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WEAK HOMOMORPHISMS BETWEEN FUNCTORIAL ALGEBRAS

Abstract. In universal algebra, homomorphisms are usually considered between algebras of the same similarity type. Different from that, the notion of a weak homomorphism, as introduced by E. Marczewski in 1961, does not depend on a signature, but only on the clones of term operations generated by the examined algebras. We generalize this idea by defining weak homomorphisms between F_1- and F_2-algebras, where F_1 and F_2 denote not necessarily equal endofunctors of the category of sets. The aim is to show that, in many respects, weak homomorphisms behave very similarly to proper homomorphisms–without restricting the scope of considerations by the necessity of a common type. For instance, concerning a set $\mathcal{F}$ of **Set**-endofunctors that weakly preserve kernels, the class of all algebras of types from $\mathcal{F}$ equipped with the class of all weak homomorphisms between these algebras forms a category which admits a canonical factorization structure for morphisms. Furthermore, we treat two product constructions from which the notion of a weak homomorphism naturally arises.

Introduction

In universal-algebraic considerations, the elements of an investigated collection of algebras are usually required to have a fixed common signature and so the notion of a homomorphism is defined only for those situations. However, this kind of restriction is rather unnecessary in a lot of investigations in which, as in the theory of completeness, the term operations of an examined algebra play the essential role.

As introduced by E. Marczewski in [9], a mapping $\varphi : A \to B$ is said to be a *weak homomorphism* from a non-indexed universal algebra $\mathcal{A} = (A, F)$ to another one $\mathcal{B} = (B, G)$ if, for each n-ary fundamental operation $f \in F$ ($n \in \mathbb{N}$), there exists an n-ary term operation g of $\mathcal{B}$ such that

$$\varphi(f(a_1, \dots, a_n)) = g(\varphi(a_1), \dots, \varphi(a_n))$$

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holds for all $(a_1, \ldots, a_n) \in A^n$ and vice versa, for every n-ary fundamental operation $g \in G$ ($n \in \mathbb{N}$), there is an n-ary term operation f of $\mathcal{A}$ satisfying the same condition. Those weak homomorphisms, in particular weak endo- and automorphisms, were investigated under various aspects, especially by K. Głazek. (For more details we refer to [3]–[7].)

Furthermore, there is a well-known and very fruitful category-theoretic generalization of universal algebra: Concerning a **Set**-endofunctor F, an F-algebra is an ordered pair $\mathcal{A} = (A, \alpha)$ consisting of some set A and a map $\alpha : F(A) \to A$. A lot of results from universal algebra have already been proven to be still valid in this much more abstract situation. (More information can be found in [2], for instance.) In the present paper, we want to contribute to this development by introducing the notion of a weak homomorphism between differently typed functorial algebras. In the case of universal algebras, our definition will coincide with E. Marczewski's concept. Moreover, in many respects, weak homomorphisms behave like usual homomorphisms: For example, it turns out that kernels of weak homomorphisms are congruence relations and that weakly homomorphic images and preimages of subalgebras are subalgebras, too. In this work, we will show that, concerning a set $\mathcal{F}$ of **Set**-endofunctors which weakly preserve kernels, the category $\mathbf{Set}^{\mathcal{F}}$ consisting of all algebras of types from $\mathcal{F}$ as objects and the weak homomorphisms between them as morphisms admits a canonical factorization structure for morphisms, which is quite interesting for the investigation of certain reflexive subcategories of $\mathbf{Set}^{\mathcal{F}}$. Finally, we will present two suggesting ways to construct special products of functorial algebras whereby the canonical projections become weak homomorphisms.

The sum of these results substantiates that the introduced concept of weak homomorphisms between differently typed functorial algebras is, in fact, a useful and promising idea.

1. Basic notions and notations

We assume that the reader is familiar with the basics of category theory. The category we will deal with is **Set**, the category of sets. In this section, we want to address some notational issues and sum up just a few essential properties of functorial algebras, which will be used in further considerations.

Notations 1.1. Let X and Y be arbitrary sets, $S \subseteq X$, $T \subseteq Y$ and $\varphi : X \to Y$. The *graph of* φ is given by $\varphi^\bullet := \{(x, \varphi(x)) \mid x \in X\}$, the *kernel of* φ by $\ker \varphi := \{(x_1, x_2) \in X \times X \mid \varphi(x_1) = \varphi(x_2)\}$, the *image of* S *under* φ by $\varphi[S] := \{\varphi(s) \mid s \in S\}$ and the *preimage of* T *under* φ by $\varphi^{-1}[T] := \{x \in X \mid \varphi(x) \in T\}$. Furthermore, we define $\iota_S^X : S \to X : s \mapsto s$, $\dot{\varphi} : X \to \varphi[X] : x \mapsto \varphi(x)$ and $\varphi|_S := \varphi \circ \iota_S^X : S \to Y$.

The following lemma is well-known to be equivalent to the Axiom of Choice.

Lemma 1.2. ([8]) *Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor and $\varphi : X \to Y$ a surjective map. Then $F\varphi : F(X) \to F(Y)$ is surjective, too.*

In this work, we want to assume the Axiom of Choice or, equivalently, the validity of 1.2. However, as a concluding remark, we will discuss how to replace the assumption of the Axiom of Choice by requiring some additional property of the functors under consideration.

Throughout the present paper, $F, F_1, \dots, F_4$ will denote arbitrary endofunctors of the category of sets, i.e. functors from **Set** to itself.

Definition 1.3. ([2]) A *(functorial) algebra of type F* or an *F-algebra* is an ordered pair $\mathcal{A} = (A, \alpha)$ consisting of a set A and a map $\alpha : F(A) \to A$, where A is called the *carrier* and α the *structure map of $\mathcal{A}$*.

Definition 1.4. ([2]) Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be algebras of type F and consider a map $\varphi : A \to B$. We say that φ is a *homomorphism from $\mathcal{A}$ to $\mathcal{B}$* or $\varphi : \mathcal{A} \to \mathcal{B}$ is a *homomorphism*, respectively, if $\varphi \circ \alpha = \beta \circ F\varphi$ holds, i.e. the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F\varphi} & F(B) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\varphi} & B \end{array}$$

commutes.

