recall that a set $Y \subseteq X_n$ is called convex if $x, y \in Y$, $z \in X_n$, and $x \leq z \leq y$ implies $z \in Y$. Moreover, an $\alpha \in T_n$ is called convex if the $\ker \alpha$-classes are convex and a partition of X_n is called convex if each of its elements is convex.

Remark 15. Each $\alpha \in OE_n$ is convex, i.e. $X_n/\ker \alpha$ is a convex partition of X_n.

We want to describe the bands within OE_n using Proposition 14.

Notation 16. Let Π and Φ be convex partitions of X_n. Then:

1. $A_{\Phi,\Pi}$ will denote the set of all $A \in \Pi$ such that $B \subseteq A$ for some $B \in \Phi$.
2. $B_{\Phi,\Pi}$ will denote all $B \in \Phi$ such that B is the union of elements of Π.

If Π and Φ are convex partitions of X_n such that each $B \in \Phi$ is the union of elements of Π or B is a subset of some $A \in \Pi$ then $A_{\Phi,\Pi} \cup B_{\Phi,\Pi}$ is a partition of X_n.

Definition 17. Let $\mathbf{\Omega}$ be a set of all convex partitions of X_n such that for $\Pi, \Phi \in \mathbf{\Omega}$:

1. Each $B \in \Phi$ is a union of elements of Π or it is contained in some $A \in \Pi$.
2. $A_{\Phi,\Pi} \cup B_{\Phi,\Pi} \in \mathbf{\Omega}$.

Denote by $\mathbf{\Omega_e}$ the set of all such sets $\mathbf{\Omega}$.

Lemma 18. *Let $\mathbf{\Omega}$ be a set of convex partitions of X_n. Then $\mathbf{\Omega} \in \mathbf{\Omega_e}$ if and only if $\mathbf{\Omega} \in \mathbf{\Omega_d}$.*

Proof. Suppose that $\mathbf{\Omega} \in \mathbf{\Omega_e}$. Let $\Pi, \Phi \in \mathbf{\Omega}$ and $B \in \Phi$ with $B_\Pi \neq \emptyset$. If B is the union of elements of Π then $B_\Pi = \bigcup\{A \in \Pi \mid A^* \in B\} = B \in B_{\Phi,\Pi}$, i.e. $B_\Pi^* = B^*$.

Suppose that $B \subseteq A$ for some $A \in \Pi$. Since $B_\Pi \neq \emptyset$, it follows that $B_\Pi = A \in A_{\Phi,\Pi}$. Moreover, $B_\Pi = A$ implies $B_\Pi^* = A^*$. From $B \subseteq A$ it follows $B^* \leq A^*$ and $A^* \in B$ implies $A^* \leq B^*$, i.e. $A^* = B^*$ and thus $B_\Pi^* = B^*$. We have shown that $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} \subseteq A_{\Phi,\Pi} \cup B_{\Phi,\Pi}$ and $B_\Pi^* = B^*$ for all $B \in \Phi$ with $B_\Pi \neq \emptyset$. If $B \in B_{\Phi,\Pi}$ then $B \in \Phi$ and $B = \bigcup\{A \in \Pi \mid A^* \in B\} = B_\Pi \neq \emptyset$. If $A \in A_{\Phi,\Pi}$ then $A \in \Pi$ and there is

$B \in \Phi$ with $B \subseteq A$. Because $\mathbf{\Omega} \in \mathbf{\Omega_e}$, for any $\widehat{B}$ of the partition Φ, $\widehat{B} \cap A = \emptyset$ or $\widehat{B} \subseteq A$. So, A is the union of elements of Φ. Thus there is $B \in \Phi$ with $B \subseteq A$ and $A^* \in B$. Since A is the only element of Π with $A^* \in B$, we have $B_\Pi = A \neq \emptyset$. This shows the converse inclusion $A_{\Phi,\Pi} \cup B_{\Phi,\Pi} \subseteq \{B_\Pi \mid B \in \Phi, B_\Pi \neq \emptyset\}$. Consequently, $\{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} = A_{\Phi,\Pi} \cup B_{\Phi,\Pi} \in \mathbf{\Omega}$. This shows that $\mathbf{\Omega} \in \mathbf{\Omega_d}$.

