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POLYNOMIALLY REPRESENTABLE SEMIRINGS

Abstract. We characterize semirings which can be represented by an algebra of binary polynomials of the form $a \cdot x + y$ where the operations are compositions of functions. Furthermore, we classify, which algebras with two binary and two nullary operations (satisfying some natural identities) can be represented in this way, and how these algebras are related to semirings.

Several attempts have been done to represent a given algebra by means of functions of a special sort. The best known is the so-called Cayley Theorem which states that every group G can be represented as a group S_G of permutations of the support of G; here the group operation in S_G is the usual composition of functions. In fact, the same representation exists for an arbitrary monoid. It was shown by S. L. Bloom, Z. Ésik and E. G. Manes that every Boolean algebra can be represented by certain binary functions, so-called guard functions, where the operations are function compositions, see [1] and [2]. A similar approach was used by the first author [3] for a representation of so-called q-algebras.

Using another sort of binary functions, it was shown that also distributive lattices (see [6]) or bounded lattices with an antitone involution (see [4]) can be represented by functions where the corresponding operations are expressed via function compositions. Later on, this approach was applied for so-called action algebras [7] and it was generalized for algebras having an arbitrary number of binary operations [5]. Other Cayley-like representations are contained in [11] (Ch. 3, Theorem 1.51) and in particular in [10], where one can find general methods for creating such representations and further references.

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The aim of this paper is to characterize semirings for which the representation by binary polynomials can be realized via a given assignment. Moreover, the study is extended for algebras with two binary and two nullary operations.

In what follows, we will consider algebras $\mathcal{A} = (A; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$. We use the notational convention that the operation $\cdot$ has a higher priority than $+$, i.e., we will omit brackets concerning $\cdot$ and write e.g. $x \cdot y + z$ instead of $(x \cdot y) + z$. For $a \in A$ we define a binary polynomial function $f_a(x, y)$ on A by

$$f_a(x, y) = a \cdot x + y.$$

Denote by $F(A)$ the set $\{f_a(x, y); a \in A\}$. On $F(A)$ we introduce the following function compositions:

$$(f_a \oplus f_b)(x, y) = f_a(x, f_b(x, y)),$$
$$(f_a \circ f_b)(x, y) = f_a(f_b(x, y), y).$$

The algebra

$$\mathcal{F}(\mathcal{A}) = (F(A); \oplus, \circ, f_0, f_1)$$

will be called the *function algebra assigned to* $\mathcal{A}$.

If we assume that $\mathcal{A} = (A; +, \cdot, 0, 1)$ satisfies the identities $1 \cdot x = x$, $0 \cdot x = 0$ and $0 + x = x$ then clearly $f_0(x, y) = y$ and $f_1(x, y) = x + y$.

The following definition is taken from [8].

Definition 1. A *semiring* is an algebra $\mathcal{A} = (A; +, \cdot, 0, 1)$ such that

(i) $+$ is associative and commutative,
(ii) $\cdot$ is associative,
(iii) $+$ and $\cdot$ satisfy the left- and right-distributive laws

$$x \cdot (y + z) = x \cdot y + x \cdot z,$$
$$(x + y) \cdot z = x \cdot z + y \cdot z,$$

(iv) $+$ and $\cdot$ satisfy the idenities $x + 0 = x$,

$$x \cdot 1 = x = 1 \cdot x,$$
$$x \cdot 0 = 0 = 0 \cdot x.$$

A semiring $\mathcal{A}$ is called *simple* (see [8] and [9]) if it satisfies the identity

$$x + 1 = 1.$$

In what follows, by φ will be denoted the mapping from A into $F(A)$ definded by

$$\varphi(a) = f_a.$$

At the first glance, φ is surjective.

Our aim is to show that a semiring $\mathcal{A}$ is simple if and only if φ is an isomorphism from $\mathcal{A}$ onto $\mathcal{F}(\mathcal{A})$. In fact, we will prove a more general result and get the semiring case as a corollary.

Lemma 1. *Let $\mathcal{A} = (A; +, \cdot, 0, 1)$ be an algebra of type $(2, 2, 0, 0)$ satisfying $x \cdot 1 = x$ and $x + 0 = x$. Then φ is a bijection from A to $F(A)$.*

Proof. For $a, b, \in A$, $a \neq b$, we have

$$f_a(1, 0) = a \cdot 1 + 0 = a \neq b = b \cdot 1 + 0 = f_b(1, 0),$$

thus $f_a \neq f_b$, i.e., φ is injective. ■

Lemma 2. *Let $\mathcal{A}$ be an algebra of type $(2, 2, 0, 0)$, then φ is a homomorphism from $\mathcal{A}$ to $\mathcal{F}(\mathcal{A})$ if and only if for all $a, b, x, y, \in A$:*

(1) $a \cdot x + (b \cdot x + y) = (a + b) \cdot x + y$ *and*
(2) $a \cdot (b \cdot x + y) + y = (a \cdot b) \cdot x + y$.