In the rest of this section, we will repeat some well-known results whose proofs are not difficult and can be found in [2]. In particular, whereas the statements (i) and (ii) of the following proposition imply that, given a **Set**-endofunctor F, the class of all F-algebras equipped with the class of all homomorphisms between these structures forms a category, which we want to denote by $\mathbf{Set}^F$, from (iii) we can deduce that the isomorphisms in $\mathbf{Set}^F$ are precisely the bijective homomorphisms.

Proposition 1.5. ([2]) *Let $\mathcal{A} = (A, \alpha)$, $\mathcal{B} = (B, \beta)$ and $\mathcal{C} = (C, \gamma)$ be algebras of type F, $\varphi : A \to B$ and $\psi : B \to C$. Then the following statements hold:*

(i) $\mathrm{id}_A : \mathcal{A} \to \mathcal{A}$ *is a homomorphism.*
(ii) *If $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are homomorphisms, then $\psi \circ \varphi : \mathcal{A} \to \mathcal{C}$ is a homomorphism, too.*
(iii) *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a bijective homomorphism, then $\varphi^{-1} : \mathcal{B} \to \mathcal{A}$ is a homomorphism.*

(iv) *If* $\psi\circ\varphi : \mathcal{A} \to \mathcal{C}$ *and* $\varphi : \mathcal{A} \to \mathcal{B}$ *are homomorphisms and* φ *is surjective, then* ψ *is a homomorphism from* $\mathcal{B}$ *into* $\mathcal{C}$*.*
(v) *If* $\psi \circ \varphi : \mathcal{A} \to \mathcal{C}$ *and* $\psi : \mathcal{B} \to \mathcal{C}$ *are homomorphisms and* ψ *is injective, then* φ *is a homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$*.*

Note that 1.2 is needed to prove 1.5(iv).

Definition 1.6. ([2]) Let $\mathcal{A} = (A, \alpha)$ be an algebra of type F. Then we define

$$\mathrm{Sub}_F(\mathcal{A}) := \{S \subseteq A \mid \exists \sigma \in S^{F(S)} : \iota_S^A \circ \sigma = \alpha \circ F(\iota_S^A)\}.$$

A subset $S \subseteq A$ is called *closed in* $\mathcal{A}$ if $S \in \mathrm{Sub}_F(\mathcal{A})$, i.e. if there exists a structure map $\sigma : F(S) \to S$ such that the canonical injection ι_S^A is a homomorphism from $\mathcal{S} = (S, \sigma)$ to $\mathcal{A}$. In this case, $\mathcal{S}$ is said to be a *subalgebra of* $\mathcal{A}$.

Definition 1.7. ([2]) Let $(\mathcal{A}_i)_{i\in I}$ be a family of F-algebras, where $\mathcal{A}_i = (A_i, \alpha_i)$ for $i \in I$. The *direct product of the family* $(\mathcal{A}_i)_{i\in I}$ is defined to be the F-algebra

$$\Pi_{j\in I}\mathcal{A}_j := (\Pi_{j\in I}A_j, \gamma),$$

where $\gamma : F(\Pi_{j\in I}A_j) \to \Pi_{j\in I}A_j$ is the unique map such that for each $i \in I$ the projection $\pi_i : \Pi_{j\in I}A_j \to A_i$ is a homomorphism from $(\Pi_{j\in I}A_j, \gamma)$ into $\mathcal{A}_i$. Furthermore, for a set I and F-algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ we define $\mathcal{A}^I := \Pi_{j\in I}\mathcal{A}_j$ with $\mathcal{A}_i := \mathcal{A}$ for each $i \in I$ and $\mathcal{B} \times \mathcal{C} := \Pi_{j\in J}\mathcal{D}_j$ with $J := \{0, 1\}$, $\mathcal{D}_0 := \mathcal{B}$, $\mathcal{D}_1 := \mathcal{C}$.

Since we will introduce weak homomorphisms between functorial algebras of different types as mappings equipped with suitable structures on their epi-mono-factorization in **Set**, the next statement is essential for all further considerations.

Proposition 1.8. ([2]) *Let* $\mathcal{A} = (A, \alpha)$ *and* $\mathcal{B} = (B, \beta)$ *be* F*-algebras and* $\varphi : \mathcal{A} \to \mathcal{B}$ *a homomorphism. Consider a factorization* $\varphi = \psi \circ \pi$ *of* φ *into a surjective map* $\pi : A \to Q$ *and an injective map* $\psi : Q \to B$ *for some set* Q*. Then there is a unique structure map* $\gamma : F(Q) \to Q$ *which makes* π *a homomorphism from* $\mathcal{A}$ *to* $\mathcal{Q} = (Q, \gamma)$*. Additionally,* ψ *is a homomorphism from* $\mathcal{Q}$ *to* $\mathcal{B}$*.*

For later use, we want to recall the following result, too.

Proposition 1.9. ([2]) *Consider algebras* $\mathcal{A} = (A, \alpha)$ *and* $\mathcal{B} = (B, \beta)$ *of type* F *and a homomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$*. Then the following statements hold:*

(i) *For each* $U \in \mathrm{Sub}_F(\mathcal{A})$*, we have* $\varphi[U] \in \mathrm{Sub}_F(\mathcal{B})$*.*
(ii) *For each* $U \in \mathrm{Sub}_F(\mathcal{B})$*, we have* $\varphi^{-1}[U] \in \mathrm{Sub}_F(\mathcal{A})$*.*

Whereas the first statement of 1.9 is a simple consequence of 1.8, the second one can be proven by describing preimages as pullbacks.

2. Algebraic equivalence

In this section, the second most important concept of the present work, namely the notion of *algebraically equivalent algebras*, is introduced and will be related to some structural properties. But in the first instance, we will investigate two existence problems concerning usual homomorphisms. Since some constructions will necessitate that the concerned functors transform kernels into weak kernels, we start with the following definition.

