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HYPER-PSEUDOFORMULAS AND M-SOLID ORDERED PSEUDOVARIETIES

Abstract. In 2009 K. Denecke and J. Koppitz proved that for a monoid $\mathcal{M}$ of hypersubstitutions M-solid positive varieties of tree languages correspond to M-solid ordered pseudovarieties. In this paper, we will characterize M-solid ordered pseudovarieties in a similar way in which in [14] M-solid varieties, in [3] M-solid quasivarieties, in [11] M-solid pseudovarieties and in [12] M-solid algebraic systems were characterized. The main idea is to show, that we have two Galois-connections and a conjugate pair of additive closure operators. Then we can apply the general theory of conjugate pairs of additive closure operators.

1. Introduction

In [17] and [9] M-solid varieties of tree languages and M-solid positive varieties of tree languages were characterized by M-solid pseudovarieties and by M-solid ordered pseudovarieties of finite algebras and of finite ordered algebras, respectively. The theory of M-solid pseudovarieties was developed in [5] and in [11]. The aim of this paper is to apply the theory of conjugate pairs of additive closure operators (see [7]) to get a characterization of M-solid ordered pseudovarieties. Since M-solid ordered pseudovarieties are finite model classes of certain sets of hyper-pseudoformulas, we can use ideas from [12]. First we want to repeat some basic concepts on finite ordered algebras. To describe classes of finite ordered algebras as model classes of logical sentences we need the concept of an implicit operation. This will be introduced at the end of this section.

An *ordered algebra* of type τ is a triple $\mathcal{A}^{\leq} := (A; (f_i^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$ consisting of a set A, an indexed set $(f_i^{\mathcal{A}})_{i\in I}$ of operations defined on A, where $f_i^{\mathcal{A}} : A^{n_i} \to A$ is n_i-ary, and a partial order relation $\leq_{\mathcal{A}}$ on A, which is compatible with all the operations $(f_i^{\mathcal{A}})_{i\in I}$, i.e. if $a_1 \leq_{\mathcal{A}} b_1, \ldots, a_{n_i} \leq_{\mathcal{A}} b_{n_i}$, then

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$f_i^{\mathcal{A}}(a_1,\ldots,a_{n_i}) \leq_{\mathcal{A}} f_i^{\mathcal{A}}(b_1,\ldots,b_{n_i})$. Let $Alg^{\leq}_{fin}(\tau)$ be the class of all finite ordered algebras of type τ. Every ordered algebra is an algebraic system (see [1]) of type $(\tau,(2))$ (see [12]).

A simple example of an ordered algebra of type $((2),(2))$ is given by $\mathcal{A}^{\leq} := (\{0,1,2\}; max, min, \leq_{\mathcal{A}})$, where $\leq_{\mathcal{A}}$ is the usual order on the set of integers $\{0,1,2\}$ and where max and min denote the maximum and the minimum with respect to this order.

Definition 1.1. Let $\mathcal{A}^{\leq} = (A; (f_i^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$ and $\mathcal{B}^{\leq} = (B; (f_i^{\mathcal{B}})_{i\in I}, \leq_{\mathcal{B}})$ be ordered algebras of type τ. We say that $\mathcal{A}^{\leq}$ is an ordered subalgebra of $\mathcal{B}^{\leq}$ if $(A; (f_i^{\mathcal{A}})_{i\in I})$ is a subalgebra of $(B; (f_i^{\mathcal{B}})_{i\in I})$ and $\leq_{\mathcal{A}}$ is the restriction of $\leq_{\mathcal{B}}$ onto A^2, i.e. $\leq_{\mathcal{A}} := \leq_{\mathcal{B}} |A^2$. A mapping $h: A \to B$ is said to be an ordered homomorphism $h: \mathcal{A}^{\leq} \to \mathcal{B}^{\leq}$ of $\mathcal{A}^{\leq}$ to $\mathcal{B}^{\leq}$ if h is an algebra homomorphism and if from $a \leq_{\mathcal{A}} b$ there follows $h(a) \leq_{\mathcal{B}} h(b)$ for all $a, b \in A$. Let $(\mathcal{A}_j^{\leq})_{j\in J}$ be a family of ordered algebras of the same type τ. Then the ordered direct product of the ordered algebras $\mathcal{A}_j^{\leq} = (A_j; (f_i^{\mathcal{A}_j})_{i\in I}, \leq_{\mathcal{A}_j})$ is the direct product of the family of underlying algebras together with the product relation $\leq_{\mathcal{P}} := \bigotimes_{j\in J} \leq_{\mathcal{A}_j}$ defined by $\leq_{\mathcal{P}} := \{((a_j)_{j\in J}, (b_j)_{j\in J}) |\ a_j \leq_{\mathcal{A}_j} b_j,\ j \in J\}$. Let $H^{\leq}, S^{\leq}, P^{\leq}$ and $P^{\leq}_{fin}$ be the operators of taking arbitrary ordered homomorphic images, ordered subalgebras, ordered direct products and finite ordered direct products. Then a class $\mathcal{K} \subseteq Alg^{\leq}_{fin}(\tau)$ is called an ordered pseudovariety of type τ if $\mathcal{K} = H^{\leq}S^{\leq}P^{\leq}_{fin}(\mathcal{K})$; i.e. if $\mathcal{K}$ is closed under these operators.

Pseudovarieties of type τ are classes of finite algebras of type τ which are closed under homomorphic images, subalgebras and finite direct products. For a logical characterization of pseudovarieties Reiterman used the concept of a pseudoidentity which is based on implicit operations (see [8]).

Definition 1.2. An n-ary implicit operation on a pseudovariety V of type τ is given by a V-indexed family $\pi := (\pi_{\mathcal{A}})_{\mathcal{A}\in V}$ satisfying the following conditions :

(i) $\pi_{\mathcal{A}} : A^n \to A$ is an n-ary operation on A for each $\mathcal{A} \in V$ and
(ii) for each homomorphism $h : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}, \mathcal{B} \in V$ there holds $h(\pi_{\mathcal{A}}(a_1,\ldots a_n)) = \pi_{\mathcal{B}}(h(a_1),\ldots,h(a_n))$ for every $a_1,\ldots,a_n \in A$, i.e. the following diagram commutes:

$$\begin{array}{ccc} A^n & \xrightarrow{\pi_{\mathcal{A}}} & A \\ \Big\downarrow h^n & (=) & \Big\downarrow h \\ B^n & \xrightarrow{\pi_{\mathcal{B}}} & B \end{array}$$

Here $h^n : A^n \to B^n$ is defined by $(a_1, \dots, a_n) \mapsto (h(a_1), \dots, h(a_n))$ for every $(a_1, \dots, a_n) \in A^n$. The set of all n-ary implicit operations on V is denoted by $\overline{\Omega}_n V$. An n-ary implicit operation on an ordered pseudovariety $V_\leq$ of type τ is defined in essentially the same way with the difference that in (ii) we take ordered homomorphisms. This corresponds to the usual definition of implicit operations over a category of algebraic structures with morphisms as operations on the universe, which are compatible with these morphisms. For an ordered pseudovariety $V_\leq$, let $V := HSP_{fin}\{\mathcal{A} \mid \mathcal{A}^\leq \in V_\leq\}$ be the pseudovariety which we obtain, if we delete in each ordered algebra from $V_\leq$ the partial order relation. Let $\overline{\Omega}_n V_\leq$ be the set of all n-ary implicit operations on the ordered pseudovariety. Since every ordered homomorphism $h : \mathcal{A}^\leq \to \mathcal{B}^\leq$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}, \mathcal{B}$ are usual algebras of type τ belonging to $\mathcal{A}^\leq$ and $\mathcal{B}^\leq$, respectively, we have $\overline{\Omega}_n V \subseteq \overline{\Omega}_n V_\leq$.