Proof. φ is a homomorphism if and only if $f_a \oplus f_b = f_{a+b}$ and $f_a \circ f_b = f_{a \cdot b}$. The first equation is equivalent to (1), and the second one is equivalent to (2). ■

Lemma 3. *Let $\mathcal{A}$ be an algebra of type $(2, 2, 0, 0)$, suppose that $+$ and $\cdot$ are associative and $\mathcal{A}$ satisfies*

(3) $(x + y) \cdot z = x \cdot z + y \cdot z$ *and*
(4) $x \cdot (y + z) + z = x \cdot y + z$.

Then φ is a homomorphism. (Note that (3) is the right distributive law.)

Proof. We have

$$a \cdot x + (b \cdot x + y) = (a \cdot x + b \cdot x) + y = (a + b) \cdot x + y$$

and

$$a \cdot (b \cdot x + y) + y = a \cdot (b \cdot x) + y = (a \cdot b) \cdot x + y,$$

i.e., (1) and (2) hold. ■

Now we are ready to state our main result:

Theorem 1. *Let $\mathcal{A}= (A; +, \cdot, 0, 1)$ be an algebra of type $(2, 2, 0, 0)$ satisfying $x \cdot 1 = x$ and $x + 0 = x$. Then the mapping $\varphi : A \to F(A)$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{F}(\mathcal{A})$ if and only if $+$ and $\cdot$ are associative and $\mathcal{A}$ satisfies (3) and (4).*

Proof. The "if" part follows from Lemma 1 and 3. In order to show the "only if" part, we use Lemma 2.

For $x = 1$ in (1), we have

$$a + (b + y) = a \cdot 1 + (b \cdot 1 + y) = (a + b) \cdot 1 + y = (a + b) + y,$$

i.e., $+$ is associative.

For $y = 0$ in (1), we have

$$a \cdot x + b \cdot x = a \cdot x + (b \cdot x + 0) = (a + b) \cdot x + 0 = (a + b) \cdot x,$$

i.e., (3) holds.

For $y = 0$ in (2), we have

$$a \cdot (b \cdot x) = a \cdot (b \cdot x + 0) + 0 = (a \cdot b) \cdot x + 0 = (a \cdot b) \cdot x,$$

i.e., $\cdot$ is associative.

For $x = 1$ in (2), we have

$$a \cdot (b + y) + y = a \cdot (b \cdot 1 + y) + y = (a \cdot b) \cdot 1 + y = (a \cdot b) + y,$$

i.e., (4) holds. ∎

Lemma 4. *Let $\mathcal{A}$ be an algebra of type* $(2, 2, 0, 0)$.

(a) *Suppose that $\mathcal{A}$ satisfies* (4), $0 + x = x$, $x \cdot 0 = 0$ *and* $x \cdot 1 = x$, *then $\mathcal{A}$ satisfies* $x + 1 = 1$.
(b) *If $+$ is associative, $\mathcal{A}$ satisfies the right distributive law* (3), *the left distributive law* $x \cdot (y + z) = x \cdot y + x \cdot z$, $1 \cdot x = x$ *and* $x + 1 = 1$, *then $\mathcal{A}$ satisfies* (4).

Proof. (a) Taking $y = 0$ in (4), we get

$$x \cdot z + z = x \cdot (0 + z) + z = x \cdot 0 + z = 0 + z = z.$$

Taking $z = 1$ yields $x \cdot 1 + 1 = 1$, i.e., $x + 1 = 1$.

(b) We have

$$\begin{aligned} x \cdot (y + z) + z &= (x \cdot y + x \cdot z) + 1 \cdot z = x \cdot y + (x \cdot z + 1 \cdot z) \\ &= x \cdot y + (x + 1) \cdot z = x \cdot y + 1 \cdot z = x \cdot y + z, \end{aligned}$$

i.e., (4). ∎

From Theorem 1 and Lemma 4 we get the

Corollary. *Let $\mathcal{A}$ be a semiring. Then φ is an isomorphism from $\mathcal{A}$ onto $\mathcal{F}(\mathcal{A})$ if and only if $\mathcal{A}$ is simple* (*i.e., $\mathcal{A}$ satisfies* $x + 1 = 1$).

Due to Lemma 4, the question arises if the identity (4) together with associativity of $+$ and the identities $0 + x = x$, $x \cdot 0 = 0$ and $x \cdot 1 = x$ implies left-distributivity. The following example shows that it is not the case.

Example. Consider a three-element set $A = \{0, a, 1\}$ and the algebra $\mathcal{A} = (A; +, \cdot, 0, 1)$ where the operations $+$ and $\cdot$ are determinded by the tables:

$+$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

$\cdot$	0	a	1
0	0	a	0
a	0	a	a
1	0	a	1

It is an easy exercise to verify that $\mathcal{A}$ satisfies all the identities of Theorem 1 and, furthermore, $x + y = y + x$, $x + x = x$, $x \cdot x = x$, $x + 1 = 1$, $1 \cdot x = x$, $x \cdot 0 = 0$. However, $\mathcal{A}$ does not satisfy left-distributivity since e.g.

$$0 \cdot (1 + a) = 0 \cdot 1 = 0 \neq a = 0 + a = 0 \cdot 1 + 0 \cdot a.$$

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