Definition 2.1. Let F be an endofunctor of **Set**. F *weakly preserves kernels* if F transforms kernels into weak kernels, i.e. for every map $\varphi : A \to B$ the following condition is fulfilled: Whenever P is a set with mappings $\psi_1, \psi_2 : P \to F(A)$ such that $F\varphi \circ \psi_1 = F\varphi \circ \psi_2$, then there exists a (not necessarily unique) map $\sigma : P \to F(\ker \varphi)$ satisfying $\psi_i = F\pi_i \circ \sigma$ for each $i \in \{1, 2\}$, where $\pi_1, \pi_2 : \ker \varphi \to A$ denote the canonical projections.

Examples 2.2.

(i) Let $\underline{\Omega} = (\Omega, \mathrm{ar})$ be an algebraic type, i.e. a set Ω equipped with a function $\mathrm{ar} : \Omega \to \mathbb{N}$. Define the functor $F_{\underline{\Omega}} : \mathbf{Set} \to \mathbf{Set}$ by

$$F_{\underline{\Omega}}(X) := \bigcup_{\omega\in\Omega} \{\omega\} \times X^{\mathrm{ar}(\omega)}$$

for every set X and $F_{\underline{\Omega}}\varphi : F_{\underline{\Omega}}(X) \to F_{\underline{\Omega}}(Y)$ with

$$(F_{\underline{\Omega}}\varphi)(\omega, (x_1, \ldots, x_{\mathrm{ar}(\omega)})) := (\omega, (\varphi(x_1), \ldots, \varphi(x_{\mathrm{ar}(\omega)})))$$

for every map $\varphi : X \to Y$. Then $F_{\underline{\Omega}}$ weakly preserves kernels.

(ii) The power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ does not preserve kernels. However, $\mathcal{P}$ weakly preserves kernels.

Weak preservation of kernels is of certain interest for our purposes: Given an algebra $\mathcal{A} = (A, \alpha)$ of type F and a map $\varphi : A \to B$, we would like to know whether there exists a suitable structure map $\beta : F(B) \to B$ such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B, \beta)$. In case F weakly preserves kernels, it is possible to describe this problem in terms of subalgebras of direct products. This will be the content of 2.4. In order to verify 2.4, we want to make use of the following lemma.

Lemma 2.3. *Let $(\mathcal{A}_i)_{i\in I}$ be a family of F-algebras, where $\mathcal{A}_i = (A_i, \alpha_i)$ and $\pi_i : \Pi_{j\in I}A_j \to A_i$ denote*s *the canonical projection for $i \in I$. Moreover, let $S \subseteq \Pi_{j\in I}A_j$ and $\sigma : F(S) \to S$. Then the following are equivalent:*

(i) *(S, σ) is a subalgebra of $\Pi_{j\in I}\mathcal{A}_j$.*

(ii) *For each $i \in I$, the restricted projection $\pi_i|_S : S \to A_i$ is a homomorphism from (S, σ) to $\mathcal{A}_i$.*

The proof of 2.3 is a standard application of the universal product property in **Set**.

Lemma 2.4. *Let $\mathcal{A} = (A, \alpha)$ be an algebra of type F and $\varphi : A \to B$. Then the following statements hold:*

(i) *If there exists a map $\beta : F(B) \to B$ such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B, \beta)$, then $\ker \varphi$ is closed in $\mathcal{A} \times \mathcal{A}$.*
(ii) *Let F weakly preserve kernels. If $\ker \varphi$ is closed in $\mathcal{A} \times \mathcal{A}$, then there exists a map $\beta : F(B) \to B$ such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B, \beta)$.*

Proof. Let $\pi_1, \pi_2 : \ker \varphi \to A$ and $\pi_1^*, \pi_2^* : \ker(F\varphi) \to F(A)$ denote the canonical projections.

(i) If $\beta : F(B) \to B$ is a map such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B, \beta)$, then a straightforward application of the fact that $(\ker \varphi, \pi_1, \pi_2)$ is a pullback of φ with itself in **Set** yields the existence of a (unique) map $\gamma : F(\ker \varphi) \to \ker \varphi$ satisfying $\pi_1 \circ \gamma = \alpha \circ F\pi_1$ and $\pi_2 \circ \gamma = \alpha \circ F\pi_2$. By 2.3, $\ker \varphi$ is closed in $\mathcal{A} \times \mathcal{A}$.

(ii) Conversely, consider some $\gamma : F(\ker \varphi) \to \ker \varphi$ such that $(\ker \varphi, \gamma)$ is a subalgebra of $\mathcal{A} \times \mathcal{A}$. Since F weakly preserves kernels and it holds $F\varphi \circ \pi_1^* = F\varphi \circ \pi_2^*$, there is a mapping $\tau : \ker(F\varphi) \to F(\ker \varphi)$ such that $\pi_1^* = F\pi_1 \circ \tau$ and $\pi_2^* = F\pi_2 \circ \tau$. As, for each pair $(a_1, a_2) \in \ker(F\varphi)$, it follows

$$\begin{aligned}(\varphi \circ \alpha)(a_1) &= (\varphi \circ \alpha \circ \pi_1^*)(a_1, a_2) = (\varphi \circ \alpha \circ F\pi_1 \circ \tau)(a_1, a_2) \\ &= (\varphi \circ \pi_1 \circ \gamma \circ \tau)(a_1, a_2) = (\varphi \circ \pi_2 \circ \gamma \circ \tau)(a_1, a_2) \\ &= (\varphi \circ \alpha \circ F\pi_2 \circ \tau)(a_1, a_2) = (\varphi \circ \alpha \circ \pi_2^*)(a_1, a_2) = (\varphi \circ \alpha)(a_2),\end{aligned}$$

we have $\ker(F\varphi) \subseteq \ker(\varphi \circ \alpha)$ and hence there exists a map $\beta : F(B) \to B$ satisfying $(\beta|_{(F\varphi)[F(A)]})^\bullet = \{((F\varphi)(a), \varphi(\alpha(a))) \mid a \in F(A)\}$. Consequently, we have $\varphi \circ \alpha = \beta \circ F\varphi$. ∎

The next lemma almost immediately follows from the fact that, for an injective map $\varphi : A \to B$, the map $\dot{\varphi} : A \to \varphi[A]$ is a bijection.