Conversely, suppose that $\mathbf{\Omega} \in \mathbf{\Omega_d}$. Let $\Pi, \Phi \in \mathbf{\Omega}$ and $B \in \Phi$. Then there is $A \in \Pi$ with $B^* \in A$. Suppose $B \not\subseteq A$. Assume that there is $C \in \Pi$ with $B \cap C \neq \emptyset$ and $C \setminus B \neq \emptyset$. First, we assume that $C^* < y$ for all $y \in A$. Let $x := (C \setminus B)^*$. Let $B_1 \in \Phi$ with $x \in B_1$, i.e. $x = B_1^* \in C$. Since $B^* \in A$ we have $B^* \notin C$. Hence $C_\Phi^* = B_1^*$. Moreover, $B_1^* < y$ for all $y \in B$ and, in particular, $B_1^* < C^*$. Thus $C_\Phi^* \neq C^*$, a contradiction to $\mathbf{\Omega} \in \mathbf{\Omega_d}$. Now we assume that $A^* < x$ for all $x \in C$. But this is impossible since then $B^* < x$ for all $x \in C$ (because $B^* \in A$), and so $B \cap C = \emptyset$. Therefore, for each $C \in \Pi$ with $B \cap C \neq \emptyset$ holds $C \subseteq B$, i.e. B is the union of elements of Π. As above, we can show that $A_{\Phi,\Pi} \cup B_{\Phi,\Pi} = \{B_\Pi \mid B \in \Phi,\ B_\Pi \neq \emptyset\} \in \mathbf{\Omega}$. Hence, $\mathbf{\Omega} \in \mathbf{\Omega_e}$. ■

Corollary 19. *A subset S of $E(OE_n)$ is a band if and only if there is $\mathbf{\Omega} \in \mathbf{\Omega_e}$ such that*

$$S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}.$$

Proof. Let S be a band. Since $OE_n \subseteq E_n$, there is $\mathbf{\Omega} \in \mathbf{\Omega_d}$ such that $S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}$ by Proposition 14. Since $X_n/\ker \varepsilon_\Pi = \Pi$ is a convex partition of X_n for all $\Pi \in \mathbf{\Omega}$, $\mathbf{\Omega}$ is a set of convex partitions of X_n, and by Lemma 18, we can conclude $\mathbf{\Omega} \in \mathbf{\Omega_e}$. Conversely, let $S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}$ for some $\mathbf{\Omega} \in \mathbf{\Omega_e}$. Then $\mathbf{\Omega}$ is a set of convex partitions of X_n and $\mathbf{\Omega} \in \mathbf{\Omega_d}$ by Lemma 18, i.e. S is a band by Proposition 14. ■

Lemma 20. *Let $S \subseteq E(OE_n)$. If S is a band then $\{X_n/\ker \alpha \mid \alpha \in S\} \in \mathbf{\Omega_e}$.*

Proof. We have $S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}$ for some $\mathbf{\Omega} \in \mathbf{\Omega_e}$ by Corollary 19. On the other hand, we have $S = \{\varepsilon_{X_n/\ker \alpha} \mid \alpha \in S\}$ since $\varepsilon_{X_n/\ker \alpha} = \alpha$ for every $\alpha \in S$. This gives $\{X_n/\ker \alpha \mid \alpha \in S\} = \mathbf{\Omega} \in \mathbf{\Omega_e}$. ■

Finally, we want to characterize the maximal bands (with respect to the inclusion) within OE_n.

Definition 21. Let $\mathcal{V}$ be a set of convex subsets of X_n with

1. $X_n \in \mathcal{V}$.
2. If $A \in \mathcal{V}$, different from a singleton set, then there is a partition $\{A_1, A_2\}$ of A with $A_1, A_2 \in \mathcal{V}$.
3. If $A, B \in \mathcal{V}$ then $A \cap B \in \{A, B, \emptyset\}$.