2. Pseudoformulas

An ordered algebra can be regarded as an algebraic system with an indexed set $(f_i^{\mathcal{A}})_{i\in I}$ of operations of type $\tau = (n_i)_{i\in I}$ and one binary relation. In general, an algebraic system $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ of type (τ, τ') consists of a set A, an indexed sequence $(f_i^{\mathcal{A}})_{i\in I}$ of fundamental operations, where $f_i^{\mathcal{A}}$ is n_i-ary and an indexed sequence $(\gamma_j^{\mathcal{A}})_{j\in J}$ of fundamental relations, where $\gamma_j^{\mathcal{A}} \subseteq A^{n_j}$. Here τ and τ' are the sequences $\tau = (n_i)_{i\in I}$ and $\tau' = (n_j)_{j\in J}$, respectively. Therefore ordered algebras are algebraic systems of type $(\tau, (2))$. Classes of algebraic systems can be defined as model classes of sets of formulas and pseudovarieties are model classes of sets of pseudoidentities. For a logical description of classes of finite algebraic systems we introduce the concept of a pseudoformula. We will define pseudoformulas of type $(\tau, (2))$, but our definition can be generalized to an arbitrary type (τ, τ').

Definition 2.1. Let $V_\leq$ be an ordered pseudovariety of type τ and let $n \geq 1$ be a natural number. An n-ary pseudoformula of type $(\tau, (2))$ on $V_\leq$ is defined in the following inductive way:

(i) If π_1, π_2 are n-ary implicit operations on $V_\leq$, then the equation $\pi_1 \approx \pi_2$ is an n-ary pseudoformula of type $(\tau, (2))$ on $V_\leq$.
(ii) If π_1, π_2 are n-ary implicit operations on $V_\leq$ and if γ is a binary relational symbol, then $\gamma(\pi_1, \pi_2)$ is an n-ary pseudoformula of type $(\tau, (2))$ on $V_\leq$.
(iii) If PF is an n-ary pseudoformula of type $(\tau, (2))$ on $V_\leq$, then $\neg(PF)$ is an n-ary pseudoformula of type $(\tau, (2))$ on $V_\leq$.
(iv) If PF_1 and PF_2 are n-ary pseudoformulas of type $(\tau, (2))$ on $V_\leq$, then $PF_1 \vee PF_2$ is an n-ary pseudoformula of type $(\tau, (2))$ on $V_\leq$.

(v) If PF is an n-ary pseudoformula of type $(\tau,(2))$ on $V_{\leq}$, and $x_i \in X_n$, then $\exists x_i(PF)$ is an n-ary pseudoformula of type $(\tau,(2))$ on $V_{\leq}$.

Let $\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}_n V_{\leq})$ be the set of all n-ary pseudoformulas of type $(\tau,(2))$ on $V_{\leq}$ and let $\mathcal{PF}_{(\tau,(2))}(\overline{\Omega} V_{\leq}) := \bigcup_{n\geq 1} \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}_n V_{\leq})$ be the set of all pseudoformulas of type $(\tau,(2))$ on $V_{\leq}$.

Let PF be a pseudoformula of type $(\tau,(2))$ on $V_{\leq}$ and let $\mathcal{A}^{\leq} := (A;(f_i^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$ be a finite ordered algebra of type τ. Then we define the satisfaction of a pseudoformula of type $(\tau,(2))$ on $V_{\leq}$ by the ordered algebra $\mathcal{A}^{\leq}$, written as $\mathcal{A}^{\leq} \underset{p.s.}{\models} PF$.

Definition 2.2. Let $V_{\leq}$ be an ordered pseudovariety of type τ and let PF be a pseudoformula of type $(\tau,(2))$.

(i) If PF is an equation $\pi_1 \approx \pi_2$, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \pi_1 \approx \pi_2 :\Leftrightarrow (\pi_1)_{\mathcal{A}^{\leq}} = (\pi_2)_{\mathcal{A}^{\leq}}$.

(ii) If PF has the form $\gamma(\pi_1,\pi_2)$ for the binary relational symbol γ and implicit operations π_1,π_2 on $V_{\leq}$, then

$$\mathcal{A}^{\leq} \underset{p.s.}{\models} \gamma(\pi_1,\pi_2) :\Leftrightarrow \gamma^{\mathcal{A}^{\leq}}((\pi_1)_{\mathcal{A}^{\leq}}, (\pi_2)_{\mathcal{A}^{\leq}}) \text{ is true in } \mathcal{A}^{\leq}$$

(This means that for every $(a_1,\ldots,a_n) \in A^n$ we have $(\pi_1)_{\mathcal{A}^{\leq}}(a_1,\ldots,a_n) \leq_{\mathcal{A}} (\pi_2)_{\mathcal{A}^{\leq}}(a_1,\ldots,a_n)$).

(iii) If the pseudoformula has the form $\neg(PF)$ and if we inductively assume that $\mathcal{A}^{\leq} \underset{p.s.}{\models} PF$ is already defined, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \neg(PF) :\Leftrightarrow \neg(\mathcal{A}^{\leq} \underset{p.s.}{\models} PF)$.

(iv) If the pseudoformula has the form $PF_1 \vee PF_2$ and if we inductively assume that $\mathcal{A}^{\leq} \underset{p.s.}{\models} PF_1$, $\mathcal{A}^{\leq} \underset{p.s.}{\models} PF_2$ are already defined, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} PF_1 \vee PF_2 :\Leftrightarrow \mathcal{A}^{\leq} \underset{p.s.}{\models} PF_1 \vee \mathcal{A}^{\leq} \underset{p.s.}{\models} PF_2$.

(v) If the pseudoformula has the form $\exists x_i(PF)$ and if we inductively assume that $\mathcal{A}^{\leq} \underset{p.s.}{\models} PF$ is already defined, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \exists x_i(PF) :\Leftrightarrow \exists x_i(\mathcal{A}^{\leq} \underset{\text{p.s.}}{\models} \text{PF})$.