Lemma 2.5. *Let $\mathcal{B} = (B, \beta)$ be an algebra of type F and $\varphi : A \to B$ injective. There exists a structure map $\alpha : F(A) \to A$ such that φ is a homomorphism from $\mathcal{A} = (A, \alpha)$ to $\mathcal{B}$ if and only if $\varphi[A]$ is closed in $\mathcal{B}$.*

There is no adequate generalization of universal-algebraic *term functions* relating to arbitrary F-algebras. However, from [10] we know that two universal algebras with a common carrier are *term equivalent*, i.e. they generate the same term functions, if and only if their infinitary invariants coincide. This characterization is captured by the following definition.

Definition 2.6. Let $\mathcal{A}_1 = (A, \alpha_1)$ be an F_1-algebra and $\mathcal{A}_2 = (A, \alpha_2)$ an F_2-algebra on a common carrier A. Then we call $\mathcal{A}_1$ and $\mathcal{A}_2$ *algebraically equivalent* and write $\mathcal{A}_1{}^{F_1}{\equiv}^{F_2}\mathcal{A}_2$ if, for each set I, it holds

$$\mathrm{Sub}_{F_1}(\mathcal{A}_1^I) = \mathrm{Sub}_{F_2}(\mathcal{A}_2^I).$$

As one might expect, the notion of algebraically equivalent structures is compatible with homomorphic images and substructures.

Lemma 2.7. *Let $\mathcal{A}_1 = (A, \alpha_1)$ and $\mathcal{B}_1 = (B, \beta_1)$ be algebras of type F_1, $\mathcal{A}_2 = (A, \alpha_2)$ and $\mathcal{B}_2 = (B, \beta_2)$ algebras of type F_2. If there exists a surjective map $\varphi : A \twoheadrightarrow B$ which is a homomorphism from $\mathcal{A}_1$ to $\mathcal{B}_1$ as well as from $\mathcal{A}_2$ to $\mathcal{B}_2$, then it holds*

$$\mathcal{A}_1{}^{F_1}{\equiv}^{F_2}\mathcal{A}_2 \implies \mathcal{B}_1{}^{F_1}{\equiv}^{F_2}\mathcal{B}_2.$$

Proof. Let I denote an arbitrary set. Due to the Axiom of Choice, the map

$$\varphi^I : A^I \to B^I : (a_i)_{i\in I} \mapsto (\varphi(a_i))_{i\in I}$$

is surjective, and a simple computation shows that φ^I is a homomorphism from $\mathcal{A}_1^I$ to $\mathcal{B}_1^I$ as well as from $\mathcal{A}_2^I$ to $\mathcal{B}_2^I$. Consider some $R \in \mathrm{Sub}_{F_1}(\mathcal{B}_1^I)$. According to 1.9(ii), $(\varphi^I)^{-1}[R]$ is closed in $\mathcal{A}_1^I$. Assuming that $\mathcal{A}_1{}^{F_1}{\equiv}^{F_2}\mathcal{A}_2$, $(\varphi^I)^{-1}[R]$ is closed in $\mathcal{A}_2^I$, too. As a consequence of the surjectivity of φ^I and 1.9(i), we have $R = \varphi^I[(\varphi^I)^{-1}[R]] \in \mathrm{Sub}_{F_2}(\mathcal{B}_2^I)$. Therefore, it holds $\mathrm{Sub}_{F_1}(\mathcal{B}_1^I) \subseteq \mathrm{Sub}_{F_2}(\mathcal{B}_2^I)$. By symmetry, we obtain $\mathrm{Sub}_{F_2}(\mathcal{B}_2^I) \subseteq \mathrm{Sub}_{F_1}(\mathcal{B}_1^I)$, and hence $\mathcal{B}_1{}^{F_1}{\equiv}^{F_2}\mathcal{B}_2$. ■

Lemma 2.8. *Let $\mathcal{A}_1 = (A, \alpha_1)$ and $\mathcal{B}_1 = (B, \beta_1)$ be F_1-algebras, $\mathcal{A}_2 = (A, \alpha_2)$ and $\mathcal{B}_2 = (B, \beta_2)$ F_2-algebras. If there exists an injective map $\varphi : A \hookrightarrow B$ which is a homomorphism from $\mathcal{A}_1$ to $\mathcal{B}_1$ as well as from $\mathcal{A}_2$ to $\mathcal{B}_2$, then it holds*

$$\mathcal{B}_1{}^{F_1}{\equiv}^{F_2}\mathcal{B}_2 \implies \mathcal{A}_1{}^{F_1}{\equiv}^{F_2}\mathcal{A}_2.$$

The proof of 2.8 proceeds analogously to that of 2.7. Of course, whereas the former of the previous two lemmata necessitates the assumption of the Axiom of Choice, the latter does not.

3. Weak homomorphisms

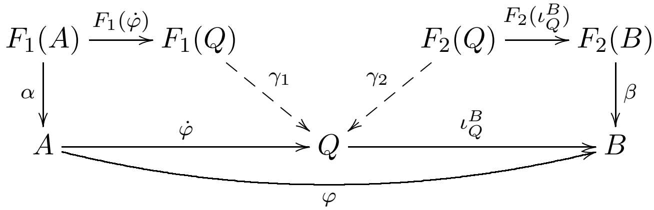
Now, everything is prepared to define and investigate weak homomorphisms between functorial algebras. Once more, we need to have a closer look at the universal algebraic situation, in particular, to reformulate E. Marczewski's definition of a weak homomorphism. Let $\mathcal{A} = (A, (f_\omega)_{\omega\in\Omega})$ and $\mathcal{B} = (B, (g_{\omega'})_{\omega'\in\Omega'})$ be two universal algebras (not necessarily of the same similarity type) and $\varphi : A \to B$. Then it is straightforward to verify that φ is a weak homomorphism from $\mathcal{A}$ to $\mathcal{B}$ in the sense of E. Marczewski if and only if there exist families $(f_\omega^*)_{\omega\in\Omega}$ and $(g_{\omega'}^*)_{\omega'\in\Omega'}$ of finitary operations on $\varphi[A]$ such that