Denote by $\mathcal{D}(X_n)$ the set of all such sets $\mathcal{V}$.

Notation 22. For $\mathcal{V} \in \mathcal{D}(X_n)$, let $X_n(\mathcal{V})$ be the set of all partitions of X_n with elements of $\mathcal{V}$ and we put

$$S(\mathcal{V}) := \{\varepsilon_\Pi \mid \Pi \in X_n(\mathcal{V})\}.$$

Lemma 23. *Let α, β be elements of a band S within OE_n. Moreover, let A and B be a $\ker \alpha$-class and a $\ker \beta$-class, respectively. Then $A \cap B \in \{A, B, \emptyset\}$.*

Proof. Since S is a band within OE_n, there is $\mathbf{\Omega} \in \mathbf{\Omega_e}$ such that $S = \{\varepsilon_\Pi \mid \Pi \in \mathbf{\Omega}\}$ by Corollary 19. Note that both $X_n/\ker \alpha$ as well as $X_n/\ker \beta$ belong to $\mathbf{\Omega}$. Hence, A is the union of $\ker \beta$-classes $C_1, \ldots, C_p$ or A is a (proper) subset of a $\ker \beta$-class C. Suppose that A is the union of $\ker \beta$-classes $C_1, \ldots, C_p$. If $B \in \{C_1, \ldots, C_p\}$ then $A \cap B = B$. If $B \notin \{C_1, \ldots, C_p\}$ then $A \cap B = \emptyset$. Now, suppose that A is a (proper) subset of a $\ker \beta$-class C. If $B = C$ then $A \cap B = A$. If $B \neq C$ then $A \cap B = \emptyset$. ∎

Proposition 24. *A set $S \subseteq E(OE_n)$ is a maximal band within OE_n if and only if there is $\mathcal{V} \in \mathcal{D}(X_n)$ with $S(\mathcal{V}) = S$.*

Proof. Let S be a maximal band within OE_n. Then we put

$$\mathcal{V} := \bigcup \{X_n/\ker \delta \mid \delta \in S\}$$

and show that $\mathcal{V} \in \mathcal{D}(X_n)$.

1. Let $\lambda \in OE_n$ with $\{X_n\} = X_n/\ker \lambda$. Clearly, $\lambda \in E(OE_n)$. Since $\operatorname{im} \lambda = \{n\}$ and $n \in \operatorname{im} \alpha$ for all $\alpha \in OE_n$, we have $\lambda\alpha = \alpha\lambda = \lambda$ for all $\alpha \in OE_n$. Thus $\lambda \in S$ because of the maximality of S. This shows that $X_n \in \mathcal{V}$.