The symbol $\underset{p.s.}{\models}$ defines a relation between the sets $Alg^{\leq}_{fin}(\tau)$ and $\mathcal{PF}_{(\tau,(2))}(\overline{\Omega} V_{\leq})$. From this relation we get a Galois-connection (PSM, PSF)

where

$$PSM : \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})) \to \mathcal{P}(Alg^{\leq}_{fin}(\tau)) \quad \text{and}$$
$$PSF : \mathcal{P}(Alg^{\leq}_{fin}(\tau)) \to \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq}))$$

are defined by

$$PSM(\mathcal{PF}) := \{\mathcal{A}^{\leq} \in Alg^{\leq}_{fin}(\tau) \mid \forall PF \in \mathcal{PF}\ (\mathcal{A}^{\leq} \underset{p.s.}{\models} PF)\} \text{ and}$$
$$PSF(\mathcal{K}) := \{PF \in \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq}) \mid \forall \mathcal{A}^{\leq} \in \mathcal{K}\ (\mathcal{A}^{\leq} \underset{p.s.}{\models} PF)\}$$

for $\mathcal{PF} \subseteq \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})$ and $\mathcal{K} \subseteq Alg^{\leq}_{fin}(\tau)$, respectively.

The fixed points under the closure operators $PSMPSF$ and $PSFPSM$, respectively are called pseudo-model classes and pseudo-theories of type $(\tau,(2))$. They form two complete lattices which are dually isomorphic to each other. It is routine work to prove that every pseudo-model class of type $(\tau,(2))$ is an ordered pseudovariety of type τ. For the opposite direction we may apply Theorem 3.3 from [6] which is formulated for admissible pseudovarieties of algebraic systems $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ of type (τ,τ'). Moreover, it is assumed that there are only finitely many fundamental relations, i.e. J is finite and that the pseudovariety V is admissible, meaning that $\gamma_j^{\mathcal{A}} \neq \emptyset$ for every $j \in J$ and every $\mathcal{A} \in \mathcal{V}$. Then $W \subseteq V$ is a sub-pseudovariety of V if and only if there exists a set $\mathcal{PF}$ of pseudoformulas of the form (i) from Definition 2.1 and of the form $\gamma_j(\pi_1,\ldots,\pi_{n_j})$ with a relation symbol γ_j and implicit operations $\pi_1,\ldots,\pi_{n_j}$ over V such that $W = PSMPSF(W)$. The satisfaction of $\gamma_j(\pi_1,\ldots,\pi_{n_j})$ in an algebraic system $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$ is defined by

$$\mathcal{A} \underset{p.s.}{\models} \gamma_j(\pi_1,\ldots,\pi_{n_j}) :\Leftrightarrow \gamma_j^{\mathcal{A}}((\pi_1)_{\mathcal{A}},\ldots,(\pi_{n_j})_{\mathcal{A}}) \text{ is true in } \mathcal{A}.$$

This generalizes (ii) from Definition 2.2. Using this result we have the following Birkhoff-type-characterization of ordered pseudovarieties.

Theorem 2.3. *A class $\mathcal{K} \subseteq Alg^{\leq}_{fin}(\tau)$ is an ordered pseudovariety of type τ if and only if $\mathcal{K} = PSMPSF(\mathcal{K})$.*

We remark that in Definition 2.1 and in Definition 2.2 the ordered pseudovariety can be replaced by a class V of algebraic systems of type (τ,τ') which is closed under taking of homomorphisms, subsystems, and finite direct products of algebraic systems (see [1]). Definition 2.1(ii) can be replaced by

(ii′) If $\pi_1,\ldots,\pi_{n_j}$ are n-ary implicit operations on V, and if γ_j is an n_j-ary relation symbol, then $\gamma_j(\pi_1,\ldots,\pi_{n_j})$ is an n-ary pseudoformula of type (τ,τ') on V.

Definition 2.2(ii) will be replaced by

(ii′) If PF has the form $\gamma_j(\pi_1,\ldots,\pi_{n_j})$ for the n_j-ary relational symbol γ_j and implicit operations $\pi_1,\ldots,\pi_{n_j}$, then

$$\mathcal{A} \underset{p.s.}{\models} \gamma_j(\pi_1,\ldots,\pi_{n_j}) :\Leftrightarrow \gamma_j^{\mathcal{A}}((\pi_1)_{\mathcal{A}},\ldots,(\pi_{n_j})_{\mathcal{A}}) \text{ is true in } \mathcal{A}.$$

If τ' is a finite type of relations and $\gamma_j^{\mathcal{A}} \neq \emptyset$ for all $j \in J$, Theorem 3.3 from [6] can be applied also in this case. Let $Algsys_{fin}(\tau,\tau')$ be the class of all finite algebraic systems of type (τ,τ').

Theorem 2.4. *A class $\mathcal{K} \subseteq Algsys^{\leq}_{fin}(\tau,\tau')$ of finite algebraic systems with a finite set of relations and such that $\gamma_j^{\mathcal{A}} \neq \emptyset$ for all $j \in J$ and all $\mathcal{A} \in Algsys^{\leq}_{fin}(\tau,\tau')$ is closed under homomorphisms, subsystems and finite direct products of algebraic systems if and only if $\mathcal{K} = PSMPSF(\mathcal{K})$.*

3. Hyper-pseudoformulas

Hypersubstitutions of type τ are introduced in [10] with the aim to define hyperidentities, i.e. identities which are defined for algebras of the corresponding type in the stronger sense that they are valid after substituting the occurring operation symbols by terms (see [4] and [7]). Let $(f_i)_{i\in I}$ be an indexed set of operation symbols and let $(\gamma_j)_{j\in J}$ be an indexed set of relation symbols where $n_i \in N \setminus \{0\}$ is the arity of f_i and $n_j \in N \setminus \{0\}$ is the arity of γ_j. Then the set $W_\tau(X_n)$ of all n-ary terms of type τ is defined using an n-element set $X_n = \{x_1,\ldots,x_n\}$ of individual variables in the usual way saying that each $x_i \in X_n$ is an n-ary term and if $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ. Then the set of all n-ary formulas $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ of type (τ,τ') is defined in a similar way as we defined pseudoformulas in Definition 2.1, with the difference that in (i) we take terms t_1, t_2 and obtain an identity $t_1 \approx t_2$ and in (ii) we define n-ary formulas $\gamma_j(t_1,\ldots,t_{n_j})$ using n-ary terms $t_1,\ldots,t_{n_j}$ and an n_j-ary relation symbol γ_j. The result is the set $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ of all n-ary formulas of type (τ,τ') and the set $\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup\limits_{n\geq 1} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ of all formulas of type (τ,τ'). Let $p_1 : W_\tau(X)^2 \to W_\tau(X)$ be the first projection defined on $W_\tau(X)^2$. In [12] hypersubstitutions for algebraic systems were defined in the following way:

Definition 3.1. Any mapping $\sigma : \{f_i | i \in I\} \cup \{\gamma_j | j \in J\} \to W_\tau(X) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ which maps operation symbols to terms preserving arities and relation symbols to formulas preserving arities is called a hypersubstitution for algebraic systems of type (τ,τ'). Let $Hyprel(\tau,\tau')$ be the set of all hypersubstitutions for algebraic systems of type (τ,τ').