(a) $\dot\varphi$ is a homomorphism from $\mathcal{A}$ to $(\varphi[A],(f_\omega^*)_{\omega\in\Omega})$,
(b) $\iota_{\varphi[A]}^B$ is a homomorphism from $(\varphi[A],(g_{\omega'}^*)_{\omega'\in\Omega'})$ to $\mathcal{B}$,
(c) $(\varphi[A],(f_\omega^*)_{\omega\in\Omega})$ and $(\varphi[A],(g_{\omega'}^*)_{\omega'\in\Omega'})$ are term equivalent.

This characterization enables us to generalize the notion of a weak homomorphism to the level of arbitrary functorial algebras, where we have to use usual homomorphisms and algebraic equivalence. In the following definition, (i) corresponds to (c), whereas (ii) is equivalent to the conjunction of (a) and (b) for the universal-algebraic case.

Definition 3.1. Let $\mathcal{A}=(A,\alpha)$ be an F_1-algebra and $\mathcal{B}=(B,\beta)$ an F_2-algebra and consider a map $\varphi: A\to B$. We say that φ is a *weak homomorphism from $\mathcal{A}$ to $\mathcal{B}$* or $\varphi:\mathcal{A}\to\mathcal{B}$ is a *weak homomorphism*, respectively, if, on the carrier $Q:=\varphi[A]$, there exist an F_1-algebra $\mathcal{Q}_1=(Q,\gamma_1)$ and an F_2-algebra $\mathcal{Q}_2=(Q,\gamma_2)$ such that the following conditions are satisfied:

(i) It holds $\mathcal{Q}_1\,{}^{F_1}\!\!\equiv^{F_2}\mathcal{Q}_2$.
(ii) $\dot\varphi:\mathcal{A}\to\mathcal{Q}_1$ and $\iota_Q^B:\mathcal{Q}_2\to\mathcal{B}$ are homomorphisms, i.e. the diagram



commutes.

Remarks 3.2.

(i) Since in **Set** different epi-mono-factorizations of the same morphism are isomorphic, in this definition we can replace the canonical factorization $\varphi=\iota_{\varphi[A]}^B\circ\dot\varphi$ by any factorization $\varphi=\psi\circ\pi$ of φ into a surjective map $\pi: A\to Q$ and an injective map $\psi: Q\to B$ for some set Q.

(ii) In the case of universal algebras, this definition of a weak homomorphism coincides with the definition given by E. Marczewski.

(iii) According to 1.8, any homomorphism from an F-algebra $\mathcal{A}$ into an F-algebra $\mathcal{B}$ is a weak homomorphism from $\mathcal{A}$ to $\mathcal{B}$. In particular, $\mathrm{id}_A:\mathcal{A}\to\mathcal{A}$ is a weak homomorphism for any F-algebra $\mathcal{A}=(A,\alpha)$.

Proposition 3.3. *Let $\mathcal{A}=(A,\alpha)$ be an F_1-algebra, $\mathcal{B}=(B,\beta)$ an F_2-algebra, $\mathcal{B}^*=(B,\beta^*)$ an F_3-algebra and $\varphi: A\to B$. Assume that $\varphi:\mathcal{A}\to\mathcal{B}$ is a weak homomorphism. Then the following statements hold:*

(i) *For each* $U \in \mathrm{Sub}(\mathcal{A})$*, we have* $\varphi[U] \in \mathrm{Sub}(\mathcal{B})$*.*
(ii) *For each* $U \in \mathrm{Sub}(\mathcal{B})$*, we have* $\varphi^{-1}[U] \in \mathrm{Sub}(\mathcal{A})$*.*
(iii) *The kernel of* φ *is a congruence relation on* $\mathcal{A}$*, i.e.* $\ker\varphi$ *is the kernel of a proper homomorphism with domain* $\mathcal{A}$*.*
(iv) *If* φ *is bijective, then* φ^{-1} *is a weak homomorphism from* $\mathcal{B}$ *to* $\mathcal{A}$*.*
(v) *If* $\mathcal{B}{}^{F_2}{\equiv}^{F_3}\mathcal{B}^*$*, then* $\varphi : \mathcal{A} \to \mathcal{B}^*$ *is a weak homomorphism.*

Proof. Let $\mathcal{Q}_1 = (Q, \gamma_1)$ be an F_1-algebra and $\mathcal{Q}_2 = (Q, \gamma_2)$ an F_2-algebra on $Q := \varphi[A]$ such that $\mathcal{Q}_1{}^{F_1}{\equiv}^{F_2}\mathcal{Q}_2$ and $\dot\varphi : \mathcal{A} \to \mathcal{Q}_1$, $\iota_Q^B : \mathcal{Q}_2 \to \mathcal{B}$ are homomorphisms.

(i) Let $U \in \mathrm{Sub}(\mathcal{A})$. From 1.9(i), we can infer that $\dot\varphi[U]$ is closed in $\mathcal{Q}_1$. Therefore, $\mathcal{Q}_1{}^{F_1}{\equiv}^{F_2}\mathcal{Q}_2$ implies that $\dot\varphi[U]$ is closed in $\mathcal{Q}_2$. A second application of 1.9(i) shows $\varphi[U] = \iota_Q^B[\dot\varphi[U]] \in \mathrm{Sub}(\mathcal{B})$.

(ii) Analogously to (i), this statement is deduced from 1.9(ii).

(iii) This follows from $\ker\varphi = \ker(\iota_Q^B \circ \dot\varphi) = \ker\dot\varphi$.