2. Let $A \in \mathcal{V}$ be different from a singleton set. Then there is $\sigma \in S$ with $A \in X_n/\ker \sigma$. Assume that for all $\alpha \in S$ and all proper partitions Π of A, $\Pi \nsubseteq X_n/\ker \alpha$. Let $\{A_1, A_2\}$ be a partition of A. Let β be the idempotent transformation with $X_n/\ker \beta = (X_n/\ker \sigma \setminus \{A\}) \cup \{A_1, A_2\}$. Let $\gamma \in S$. By Lemma 20, we have $\mathbf{\Omega} := \{X_n/\ker \delta \mid \delta \in S\} \in \mathbf{\Omega_e}$. Hence, A is the union of $\ker \gamma$-classes or it is a subset of a $\ker \gamma$-class. Assume that A is the union of $\ker \gamma$-classes $\hat{A}_1, \ldots, \hat{A}_p$. Then $\{\hat{A}_1, \ldots, \hat{A}_p\}$ is a partition of A with $\hat{A}_i \in X_n/\ker \gamma$ for all $1 \leq i \leq p$, a contradiction to the assumption about A. So, A is a subset of a $\ker \gamma$-class $\widetilde{A}$. Hence, both A_1 and A_2 are subsets of $\widetilde{A}$. We put $\overline{\gamma} := X_n/\ker \gamma$, $\overline{\sigma} := X_n/\ker \sigma$, and $\overline{\beta} := X_n/\ker \beta$. Since both A_1 and A_2 are subsets of a $\ker \gamma$-class $\widetilde{A}$, it is easy to check that $A_{\overline{\gamma},\overline{\sigma}} \cup B_{\overline{\gamma},\overline{\sigma}} = A_{\overline{\gamma},\overline{\beta}} \cup B_{\overline{\gamma},\overline{\beta}}$ and $A_{\overline{\sigma},\overline{\gamma}} \cup B_{\overline{\sigma},\overline{\gamma}} = A_{\overline{\beta},\overline{\gamma}} \cup B_{\overline{\beta},\overline{\gamma}}$. Because of $A_{\overline{\gamma},\overline{\sigma}} \cup B_{\overline{\gamma},\overline{\sigma}} \in \mathbf{\Omega}$ (since $\mathbf{\Omega} \in \mathbf{\Omega_e}$), we have $A_{\overline{\gamma},\overline{\beta}} \cup B_{\overline{\gamma},\overline{\beta}} \in \mathbf{\Omega} \subseteq$

$\{X_n/\ker\delta \mid \delta \in S \cup \{\beta\}\}$, and $A_{\overline{\beta},\overline{\gamma}} \cup B_{\overline{\beta},\overline{\gamma}} \in \mathbf{\Omega} \subseteq \{X_n/\ker\delta \mid \delta \in S \cup \{\beta\}\}$ by the same argument. Since $\{X_n/\ker\delta \mid \delta \in S\} \in \mathbf{\Omega_e}$, the foregoing argument shows that also $\{X_n/\ker\delta \mid \delta \in S \cup \{\beta\}\} \in \mathbf{\Omega_e}$ and thus $S \cup \{\beta\} = \{\varepsilon_{X_n/\ker\delta} \mid \delta \in S \cup \{\beta\}\}$ is a band within OE_n by Corollary 19, where $\beta \notin S$ by the assumption. This contradicts the maximality of S.

Hence, there is an $\alpha \in S$ such that there is a proper partition $\{A_1, \dots, A_p\}$ $(2 \leq p \in \mathbb{N})$ of A, which is a subset of $X_n/\ker\alpha$. We can choose α such that p is minimal. We put $B_1 := A_1$ and $B_2 := A \setminus A_1$. Let $\gamma \in S$. Then A is a subset of a $\ker\gamma$-class or A is the union of $\ker\gamma$-classes. If A is a subset of a $\ker\gamma$-class B then both B_1 and B_2 are subsets of B. Suppose that A is the union of $\ker\gamma$-classes $\widetilde{B}_1, \dots, \widetilde{B}_q$ $(2 \leq q)$. Assume that there is a $\widetilde{B}_i$ $(1 \leq i \leq q)$ such that $\widetilde{B}_i$ is the union of $\ker\alpha$-classes $A_r, \dots, A_s$ $(1 \leq r < s \leq p)$. Let us consider the transformation $\gamma\alpha \in S$. Then it is easy to verify that there are at most $p - s + r$ different $\ker\gamma\alpha$ -classes which are contained in A. These $\ker\gamma\alpha$-classes provide a partition of A with less than p classes. This contradicts the minimality of p. Hence each $\widetilde{B}_i$ $(1 \leq i \leq q)$ is contained in a $\ker\alpha$-class A_{k_i} $(1 \leq k_i \leq p)$, which implies that each A_i $(1 \leq i \leq p)$ is the union of $\ker\gamma$-classes (within $\widetilde{B}_1, \dots, \widetilde{B}_q$). In particular, B_1 is the union of the $\ker\gamma$-classes $\widetilde{B}_i$ with $\widetilde{B}_i \subseteq A_1$ and B_2 is the union of the $\ker\gamma$-classes $\widetilde{B}_i$ with $\widetilde{B}_i \not\subseteq A_1$.