Let $Hyprel(\tau,\tau')$ be the collection of all hypersubstitutions for algebraic systems of type (τ,τ'). We defined the extension $\widehat{\sigma} : W_\tau(X)\cup\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) \to W_\tau(X)\cup\mathcal{F}_{(\tau,\tau')}(W_\tau(X))$ and proved that the set $Hyprel(\tau,\tau')$ together with a binary operation $\circ_r$ defined by $\sigma_1 \circ_r \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, and the hypersubstitution σ_{id} mapping f_i to the term $f_i(x_1,\ldots,x_{n_i}), i \in I$ and the relation symbols γ_l to the formula $\gamma_l(x_1,\ldots,x_{n_l})$, $l \in J$ forms the monoid $(Hyprel(\tau,\tau'),\circ_r,\sigma_{id})$.

Let $V_\leq$ be an ordered pseudovariety of type τ. As for pseudovarieties on the set $\overline{\Omega}_n V_\leq$ one can define a topology such that $\overline{\Omega}_n V_\leq$ equipped with operations $f_i^{\overline{\Omega}_n V_\leq}, i \in I$, defined by

$$f_i^{\overline{\Omega}_n V_\leq}(\pi_1,\ldots,\pi_{n_i})_{\mathcal{A}}(a_1,\ldots,a_n) := f_i^{\mathcal{A}^\leq}((\pi_1)_{\mathcal{A}^\leq}(a_1,\ldots,a_n),\ldots,(\pi_{n_j})_{\mathcal{A}^\leq}(a_1,\ldots,a_n))$$

for all $a_1,\ldots,a_n \in A$ becomes a Hausdorff topological algebra which is compact and totally disconnected. This algebra can be regarded as an ordered algebra if we add the partial order $\triangle_{\overline{\Omega}_n V_\leq}$. Then for every $\pi \in \overline{\Omega}_n V_\leq$ there exists a sequence $((t_k^{\mathcal{A}})_{\mathcal{A}\in V})_{k\geq 1}$ where t_k are terms of type τ such that $\pi = \lim\limits_{k\to\infty}(t_k^{\mathcal{A}})_{\mathcal{A}\in V}$. Let $Hyp(\tau)$ be the set of all usual hypersubstitutions, and let $\circ_h$ denote the product of usual hypersubstitutions. Then for every hypersubstitution $\sigma_H \in Hyp(\tau)$ as in [11] a mapping $\bar{\sigma}_H : \overline{\Omega}_n V_\leq \to \overline{\Omega}_n V_\leq$ can be defined by $\bar{\sigma}_H(\pi) := \lim\limits_{k\to\infty}(\hat{\sigma}_H[t_k]^{\mathcal{A}})_{\mathcal{A}\in V}$. This mapping will be used to define hypersubstitutions for pseudoformulas of type $(\tau,(2))$.

Definition 3.2. Let $V_\leq$ be an ordered pseudovariety of type τ and $\mathcal{A}^\leq \in V_\leq$. Let $\sigma \in Hyprel(\tau,(2))$. Then we define a mapping $\sigma^* : \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_\leq) \to \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_\leq)$ inductively as follows:

(i) $\sigma^*[\pi_1 \approx \pi_2] := \bar{\sigma}_H(\pi_1) \approx \bar{\sigma}_H(\pi_2)$,
(ii) $\sigma^*[\gamma(\pi_1,\pi_2)] := \gamma(\bar{\sigma}_H(\pi_1),\bar{\sigma}_H(\pi_2))$,
(iii) $\sigma^*[\neg(PF)] := \neg(\sigma^*[PF])$,
(iv) $\sigma^*[PF_1 \vee PF_2] := \sigma^*[PF_1] \vee \sigma^*[PF_2]$,
(v) $\sigma^*[\exists x_i(PF)] := \exists x_i(\sigma^*[PF])$.

This definition can be generalized to classes of finite algebraic systems closed under homomorphic images, subsystems and finite direct products of algebraic systems.

Definition 3.3. Let $V_\leq$ be an ordered pseudovariety of type τ and $\mathcal{A}^\leq \in V_\leq$. A pseudoformula PF in $V_\leq$ is said to be satisfied as a hyper-pseudoformula in $\mathcal{A}^\leq$ if $\bar{\sigma}_H(\pi)$ exists for all implicit operations occurring in

PF and if $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF]$ for all $\sigma \in Hyprel(\tau,(2))$. In this case we write $\mathcal{A}^{\leq} \underset{h.p.s.}{\models} PF$.

This hypersatifaction relation $\underset{h.p.s.}{\models}$ defines a second Galois-connection $(HPSM, HPSF)$ with

$$HPSM : \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})) \to \mathcal{P}(Alg^{\leq}_{fin}(\tau)) \quad \text{and}$$
$$HPSF : \mathcal{P}(Alg^{\leq}_{fin}(\tau)) \to \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq}))$$

defined by

$$HPSM(\mathcal{PF}) := \{\mathcal{A}^{\leq} \in Alg^{\leq}_{fin}(\tau) \mid \forall PF \in \mathcal{PF}\ (\mathcal{A}^{\leq} \underset{h.p.s.}{\models} PF)\} \quad \text{and}$$
$$HPSF(\mathcal{K}) := \{PF \in \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq}) \mid \forall\ \mathcal{A}^{\leq} \in \mathcal{K}\ (\mathcal{A}^{\leq} \underset{h.p.s.}{\models} PF)\},$$

respectively for $\mathcal{PF} \subseteq \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})$ and $\mathcal{K} \subseteq Alg^{\leq}_{fin}(\tau)$. If instead of all hypersubstitutions, we use hypersubstitutions from a submonoid $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ we obtain a relation $\underset{Mh.p.s.}{\models}$ and a similar Galois-connection $(H_M PSM, H_M PSF)$.

For the extension of hypersubstitutions from $Hyprel(\tau,(2))$ to pseudoformulas we have:

Lemma 3.4. *Let $\sigma_1, \sigma_2 \in Hyprel(\tau,(2))$. Then $(\sigma_1 \circ_r \sigma_2)^* = \sigma_1^* \circ \sigma_2^*$.*

Proof. We will give a proof by induction following the inductive definition of a pseudoformula PF of type $(\tau,(2))$.