(iv) If φ is bijective, then we have $\varphi = \dot\varphi$, $\iota_Q^B = \mathrm{id}_B$. By 1.5(iii), $\mathrm{id}_B : \mathcal{B} \to \mathcal{Q}_2$ and $\varphi^{-1} : \mathcal{Q}_1 \to \mathcal{A}$ are homomorphisms. With regard to 3.2(i), $\varphi^{-1} = \varphi^{-1} \circ \mathrm{id}_B$ is a suitable factorization.

(v) From $Q \in \mathrm{Sub}_{F_2}(\mathcal{B})$ and $\mathcal{B}{}^{F_2}{\equiv}^{F_3}\mathcal{B}^*$, it follows $Q \in \mathrm{Sub}_{F_3}(\mathcal{B}^*)$. Hence, there is an algebra $\mathcal{Q}_3 = (Q, \gamma_3)$ of type F_3 such that $\iota_Q^B : \mathcal{Q}_3 \to \mathcal{B}^*$ is a homomorphism. According to 2.8, $\mathcal{B}{}^{F_2}{\equiv}^{F_3}\mathcal{B}^*$ implies $\mathcal{Q}_2{}^{F_2}{\equiv}^{F_3}\mathcal{Q}_3$. Thus, we have $\mathcal{Q}_1{}^{F_1}{\equiv}^{F_3}\mathcal{Q}_3$ and, therefore, $\varphi : \mathcal{A} \to \mathcal{B}^*$ is a weak homomorphism. ∎

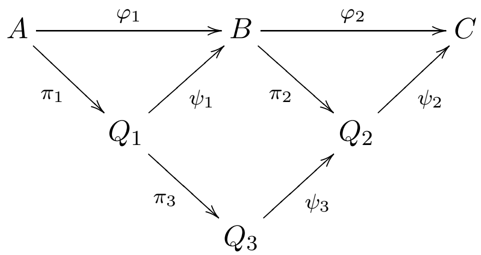
Furthermore, we might expect that the composite of two weak homomorphisms is a weak homomorphism. According the following theorem, it suffices to require that a certain one of the involved functors weakly preserves kernels.

Theorem 3.4. *Let* $\mathcal{A}$ *be an* F_1*-algebra,* $\mathcal{B}$ *an* F_2*-algebra and* $\mathcal{C}$ *an* F_3*-algebra. Assume that* F_1 *weakly preserves kernels. Whenever* $\varphi_1 : \mathcal{A} \to \mathcal{B}$ *and* $\varphi_2 : \mathcal{B} \to \mathcal{C}$ *are weak homomorphisms, then* $\varphi_2 \circ \varphi_1 : \mathcal{A} \to \mathcal{C}$ *is a weak homomorphism, too.*

Proof. In accordance with our assumptions and with regard to 3.2(i), we consider factorizations $\varphi_1 = \psi_1 \circ \pi_1$, $\varphi_2 = \psi_2 \circ \pi_2$ of φ_1, φ_2 into surjective maps $\pi_1 : A \to Q_1$, $\pi_2 : B \to Q_2$ and injective maps $\psi_1 : Q_1 \to B$, $\psi_2 : Q_2 \to C$, for some sets Q_1, Q_2, as well as an F_1-algebra $\mathcal{Q}_{11} = (Q_1, \gamma_{11})$, F_2-algebras $\mathcal{Q}_{12} = (Q_1, \gamma_{12})$, $\mathcal{Q}_{22} = (Q_2, \gamma_{22})$, and an F_3-algebra $\mathcal{Q}_{23} = (Q_2, F_3, \gamma_{23})$ such that:

(1) It holds $\mathcal{Q}_{11}{}^{F_1}{\equiv}^{F_2}\mathcal{Q}_{12}$ and $\mathcal{Q}_{22}{}^{F_2}{\equiv}^{F_3}\mathcal{Q}_{23}$.
(2) $\pi_1 : \mathcal{A} \to \mathcal{Q}_{11}$, $\psi_1 : \mathcal{Q}_{12} \to \mathcal{B}$, $\pi_2 : \mathcal{B} \to \mathcal{Q}_{22}$ and $\psi_2 : \mathcal{Q}_{23} \to \mathcal{C}$ are homomorphisms.

By 1.5(ii), $\pi_2 \circ \psi_1 : \mathcal{Q}_{12} \to \mathcal{Q}_{22}$ is a homomorphism, too. Let $\pi_2 \circ \psi_1 = \psi_3 \circ \pi_3$ be a factorization of $\pi_2 \circ \psi_1$ into a surjective map $\pi_3 : Q_1 \to Q_3$ and an injective map $\psi_3 : Q_3 \to Q_2$. Evidently, the diagram