Let $\mu \in E(OE_n)$ with $\{B_1, B_2\} \cup \{\{a\} \mid a \in X_n \setminus A\}$ as the set of $\ker\mu$-classes and let us consider the set $\widetilde{S} := S \cup \{\mu\delta \mid \delta \in S\}$ and $\widetilde{\mathbf{\Omega}} := \{X_n/\ker\delta \mid \delta \in \widetilde{S}\}$. Let $\delta \in S$. If A is a subset of a $\ker\delta$-class then $\mu\delta = \delta$. If A is the union of $\ker\delta$-classes then

$$X_n/\ker\mu\delta = \{D \in X_n/\ker\delta \mid D \not\subseteq A\} \cup \{B_1, B_2\}$$

since B_1 as well as B_2 is the union of $\ker\delta$-classes by the previous observation. Since, for every $\gamma \in S$, both B_1 and B_2 are subsets of the same $\ker\gamma$-class or both are unions of $\ker\gamma$-classes, since $\{X_n/\ker\delta \mid \delta \in S\} \in \mathbf{\Omega_e}$, and since $A_{\Phi,\Pi} \cup B_{\Phi,\Pi} \in \{X_n/\ker\delta \mid \delta \in S\}$ for all $\Phi, \Pi \in \{X_n/\ker\delta \mid \delta \in S\}$, it is easy to verify that $\widetilde{\mathbf{\Omega}}$ satisfies both conditions of Definition 17, i.e. $\widetilde{\mathbf{\Omega}} \in \mathbf{\Omega_e}$. Thus, $\widetilde{S}$ is a band within OE_n (by Corollary 19) containing S. Because of the maximality of S, we have $S = \widetilde{S}$, and thus $\alpha\mu \in S$. This gives $B_1, B_2 \in \mathcal{V}$ where $A = B_1 \cup B_2$.

3. Let $A, B \in \mathcal{V}$. Then $A \cap B \in \{A, B, \emptyset\}$ by Lemma 23.

Consequently, $\mathcal{V} \in \mathcal{D}(X_n)$. We have still to show that $X_n(\mathcal{V}) \in \mathbf{\Omega_e}$. Let $\Pi, \Phi \in X_n(\mathcal{V})$ and $B \in \Phi$. Then for each $A \in \Pi$, $A \cap B \in \{A, B, \emptyset\}$. This shows that B is the union of elements of Π or B is a subset of some $A \in \Pi$. Then $A_{\Phi,\Pi} \cup B_{\Phi,\Pi} \in X_n(\mathcal{V})$ since $A_{\Phi,\Pi} \cup B_{\Phi,\Pi}$ is a partition of X_n with elements of $\mathcal{V}$. Consequently, $X_n(\mathcal{V}) \in \mathbf{\Omega_e}$ and $S(\mathcal{V}) = \{\varepsilon_\Pi \mid \Pi \in X_n(\mathcal{V})\}$ is a band by Corollary 19. In fact, from $\{X_n/\ker\delta \mid \delta \in S\} \subseteq X_n(\mathcal{V})$ it follows $S \subseteq S(\mathcal{V})$ and the maximality of S yields $S = S(\mathcal{V})$.

Conversely, let $S = S(\mathcal{V})$ for some $\mathcal{V} \in \mathcal{D}(X_n)$. Clearly, $S \subseteq E(OE_n)$. Let $\mathbf{\Omega} := X_n(\mathcal{V})$. By the argument from the previous paragraph, $\mathbf{\Omega} \in \mathbf{\Omega_e}$, so S is a semigroup within $E(OE_n)$ by Corollary 19. Now, we have to show that S is maximal. Suppose that $S \subseteq S_1$, where S_1 is a semigroup within $E(OE_n)$. Let $\alpha \in S_1$. Then, by Lemma 23, $A \cap B \in \{A, B, \emptyset\}$ for all $\ker\alpha$-classes B, all $\Pi \in \mathbf{\Omega}$, and all $A \in \Pi$.