(i) If PF is a pseudoidentity $\pi \approx \rho$, then by Definition 3.2(i),

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)^*[\pi \approx \rho] &= \overline{(\sigma_1 \circ_r \sigma_2)}_H(\pi) \approx \overline{(\sigma_1 \circ_r \sigma_2)}_H(\rho)\\ &= (\overline{(\sigma_1)_H \circ_h (\sigma_2)}_H)(\pi) \approx (\overline{(\sigma_1)_H \circ_h (\sigma_2)}_H)(\rho) \text{ by Lemma 3.4 in [12]}\\ &= ((\overline{\sigma}_1)_H \circ (\overline{\sigma}_2)_H)(\pi) \approx ((\overline{\sigma}_1)_H \circ (\overline{\sigma}_2)_H)(\rho) \quad \text{by Lemma 2.7 in [11]}\\ &= \sigma_1^*[(\overline{\sigma}_2)_H(\pi)] \approx \sigma_1^*[(\overline{\sigma}_2)_H(\rho)]\\ &= {\sigma_1}^*[(\overline{\sigma}_2)_H(\pi) \approx (\overline{\sigma}_2)_H(\rho)]\\ &= {\sigma_1}^*[{\sigma_2}^*[\pi \approx \rho]] = ({\sigma_1}^* \circ {\sigma_2}^*)[\pi \approx \rho].\end{aligned}$$

(ii) If PF has the form $\gamma(\pi, \rho)$, then

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)^*[\gamma(\pi,\rho)] &= \gamma(\overline{(\sigma_1 \circ_r \sigma_2)}_H(\pi), \overline{(\sigma_1 \circ_r \sigma_2)}_H(\rho)) \\ &= \gamma((\overline{(\sigma_1)_H \circ_h (\sigma_2)}_H)(\pi), (\overline{(\sigma_1)_H \circ_h (\sigma_2)}_H)(\rho)) \\ &= \gamma(((\overline{\sigma}_1)_H \circ (\overline{\sigma}_2)_H)(\pi), ((\overline{\sigma}_1)_H \circ (\overline{\sigma}_2)_H)(\rho)) \\ &= \gamma((\overline{\sigma}_1)_H((\overline{\sigma}_2)_H(\pi)]), (\overline{\sigma}_1)_H((\overline{\sigma}_2)_H(\rho))) \\ &= {\sigma_1}^*[\gamma((\overline{\sigma}_2)_H(\pi), (\overline{\sigma}_2)_H(\rho))] \\ &= {\sigma_1}^*[{\sigma_2}^*[\gamma(\pi,\rho)]] = ({\sigma_1}^* \circ {\sigma_2}^*)[\gamma(\pi,\rho)].\end{aligned}$$

(iii) If the pseudoformula has the form $\neg(PF)$ for a pseudoformula PF and assume that $(\sigma_1 \circ_r \sigma_2)^*[PF] = ({\sigma_1}^* \circ {\sigma_2}^*)[PF]$. Then

$$\begin{aligned}(\sigma_1 \circ_r \sigma_2)^*[\neg(PF)] &= \neg(\sigma_1 \circ_r \sigma_2)^*[PF] \quad \text{by Definition 3.2(iii)} \\ &= \neg(({\sigma_1}^* \circ {\sigma_2}^*)[PF]) \\ &= \neg({\sigma_1}^*[{\sigma_2}^*[PF]] \\ &= {\sigma_1}^*[\neg({\sigma_2}^*[PF])] \quad \text{by Definition 3.2(iii)} \\ &= {\sigma_1}^*[{\sigma_2}^*[\neg(PF)]] \quad \text{by Definition 3.2(iii)} \\ &= ({\sigma_1}^* \circ {\sigma_2}^*)[\neg(PF)].\end{aligned}$$

In cases (iv) and (v) we proceed in a similar way. ∎

In a similar way for the identity element $\sigma_{id} \in Hyprel(\tau,(2))$ we can show that $\sigma^*_{id}[PF] = PF$ for all pseudoformulas $PF \in \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})$.

4. M-solid ordered pseudovarieties

In this section we define M-solid ordered pseudovarieties as pseudovarieties of finite ordered algebras, which are closed under taking of so-called derived ordered algebras, and prove a characterization theorem for those closures. M-solid ordered pseudovarieties were used in [9] to give an Eilenberg-type characterization of M-solid positive varieties of tree languages. Let $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ be a monoid of hypersubstitutions for algebraic systems.

For a set $\mathcal{PF} \subseteq \mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})$ of pseudoformulas of type $(\tau,(2))$ we define an operator

$$\chi_M^{PF} : \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})) \to \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq}))$$

by

$$\chi_M^{PF}(\mathcal{PF}) := \{\sigma^*[PF] \mid \sigma \in M \text{ and } PF \in \mathcal{PF}\}.$$

Clearly, $\chi_M^{PF}(\mathcal{PF}) := \bigcup_{PF \in \mathcal{PF}} \chi_M^{PF}(\{PF\})$, i.e. χ_M^{PF} is completely additive.

Lemma 4.1. *For every submonoid $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ the operator χ_M^{PF} has the properties of a completely additive closure operator.*

Proof. Because of $\sigma_{id}^*[PF] = PF$ for every pseudoformula $PF \in \mathcal{PF}$, the operator χ_M^{PF} is extensive. We noticed already that χ_M^{PF} is completely additive. This implies monotonicity, i.e. $\mathcal{PF}_1 \subseteq \mathcal{PF}_2$ implies $\chi_M^{PF}(\mathcal{PF}_1) \subseteq \chi_M^{PF}(\mathcal{PF}_2)$. Extensivity gives $\chi_M^{PF}(\mathcal{PF}) \subseteq \chi_M^{PF}(\chi_M^{PF}(\mathcal{PF}))$. Conversely, let $PF_1 \in \chi_M^{PF}(\chi_M^{PF}(\mathcal{PF}))$. Then there are $\sigma_1, \sigma_2 \in M \subseteq Hyprel(\tau,(2))$ and $PF_2 \in \mathcal{PF}$ such that $PF_1 = \sigma_1^*[\sigma_2^*[PF_2]]$. By Lemma 3.4, there is a hypersubstitution in $M \subseteq Hyprel(\tau,(2))$, namely $\sigma_1 \circ_r \sigma_2$ such that $PF_1 = (\sigma_1 \circ_r \sigma_2)^*[PF_2] = \sigma_1^*[\sigma_2^*[PF_2]]$. Therefore, $PF_1 \in \chi_M^{PF}(\mathcal{PF})$ and then $\chi_M^{PF}(\chi_M^{PF}(\mathcal{PF})) \subseteq \chi_M^{PF}(\mathcal{PF})$. Altogether, we have equality. ∎