commutes. By 1.8, there is an F_2-algebra $\mathcal{Q}_{32} = (Q_3, \gamma_{32})$ such that $\pi_3 : \mathcal{Q}_{12} \to \mathcal{Q}_{32}$ and $\psi_3 : \mathcal{Q}_{32} \to \mathcal{Q}_{22}$ are homomorphisms. By 2.4(i), we obtain $\ker \pi_3 \in \mathrm{Sub}_{F_2}(\mathcal{Q}_{12} \times \mathcal{Q}_{12})$ and, because of $\mathcal{Q}_{11}{}^{F_1}{\equiv}^{F_2} \mathcal{Q}_{12}$, $\ker \pi_3 \in \mathrm{Sub}_{F_1}(\mathcal{Q}_{11} \times \mathcal{Q}_{11})$. Since F_1 weakly preserves kernels, by 2.4(ii), there is an F_1-algebra $\mathcal{Q}_{31} = (Q_3, \gamma_{31})$ such that $\pi_3 : \mathcal{Q}_{11} \to \mathcal{Q}_{31}$ is a homomorphism. As a consequence of 2.7, $\mathcal{Q}_{11}{}^{F_1}{\equiv}^{F_2} \mathcal{Q}_{12}$ implies $\mathcal{Q}_{31}{}^{F_1}{\equiv}^{F_2} \mathcal{Q}_{32}$. Analogously, by 2.5, it follows $\psi_3[Q_3] \in \mathrm{Sub}_{F_2}(\mathcal{Q}_{22})$ and, on account of $\mathcal{Q}_{22}{}^{F_2}{\equiv}^{F_3} \mathcal{Q}_{23}$, $\psi_3[Q_3] \in \mathrm{Sub}_{F_3}(\mathcal{Q}_{23})$, too. An application of 2.5 provides the existence of an F_3-algebra $\mathcal{Q}_{33} = (Q_3, \gamma_{33})$ such that $\psi_3 : \mathcal{Q}_{33} \to \mathcal{Q}_{23}$ is a homomorphism. By 2.8, $\mathcal{Q}_{22}{}^{F_2}{\equiv}^{F_3} \mathcal{Q}_{23}$ implies $\mathcal{Q}_{32}{}^{F_2}{\equiv}^{F_3} \mathcal{Q}_{33}$. Thus, we have $\mathcal{Q}_{31}{}^{F_1}{\equiv}^{F_3} \mathcal{Q}_{33}$. Finally, it remains to be remarked that $\pi_3 \circ \pi_1 : \mathcal{A} \to \mathcal{Q}_{31}$ is a surjective homomorphism and $\psi_2 \circ \psi_3 : \mathcal{Q}_{33} \to \mathcal{B}$ is an injective homomorphism. By 3.2(i), $\varphi_2 \circ \varphi_1$ is a weak homomorphism from $\mathcal{A}$ to $\mathcal{B}$. ∎

One of the main consequences of the previous theorem, in connection with 3.2(iii), is the following: For a set $\mathcal{F}$ of **Set**-endofunctors which weakly preserve kernels, the class of all algebras of types from $\mathcal{F}$ endowed with the class of all homomorphisms between these structures forms a category which, in the sequel, will be denoted by $\mathbf{Set}^{\mathcal{F}}$. According to 3.3(iv) and the fact that isomorphisms in $\mathbf{Set}^{\mathcal{F}}$ need to be bijective, the isomorphisms in $\mathbf{Set}^{\mathcal{F}}$ are exactly the bijective weak homomorphisms. We are going to investigate some further structural properties of $\mathbf{Set}^{\mathcal{F}}$ which correspond to well-known results for $\mathbf{Set}^{F}$ and, in fact, show that the chosen definition is a useful generalization. The next theorem arises from 1.5(iv), (v).

Theorem 3.5. *Let $\mathcal{A} = (A, \alpha)$ be an F_1-algebra, $\mathcal{B} = (B, \beta)$ an F_2-algebra and $\mathcal{C} = (C, \gamma)$ an F_3-algebra. Furthermore, let $\varphi : A \to B$ and $\psi : B \to C$ be maps such that $\psi \circ \varphi$ is a weak homomorphism from $\mathcal{A}$ to $\mathcal{C}$. Then the following statements hold:*

(i) *Assume that F_2 weakly preserves kernels. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective weak homomorphism, then $\psi : \mathcal{B} \to \mathcal{C}$ is a weak homomorphism.*

(ii) *If $\psi : \mathcal{B} \to \mathcal{C}$ is an injective weak homomorphism, then $\varphi : \mathcal{A} \to \mathcal{B}$ is a weak homomorphism, too.*

Proof. (i) If $\varphi : \mathcal{A} \to \mathcal{B}$ is a surjective weak homomorphism, then there exists an F_1-algebra $\mathcal{B}^* = (B, \beta^*)$ such that $\mathcal{B}^{*F_1}{\equiv}^{F_2}\mathcal{B}$ and φ is a homomorphism from $\mathcal{A}$ into $\mathcal{B}^*$. Let $Q := \psi[B] = \psi[\varphi[A]]$. Since $\psi \circ \varphi : \mathcal{A} \to \mathcal{C}$ is a weak homomorphism, there exist algebras $\mathcal{Q}_1 = (Q, \gamma_1)$ of type F_1 and $\mathcal{Q}_3 = (Q, \gamma_3)$ of type F_3 such that $\mathcal{Q}_1{}^{F_1}{\equiv}^{F_3}\mathcal{Q}_3$ and $\dot{\psi} \circ \varphi : \mathcal{A} \to \mathcal{Q}_1$, $\iota_Q^C : \mathcal{Q}_3 \to \mathcal{C}$ are homomorphisms. According to 1.5(iv), $\dot{\psi} : \mathcal{B}^* \to \mathcal{Q}_1$ is a homomorphism. By 2.4(i), it follows $\ker \dot{\psi} \in \mathrm{Sub}_{F_1}(\mathcal{B}^* \times \mathcal{B}^*)$ and, because of $\mathcal{B}^{*F_1}{\equiv}^{F_2}\mathcal{B}$, $\ker \dot{\psi} \in \mathrm{Sub}_{F_2}(\mathcal{B} \times \mathcal{B})$, too. As F_2 weakly preserves kernels, by 2.4(ii) there exists an F_2-algebra $\mathcal{Q}_2 = (Q, \gamma_2)$ such that $\dot{\psi} : \mathcal{B} \to \mathcal{Q}_2$ becomes a homomorphism. On account of 2.7, $\mathcal{B}^{*F_1}{\equiv}^{F_2}\mathcal{B}$ implies $\mathcal{Q}_1{}^{F_1}{\equiv}^{F_2}\mathcal{Q}_2$. Therefore, we have $\mathcal{Q}_2{}^{F_2}{\equiv}^{F_3}\mathcal{Q}_3$. Consequently, $\psi : \mathcal{B} \to \mathcal{C}$ is a weak homomorphism.

(ii) Analogously to the proof of (i), statement (ii) is derived from 1.5(v), 2.5 and 2.8. ∎

From the previous theorem one can easily deduce the two corollaries given below. The analogous results for morphisms in the categories **Set** and $\mathbf{Set}^F$ can be found in [8] and [2], respectively.