Let B be any $\ker\alpha$-class. Select $\Pi \in \mathbf{\Omega}$ and $A_1 \in \Pi$ such that $B \subseteq A_1$ and for all $\Phi \in \mathbf{\Omega}$ and all $C \in \Phi$, if $B \subseteq C$ then $|A_1| \leq |C|$. (Such Π and A_1 exist, since $\{X_n\} \in X_n(\mathcal{V}) = \mathbf{\Omega}$ and $B \subseteq X_n$.)

We claim that $B = A_1$. If A_1 is a singleton then the claim is obviously true. Suppose $|A_1| \geq 2$. Then $A_1 = A_2 \cup A_3$, where $A_2, A_3 \in \mathcal{V}$ and $\{A_1, A_2\}$ is a partition of A. Consider $\widetilde{\Pi} := (\Pi \setminus \{A_1\}) \cup \{A_2, A_3\} \in \mathbf{\Omega}$. Then $\varepsilon_{\widetilde{\Pi}} \in S$, and so, by Lemma 23, $B \cap A_2 \in \{B, A_2, \emptyset\}$. If $B \cap A_2 = B$ then $B \subseteq A_2$, which contradicts the minimality of A_1. If $B \cap A_2 = \emptyset$ then $B \subseteq A_3$, which, again, contradicts the minimality of A_1. Hence $B \cap A_2 = A_2$. Similarly, $B \cap A_3 = A_3$, and so $B = A_1$.

Consequently, $B \in \mathcal{V}$, which shows that $X_n/\ker\alpha \subseteq \mathcal{V}$, i.e. $X_n/\ker\alpha \in X_n(\mathcal{V})$. Thus $\varepsilon_{X_n/\ker\alpha} = \alpha \in S$ since $S = \{\varepsilon_\Pi \mid \Pi \in X_n(\mathcal{V})\}$, which implies $S = S_1$. ∎

Acknowledgements. The authors thanks the referee of this paper for their numerous valuable suggestions.

References

[1] G. Bijev, *Some applications of semigroups and computer algebra in matrix theory*, AIP Conference Proc. 946 (2007), 263–270.

[2] G. Bijev, K. Todorov, *Coregular semigroups*, Notes on Semigroups VI, Budapest (1980–1984), 1–11.

[3] G. Bijev, K. Todorov, *On the representation of abstract semigroups by transformation semigroups: computer investigations*, Semigroup Forum 43 (1991), 253–256.

[4] J. Chvalina, K. Motoushkova, *Coregularity of endomorphism monoid of unars*, Arch. Math. 20 (1984), 43–48.

[5] W. Edmond, H. Lee, *Hereditarily finitly based monoids of extensive transformations*, Algebra Universalis 61 (2009), 31–58.

[6] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, 1995.

[7] E. H. Moore, *On the reciprocal of the general algebraic matrix*, Bull. Amer. Math. Soc. 26 (1920), 394–395.

[8] K. Todorov, *On the antiinverse an coregular semigroups and some their applications*, Filomat 9(3) (1995), 947–959.

[9] M. V. Volkov, *Reflexive relations, extensive transformations and piecewise testasble languages of a given height*, Internat. J. Algebra Comput. 14 (2004), 817–827.

Corresponding Author
Ilinka Dimitrova
FACULTY OF MATHEMATICS AND NATURAL SCIENCE
SOUTH-WEST UNIVERSITY "NEOFIT RILSKI"
BLAGOEVGRAD, 2700, BULGARIA
E-mail: ilinka_dimitrova@yahoo.com

Jörg Koppitz
INSTITUTE OF MATHEMATICS
POTSDAM UNIVERSITY
POTSDAM, 14469, GERMANY
E-mail: koppitz@rz.uni-potsdam.de

Received September 27, 2010; revised version July 22, 2011.