For an ordered algebra $\mathcal{A}^{\leq} = (A; (f_i^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$ and $\sigma \in Hyprel(\tau,(2))$ we define $\sigma(\mathcal{A}^{\leq}) := (A; (\sigma_H(f_i)^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$. The following observation shows that we obtain again an ordered algebra. By induction on the complexity of the term $\sigma_H(f_i)$ we show that $\sigma_H(f_i)^{\mathcal{A}}$ preserves the partial order $\leq_{\mathcal{A}}$. Let $\sigma_H(f_i) = x_j, 1 \leq j \leq n_i$, and assume that $a_1 \leq_{\mathcal{A}} b_1, \ldots, a_{n_i} \leq_{\mathcal{A}} b_{n_i}$. Then $x_j^{\mathcal{A}}(a_1,\ldots,a_{n_i}) = e_j^{n_i,\mathcal{A}}(a_1,\ldots,a_{n_i}) = a_j \leq_{\mathcal{A}} b_j = e_j^{n_i,\mathcal{A}}(b_1,\ldots,b_{n_i}) = x_j^{\mathcal{A}}(b_1,\ldots,b_{n_i})$. Assume now that $\sigma_H(f_i) = f_l(t_1,\ldots,t_{n_l})$ and that the term operations $t_1^{\mathcal{A}},\ldots,t_{n_l}^{\mathcal{A}}$ preserve $\leq_{\mathcal{A}}$. Then we get $f_l(t_1,\ldots,t_{n_l})^{\mathcal{A}}(a_1,\ldots,a_{n_l}) = f_l^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1,\ldots,a_{n_l}),\ldots,t_{n_l}^{\mathcal{A}}(a_1,\ldots,a_{n_l})) \leq_{\mathcal{A}} = f_l^{\mathcal{A}}(t_1^{\mathcal{A}}(b_1,\ldots,b_{n_l}),\ldots,t_{n_l}^{\mathcal{A}}(b_1,\ldots,b_{n_l})) = f_l(t_1,\ldots,t_{n_l})^{\mathcal{A}}(b_1,\ldots,b_{n_l})$ since by hypothesis $t_k^{\mathcal{A}}(a_1,\ldots,a_{n_l}) \leq_{\mathcal{A}} t_k^{\mathcal{A}}(b_1,\ldots,b_{n_l})$ for $1 \leq k \leq n_l$, and since $f_l^{\mathcal{A}}$ preserves the partial order $\leq_{\mathcal{A}}$. Let $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ be any submoniod.

Then we define:

Definition 4.2. An ordered pseudovariety $V_{\leq}$ is said to be M-solid if $\sigma(\mathcal{A}^{\leq}) \in V_{\leq}$ for all $\sigma \in M$ and all $\mathcal{A}^{\leq} \in V_{\leq}$. For $\mathcal{M} = \mathcal{R}elhyp(\tau,(2))$ we call an M-solid ordered pseudovariety solid.

Remarks. In [13] we defined for an algebraic system $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{i\in I})$ and for any $\sigma \in Hyprel_{\mathcal{A}}(\tau,\tau')$ the derived algebraic system $\sigma(\mathcal{A})$ by $\sigma(\mathcal{A}) := (A; (\sigma_F(f_i)^{\mathcal{A}})_{i\in I}, (\sigma_R(\gamma_j)^{\mathcal{A}})_{j\in J})$. Here $\sigma \in Hyprel_{\mathcal{A}}(\tau,\tau')$ is defined as a pair (σ_F,σ_R) where $\sigma_F : \{f_i^{\mathcal{A}} | \ i \in I\} \to T(\mathcal{A})$ maps fundamental operations to term operations preserving arities and σ_R maps fundamental relations to elements of the relational algebra generated by $\{\gamma_j^{\mathcal{A}} | \ j \in J\}$. If the component σ_R is the identity mapping on $\{\gamma_j^{\mathcal{A}} | \ j \in J\}$ we obtain a submonoid of $Hyprel_{\mathcal{A}}(\tau,\tau')$.

For $\mathcal{K} \subseteq Alg_{fin}^{\leq}(\tau)$ and $\sigma \in Hyprel(\tau,(2))$ we define a mapping $\chi^{OA} : \mathcal{P}(Alg_{fin}^{\leq}(\tau)) \to \mathcal{P}(Alg_{fin}^{\leq}(\tau))$ by $\chi^{OA}(\mathcal{K}) := \{\sigma(\mathcal{A}^{\leq}) \mid \sigma \in Hyprel(\tau,(2)),$

$\mathcal{A}^{\leq} \in \mathcal{K}\}$. Clearly, $\chi^{OA}(\mathcal{K}) = \bigcup_{\mathcal{A}^{\leq}\in\mathcal{K}} \chi^{OA}(\mathcal{A}^{\leq})$, i.e. the operator χ^{OA} is completely additive. For a submonoid $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ we obtain a corresponding operator $\chi_M^{OA}(\mathcal{K})$ defined by $\chi_M^{OA}(\mathcal{K}) := \{\sigma(\mathcal{A}^{\leq}) \mid \sigma \in M, \mathcal{A}^{\leq} \in \mathcal{K}\}$. Then we get

Lemma 4.3. *For any submonoid $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ the operator χ_M^{OA} has the properties of a completely additive closure operator.*

Proof. Monotonicity of the operator χ_M^{OA} follows from additivity. Let σ_{id} be the identity element of the monoid $\mathcal{M}$. Then $\sigma_{id}(\mathcal{A}^{\leq}) = (A;((\sigma_{id})_H(f_i)^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}}) = (A;(f_i^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$. As a consequence, the operator χ_M^{OA} is completely additive. The inclusion $\chi_M^{OA}(\mathcal{K}) \subseteq \chi_M^{OA}(\chi_M^{OA}(\mathcal{K}))$ follows from extensivity. If $\mathcal{A}^{\leq} \in \chi_M^{OA}(\chi_M^{OA}(\mathcal{K}))$, then there are $\sigma_1, \sigma_2 \in M \subseteq Hyprel(\tau,(2))$ and there is a finite ordered algebra $\mathcal{A}'^{\leq} \in \mathcal{K}$ such that $\mathcal{A}^{\leq} = \sigma_1(\sigma_2(\mathcal{A}'^{\leq})) = (A;((\hat{\sigma}_2)_H((\sigma_1)_H(f_i)^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}}) = (A;((\sigma_2 \circ_r \sigma_1)_H(f_i)^{\mathcal{A}})_{i\in I}, \leq_{\mathcal{A}})$. Since by Lemma 3.4 in [12] $(\sigma_1 \circ_r \sigma_2)_H = (\sigma_1)_H \circ_h (\sigma_2)_H$ and since $(\sigma_1 \circ_r \sigma_2)_H = (\sigma_1)_H \circ_h (\sigma_2)_H$ we obtain a hypersubstitution $\sigma_1 \circ_r \sigma_2 \in M \subseteq Hyprel(\tau,(2))$ with $\mathcal{A}^{\leq} = (\sigma_1 \circ_r \sigma_2)(\mathcal{A}'^{\leq})$. This finishes the proof. ∎

For the ordered pseudovariety $V_{\leq}$ the pseudo-satisfaction relation $\underset{p.s.}{\models}$ relates finite ordered algebras $\mathcal{A}^{\leq}$ with sets of pseudoformulas of type $(\tau,(2))$ and defines a Galois-connection (PSM, PSF) with

$$PSM : \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq}) \to \mathcal{P}(Alg_{fin}^{\leq}(\tau)) \quad \text{and}$$
$$PSF : \mathcal{P}(Alg_{fin}^{\leq}(\tau)) \to \mathcal{P}(\mathcal{PF}_{(\tau,(2))}\overline{\Omega}V_{\leq}).$$

A second Galois-connection can be defined by using the M-hyper-pseudo-satisfaction relation $\underset{h.p.s.}{\models}$. This gives $(H_M PSM, H_M PSF)$ for any submonoid $\mathcal{M} \subseteq \mathcal{H}yprel(\tau,(2))$ with

$$H_M PSM : \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})) \to \mathcal{P}(Alg_{fin}^{\leq}(\tau)) \quad \text{and}$$
$$H_M PSF : \mathcal{P}(Alg_{fin}^{\leq}(\tau)) \to \mathcal{P}(\mathcal{PF}_{(\tau,(2))}(\overline{\Omega}V_{\leq})).$$

Moreover, we defined two additive closure operators χ_M^{OA} and χ_M^{PF}. Four more closure operators are defined by the products $PSMPSF, PSFPSM$, $H_M PSMH_M PSF$, and $H_M PSFH_M PSM$. The fixed points under these closure operators form complete lattices. We will now prove that the closure operators, χ_M^{OA} and χ_M^{PF} form a so-called conjugate pair (see [7]). Then we can apply the general theory of conjugate pairs of additive closure operators which describes the relationships between the 6 complete lattices. We need the following lemmas.

