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CENTRALIZER CLONES ARE PRESERVED BY CATEGORY EQUIVALENCES

Abstract. In this paper it will be proved that if a clone C is category equivalent to a centralizer clone then C is a centralizer clone.

1. Notations and preliminaries

Let A be a finite set and let n be a positive integer. The set of all n-ary operations on A is denoted by $O_A^{(n)}$ and $O_A := \bigcup_{n\geq 1} O_A^{(n)}$. The set of all n-ary relations on A is denoted by $Rel_A^{(n)}$ and $Rel_A := \bigcup_{n\geq 1} Rel_A^{(n)}$. For $1 \leq i \leq n$, the *n-ary i-th projection* is defined as $e_i^{n,A}(a_1, \ldots, a_n) = a_i$ for all $a_1, \ldots, a_n \in A$. For $f \in O_A^{(n)}$ and $g_1, \ldots, g_n \in O_A^{(m)}$, we define their composition to be the m-ary operation $f(g_1, \ldots, g_n)$ defined by $f(g_1, \ldots, g_n)(a_1, \ldots, a_m) = f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$ for all $a_1, \ldots, a_m \in A$.

A *clone* on A is a set of operations defined on A which contains all projections and is closed under composition. It is well-known that the intersection of an arbitrary set of clones on A is a clone on A. Thus for $F \subseteq O_A$, there is the least clone containing F, called the clone *generated* by F and it is denoted by $\langle F\rangle$.

Let $f \in O_A^{(n)}$ and $\rho \in Rel_A^{(h)}$. The operation f *preserves* ρ if $(f(a_1^1, \ldots, a_1^n), \ldots, f(a_h^1, \ldots, a_h^n)) \in \rho$ whenever $(a_1^1, \ldots, a_h^1), \ldots, (a_1^n, \ldots, a_h^n) \in \rho$.

Let $F \subseteq O_A$ and $R \subseteq Rel_A$. The set of all operations on A which preserve all relations in R is denoted by $Pol_A R$. We usually abbreviate $Pol_A R$ to $Pol R$ if there is no danger of misunderstanding. The set of all relations on A which are preserved by all operations in F is denoted by $Inv_A F$. We usually abbreviate $Inv_A F$ to $Inv F$ if there is no danger of misunderstanding.

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The pair (Pol, Inv) forms a Galois-connection between sets O_A and Rel_A (see [5]).

On the set Rel_A one can defined operations $\varsigma^A, \tau^A, \Delta^A, \circ^A, \delta_A^{\{1;2,3\}}$ and obtain algebra $\mathbf{Rel_A} = (Rel_A; \varsigma^A, \tau^A, \Delta^A, \circ^A, \delta_A^{\{1;2,3\}})$ of type $(1,1,1,2,0)$. Any subalgebra of this algebra is called a *relation algebra* (see [5]).

Let $V(\mathbf{A})$ be the variety generated by an algebra $\mathbf{A}$ and let $T(\mathbf{A})$ be the clone generated by the set of all fundamental operations of $\mathbf{A}$. In [2] and [3] the authors defined category equivalences of clones and characterized category equivalences of clones by isomorphisms of some relation algebras.

Definition 1.1. ([2]) A clone C on a set A is *category equivalent* to a clone C' on a set B if there are algebras $\mathbf{A}$ and $\mathbf{B}$ with universes A and B, respectively such that $C = T(\mathbf{A})$, $C' = T(\mathbf{B})$ and $V(\mathbf{A})$ and $V(\mathbf{B})$ are category equivalent.

Theorem 1.2. ([3]) *Two clones C and C' on finite sets A and B, respectively are category equivalent if and only if the relation algebras* $\mathbf{Inv_A C}$ *and* $\mathbf{Inv_B C'}$ *are isomorphic.*

Now we recall some concepts on centralizer clones.

For each n-ary operation f on A, the *graph* of f is an $(n+1)$-ary relation $f^\bullet := \{(a_1, \dots, a_n, y) \mid a_1, \dots, a_n, y \in A \text{ and } y = f(a_1, \dots, a_n)\}$. The concept of a graph can be naturally extended to a set F of operations by $F^\bullet := \{f^\bullet \mid f \in F\}$.

Definition 1.3. A clone C on A is a *centralizer clone* (*primitive positive clone*) if there is a subset $F \subseteq O_A$ such that $C = PolF^\bullet$.

For a survey on properties of centralizer clones see [4].

We are interested in properties of clones which are preserved under category equivalences, for instant we have:

Theorem 1.4. ([2]) *Let $Pol_A\rho$ be a maximal clone on a finite set A ($|A| > 1$) and let C be a clone on a finite set B ($|B| > 1$). If C is category equivalent to $Pol_A\rho$, then C is a maximal clone in the same class as $Pol_A\rho$.*

In [1], the author has shown that there are only three classes of maximal clones such that each clone in these classes is a centralizer clone.

Theorem 1.5. ([1]) *A maximal clone $Pol\rho$ on a finite set A ($|A| > 1$) is a centralizer clone if and only if ρ is a graph of unary constant function, or a graph of prime permutation, or a graph of linear operation.*

2. The main result

Theorem 1.4 and Theorem 1.5 imply that if a clone C is category equivalent to a maximal centralizer clone then C is a maximal centralizer clone. Therefore we claim that if a clone C is category equivalent to a centralizer clone then C is a centralizer clone. To prove this conjecture we need some properties of homomorphisms between relation algebras.

Lemma 2.1. *Let* $\mathbf{R_A} = (R_A; \varsigma^A, \tau^A, \Delta^A, \circ^A, \delta_A^{\{1;2,3\}})$ *and* $\mathbf{R_B} = (R_B; \varsigma^B, \tau^B, \Delta^B, \circ^B, \delta_B^{\{1;2,3\}})$ *be relation algebras where* $R_A \subseteq Rel_A$ *and* $R_B \subseteq Rel_B$ *and let* φ *be a homomorphism from* $\mathbf{R_A}$ *to* $\mathbf{R_B}$. *If* $\rho \in R_A$ *is the graph of an* n*-ary operation on* A, *then* $\varphi(\rho)$ *is the graph of an* n*-ary operation on* B.

Finally, we use Theorem 1.2 and Lemma 2.1 to prove the following theorem.

Theorem 2.2. *Let* C *and* C' *be clones on finite sets* A *and* B, *respectively where* C *is a centralizer clone. If* C' *is category equivalent to* C, *then* C' *is a centralizer clone.*

Proof. Since C is a centralizer clone on A, there is a set F of operations on A such that $C = PolF^{\bullet}$. Since C' is category equivalent to C and by Theorem 1.2, there is an isomorphism φ from a relation algebra **InvC** onto a relation algebra **InvC'**. Then

$$InvC' = \varphi(InvC) = \varphi(InvPolF^{\bullet}) = \varphi(\langle F^{\bullet}\rangle) = \langle \varphi(F^{\bullet})\rangle = InvPol\varphi(F^{\bullet}).$$

We have

$$C' = PolInvC' = PolInvPol\varphi(F^{\bullet}) = Pol\varphi(F^{\bullet})$$

and

$$\varphi(F^{\bullet}) = \{\varphi(\rho) | \rho \in F^{\bullet}\}$$

is a set of graphs of operations on B. These imply that C' is a centralizer clone on B. ∎

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