Lemma 4.4. *Let $V_{\leq}$ be an M-solid ordered pseudovariety of type τ, let $\sigma \in M \subseteq Hyprel(\tau, (2))$ and let $\pi \in \overline{\Omega}_n V_{\leq}$ for some $n \in N^+$. Then $(\overline{\sigma}_H(\pi))_{\mathcal{A}^{\leq}} = \pi_{\sigma(\mathcal{A}^{\leq})}$.*

Proof. Let V be the class of all finite algebras $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ where $\mathcal{A}^{\leq}$ belongs to $V_{\leq}$ and let V' be the pseudovariety generated by V. It is well-known that for every implicit operation π on V' and for every finite algebra $\mathcal{A} \in V'$ there is a term $_{\mathcal{A}}t$ such that π is the sequence $((_{\mathcal{A}}t)^{\mathcal{A}})_{\mathcal{A} \in V'}$. (see e.g. Lemma 5.1.1 in [2]). Using this result and the definition of an implicit operation (Definition 1.2) we have $\pi_{\mathcal{A}} = (_{\mathcal{A}}t)^{\mathcal{A}}$ for every $\mathcal{A} \in V'$ and therefore for every algebra $\mathcal{A} \in V \subseteq V'$. Since the term operations of $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ and of $\mathcal{A}^{\leq} = (A; (f_i^{\mathcal{A}})_{i \in I}, \leq_{\mathcal{A}})$ agree, this holds also for ordered algebras from $V_{\leq}$. This gives for $\sigma(\mathcal{A})$ the equation $\pi_{\sigma(\mathcal{A})} = (_{\sigma(\mathcal{A})}t)^{\sigma(\mathcal{A})}$ with a term $_{\sigma(\mathcal{A})}t \in W_\tau(X_n)$. Applying the theory of conjugate pairs of additive closure operators for terms (see [7]) one has $(_{\sigma(\mathcal{A})}t)^{\sigma(\mathcal{A})} = (\hat{\sigma}[_{\sigma(\mathcal{A})}t])^{\mathcal{A}}$. Then by Proposition 5.2.3 from [2] we get $(\hat{\sigma}_H[_{\sigma(\mathcal{A})}t])^{\mathcal{A}} = (\overline{\sigma}_H(\pi))_{\mathcal{A}}$. ∎

Lemma 4.5. *Let $\mathcal{A}^{\leq} = (A; (f_i^{\mathcal{A}})_{i \in I}, \leq_{\mathcal{A}})$ be a finite ordered algebra of type τ and $V_{\leq}$ be an ordered pseudovariety of type τ, $\mathcal{A}^{\leq} \in V_{\leq}$. Then for each $PF \in \mathcal{PF}_{(\tau,(2))}(\overline{\Omega} V_{\leq})$ and each $\sigma \in Hyprel(\tau, (2))$ we have*

$$\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF] \;\Leftrightarrow\; \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF.$$

Proof. We will give a proof by induction on the definition of a pseudoformula.

(i) If PF has the form $\pi \approx \rho$, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[\pi \approx \rho]$

$$\begin{array}{lll}
\Leftrightarrow & \mathcal{A}^{\leq} \underset{p.s.}{\models} \overline{\sigma}_H(\pi) \approx \overline{\sigma}_H(\rho) & \text{by Definition 3.2(i)} \\
\Leftrightarrow & (\overline{\sigma}_H(\pi))_{\mathcal{A}^{\leq}} = (\overline{\sigma}_H(\rho))_{\mathcal{A}^{\leq}} & \text{by Definition 2.2(i)} \\
\Leftrightarrow & \pi_{\sigma(\mathcal{A}^{\leq})} = \rho_{\sigma(\mathcal{A}^{\leq})} & \text{by Lemma 4.4} \\
\Leftrightarrow & \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} \pi \approx \rho & \text{by Definition 2.2(i).}
\end{array}$$

(ii) If PF has the form $\gamma(\pi, \rho)$, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[\gamma(\pi, \rho)]$

$$\begin{array}{lll}
\Leftrightarrow & \mathcal{A}^{\leq} \underset{p.s.}{\models} \gamma(\overline{\sigma}_H(\pi), \overline{\sigma}_H(\rho)) & \text{by Definition 3.2(ii)} \\
\Leftrightarrow & ((\overline{\sigma}_H(\pi))_{\mathcal{A}^{\leq}}, (\overline{\sigma}_H(\rho))_{\mathcal{A}^{\leq}}) \in \gamma^{\mathcal{A}^{\leq}} & \text{by Definition 2.2(ii)} \\
\Leftrightarrow & \pi_{\sigma(\mathcal{A}^{\leq})} \leq_{\mathcal{A}} \rho_{\sigma(\mathcal{A}^{\leq})} & \text{by Lemma 4.4} \\
\Leftrightarrow & \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} \gamma(\pi, \rho) & \text{by Definition 2.2(ii).}
\end{array}$$

(iii) If PF has the form $\neg(PF)$, and if we assume that $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF] \Leftrightarrow \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF$, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[\neg(PF)]$

$$\Leftrightarrow \quad \mathcal{A}^{\leq} \underset{p.s.}{\models} \neg(\sigma^*[PF]) \quad \text{by Definition 3.2(iii)}$$

$$\Leftrightarrow \quad \neg(\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF]) \quad \text{by Definition 2.2(iii)}$$

$$\Leftrightarrow \quad \neg(\sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF) \quad \text{by our presumption}$$

$$\Leftrightarrow \quad \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} \neg(PF) \quad \text{by Definition 2.2(iii).}$$

(iv) If PF has the form $PF_1 \vee PF_2$, and if we assume that $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF_j] \Leftrightarrow \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF_j$, $j = 1, 2$, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF_1 \vee PF_2]$

$$\Leftrightarrow \quad \mathcal{A}^{\leq} \underset{p.s.}{\models} (\sigma^*[PF_1] \vee \sigma^*[PF_2]) \quad \text{by Definition 3.2(iv)}$$

$$\Leftrightarrow \quad \mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF_1] \vee \mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF_2] \quad \text{by Definition 2.2(iv)}$$

$$\Leftrightarrow \quad \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF_1 \vee \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF_2 \quad \text{by our presumption}$$

$$\Leftrightarrow \quad \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF_1 \vee PF_2 \quad \text{by Definition 2.2(iv).}$$

(v) If PF has the form $\exists x_i(PF)$ and if we assume that $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF] \Leftrightarrow \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF$, then $\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[\exists x_i(PF)]$

$$\Leftrightarrow \quad \mathcal{A}^{\leq} \underset{p.s.}{\models} \exists x_i(\sigma^*[PF]) \quad \text{by Definition 3.2(v)}$$

$$\Leftrightarrow \quad \exists x_i(\mathcal{A}^{\leq} \underset{p.s.}{\models} \sigma^*[PF]) \quad \text{by Definition 2.2(v)}$$

$$\Leftrightarrow \quad \exists x_i(\sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} PF) \quad \text{by our presumption}$$

$$\Leftrightarrow \quad \sigma(\mathcal{A}^{\leq}) \underset{p.s.}{\models} \exists x_i(PF) \quad \text{by Definition 2.2(v).} \blacksquare$$

Now all conditions to apply the general theory of conjugate pairs of additive closure operators are satisfied, and we can characterize M-solid ordered pseudovarieties. To do this we apply the characterization theorem from [7]. Definition 4.2 means that an ordered pseudovariety is M-solid iff $\chi_M^{OA}[V_{\leq}] = V_{\leq}$.

Theorem 4.6. *Let $\mathcal{K} \subseteq Alg^{\leq}_{fin}(\tau)$ be a class of finite ordered algebras of type τ such that $\mathcal{K} = PSM(\mathcal{PF})$ for some set $\mathcal{PF}$ of pseudoformulas of* *type* $(\tau,(2))$, *i.e.* $\mathcal{K}$ *is an ordered pseudovariety. Then the following are equivalent*:

(i) $\mathcal{K} = H_M PSMH_M PSF(\mathcal{K})$,
(ii) $\chi_M^{OA}[\mathcal{K}] = \mathcal{K}$,
(iii) $PSF(\mathcal{K}) = H_M PSF(\mathcal{K})$,
(iv) $\chi_M^{PF}(PSF(\mathcal{K})) = PSF(\mathcal{K})$.

Dually the following propositions (i′), (ii′), (iii′) *and* (iv′) *are also pairwise equivalent*:

(i′) $\mathcal{PF} = H_M PSFH_M PSM(\mathcal{PF})$,
(ii′) $\chi_M^{PF}[\mathcal{PF}] = \mathcal{PF}$,
(iii′) $PSM(\mathcal{PF}) = H_M PSM(\mathcal{PF})$,
(iv′) $\chi_M^{OA}[PSM(\mathcal{PF})] = PSM(\mathcal{PF})$.

The equivalence (ii)⇔(iii) means that $\mathcal{K}$ is M-solid iff every pseudoformula is satisfied as a hyper-pseudoformula. Theorem 4.6 describes the relationships between the complete lattices defined by the fixed points of the closure operators mentioned before Lemma 4.4. From the theory of conjugate pairs of additive closure operators we obtain also the following theorem (see [7]):

Theorem 4.7. *For all* $\mathcal{K} \subseteq Alg_{fin}^{\leq}(\tau)$ *and all* $\mathcal{PF} \subseteq \mathcal{PF}_{\tau,(2)}(\overline{\Omega}V_{\leq})$ *the following properties hold:*

(i) $H_M PSF(\mathcal{K}) = PSF(\chi_M^{OA}[\mathcal{K}])$,
(ii) $H_M PSF(\mathcal{K}) \subseteq PSF(\mathcal{K})$,
(iii) $\chi_M^{PF}[H_M PSF(\mathcal{K})] = H_M PSF(\mathcal{K})$,
(iv) $\chi_M^{OA}[PSM(H_M PSF(\mathcal{K}))] = PSM(H_M PSF(\mathcal{K}))$,
(v) $H_M PSF(H_M PSM(\mathcal{PF})) = PSF(PSM(\chi_M^{PF}(\mathcal{PF})))$, *and dually*
(i′) $H_M PSM(\mathcal{PF}) = PSM(\chi_M^{PF}[\mathcal{PF}])$,
(ii′) $H_M PSM(\mathcal{PF}) \subseteq PSM(\mathcal{PF})$,
(iii′) $\chi_M^{OA}[H_M PSM(\mathcal{PF})] = H_M PSM(\mathcal{PF})$,
(iv′) $\chi_M^{PF}[PSF(H_M PSM(\mathcal{PF}))] = PSF(H_M PSM(\mathcal{PF}))$,
(v′) $H_M PSM(H_M PSF(\mathcal{K})) = PSM(PSF(\chi_M^{OA}[\mathcal{K}]))$.

As a consequence of Theorem 4.6 and Theorem 4.7 we obtain:

Corollary 4.8. *For every* $\mathcal{K} \subseteq Alg_{fin}^{\leq}(\tau)$ *with* $\mathcal{K} = PSM(\mathcal{PF})$ *for some set* $\mathcal{PF}$ *of pseudoformulas of type* $(\tau,(2))$, $\mathcal{K}$ *is an* M*-solid ordered pseudovariety iff* $\mathcal{K} = PSM(\chi_M^{PF}[\mathcal{PF}])$.

Proof. "$\Rightarrow$": Assume that $\mathcal{K}$ is M-solid. Then $\chi_M^{OA}[\mathcal{K}] = \mathcal{K}$ and then $\chi_M^{OA}[PSM(\mathcal{PF})] = \mathcal{K}$. From Theorem 4.6 (iii′) there follows $\mathcal{K} = \chi_M^{OA}[H_M PSM(\mathcal{PF})]$ and from Theorem 4.7 (iii′) we get $\mathcal{K} = H_M PSM(\mathcal{PF})$. Now (i′) from Theorem 4.7 gives $\mathcal{K} = PSM(\chi_M^{PF}[\mathcal{PF}])$.

"$\Leftarrow$": Assume that $\mathcal{K} = PSM(\chi_M^{PF}[\mathcal{PF}])$. Then by Theorem 4.7 (i′) we have that $\mathcal{K} = PSM(\mathcal{PF}) = H_M PSM(\mathcal{PF})$. By Theorem 4.6, this is equivalent to $\chi_M^{OA}[\mathcal{K}] = \chi_M^{OA}[PSM(\mathcal{PF}] = PSM(\mathcal{PF}) = \mathcal{K}$ and $\mathcal{K}$ is M-solid. ∎

Corollary 4.8 means that for checking whether an ordered pseudovariety which is given as the pseudo-model class of a set of pseudoformulas, is M-solid, it is enough to check whether these pseudoformulas are satisfied as M-hyper-pseudoformulas.

Our results show that also in the case of hyper-pseudoformulas and M-solid ordered pseudovarieties the theory of conjugate pairs of additive closure operators may be applied to get a characterization of those classes of finite algebraic systems.

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