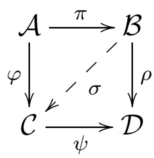
Corollary 3.6. *Let $\mathcal{A}$ be an F_1-algebra, $\mathcal{B}$ an F_2-algebra, $\mathcal{C}$ an F_3-algebra, $\pi : \mathcal{A} \to \mathcal{B}$ and $\varphi : \mathcal{A} \to \mathcal{C}$ weak homomorphisms. Assume that F_2 weakly preserves kernels and π is surjective. Then there exists a weak homomorphism $\psi : \mathcal{B} \to \mathcal{C}$ with the property $\psi \circ \pi = \varphi$ if and only if* $\ker \pi \subseteq \ker \varphi$. *In this case, ψ is uniquely determined.*

Corollary 3.7. *Consider an F_1-algebra $\mathcal{A}$, an F_2-algebra $\mathcal{B}$, an F_3-algebra $\mathcal{C}$, an F_4-algebra $\mathcal{D}$ as well as weak homomorphisms $\pi : \mathcal{A} \to \mathcal{B}$, $\varphi : \mathcal{A} \to \mathcal{C}$, $\rho : \mathcal{B} \to \mathcal{D}$ and $\psi : \mathcal{C} \to \mathcal{D}$. Assume that F_2 weakly preserves kernels, π is surjective and ψ is injective. If it holds $\rho \circ \pi = \psi \circ \varphi$, then there exists a unique weak homomorphism $\sigma : \mathcal{B} \to \mathcal{C}$ with the property $\sigma \circ \pi = \varphi$. In addition, σ satisfies $\psi \circ \sigma = \rho$.*

Let $\mathcal{F}$ be a set of **Set**-endofunctors which weakly preserve kernels, let $\mathbf{E}^{\mathcal{F}}$ denote the class of all surjective weak homomorphisms between objects of $\mathbf{Set}^{\mathcal{F}}$ and $\mathbf{M}^{\mathcal{F}}$ the class of all injective weak homomorphisms between objects of $\mathbf{Set}^{\mathcal{F}}$. Taking 3.7 into account, we observe that $(\mathbf{E}^{\mathcal{F}}, \mathbf{M}^{\mathcal{F}})$ is a *factorization structure for morphisms* in $\mathbf{Set}^{\mathcal{F}}$ (see [1]), which means that

(1) each of $\mathbf{E}^{\mathcal{F}}$ and $\mathbf{M}^{\mathcal{F}}$ is closed under composition with isomorphisms in $\mathbf{Set}^{\mathcal{F}}$,
(2) each morphism $\varphi : \mathcal{A} \to \mathcal{B}$ in $\mathbf{Set}^{\mathcal{F}}$ has a factorization $\varphi = \psi \circ \pi$ with $\pi \in \mathbf{E}^{\mathcal{F}}$ and $\psi \in \mathbf{M}^{\mathcal{F}}$ (cf. 3.1, 3.3(v)), and

(3) whenever $\pi : \mathcal{A} \to \mathcal{B}$, $\varphi : \mathcal{A} \to \mathcal{C}$, $\rho : \mathcal{B} \to \mathcal{D}$ and $\psi : \mathcal{C} \to \mathcal{D}$ are morphisms in $\mathbf{Set}^{\mathcal{F}}$ with $\pi \in \mathbf{E}^{\mathcal{F}}$, $\psi \in \mathbf{M}^{\mathcal{F}}$ and $\rho \circ \pi = \psi \circ \varphi$, then there exists a unique morphism $\sigma : \mathcal{B} \to \mathcal{C}$ in $\mathbf{Set}^{\mathcal{F}}$ such that the diagram



commutes, i.e. it holds $\sigma \circ \pi = \varphi$ as well as $\psi \circ \sigma = \rho$.

4. Products of differently typed algebras

Concerning a set $\mathcal{F}$ of endofunctors of **Set** which weakly preserve kernels, we would like to find general conditions for $\mathcal{F}$ under which $\mathbf{Set}^{\mathcal{F}}$ has certain limits. However, this is still an open problem. In this section, we want to discuss two very suggesting constructions for a product of differently typed algebras where the canonical projections become weak homomorphisms. Unfortunately, the constructed objects fail to have the universal product property.

Remarks 4.1.

(i) Let $(F_i)_{i\in I}$ be a family of **Set**-endofunctors. For every set X, we define $(\Pi_{i\in I}F_i)(X) := \Pi_{i\in I}F_i(X)$ and $(\Sigma_{i\in I}F_i)(X) := \Sigma_{i\in I}F_i(X) = \bigcup_{i\in I}\{i\} \times F_i(X)$, and for every map $\varphi : X \to Y$, we define

$$(\Pi_{i\in I}F_i)(\varphi) : (\Pi_{i\in I}F_i)(X) \to (\Pi_{i\in I}F_i)(Y) : z \mapsto ((F_i\varphi)(z(i)))_{i\in I},$$
$$(\Sigma_{i\in I}F_i)(\varphi) : (\Sigma_{i\in I}F_i)(X) \to (\Sigma_{i\in I}F_i)(Y) : (i, z) \mapsto (i, (F_i\varphi)(z)).$$

Then the assignments $\Pi_{i\in I}F_i$ and $\Sigma_{i\in I}F_i$ constitute endofunctors of **Set**.

(ii) ([6]) Let F be an endofunctor of **Set**. If there is a nonempty set X with $F(X) = \emptyset$, then F is *trivial*, i.e. for each set Y and each map f it holds $F(Y) = \emptyset$ and $(Ff)^\bullet = \emptyset$.

At first we will treat a construction concerning the functor $\Pi_{i\in I}F_i$.

Definition 4.2. Let $(F_i)_{i\in I}$ be a family of **Set**-endofunctors and $(\mathcal{A}_i)_{i\in I}$ a family of algebras such that, for each $i \in I$, $\mathcal{A}_i = (A_i, \alpha_i)$ is an F_i-algebra. For each $i \in I$, consider the canonical projection $\widetilde{\pi}_{i,A_i} : (\Pi_{k\in I}F_k)(A_i) \to F_i(A_i)$ and the structure map $\delta_i := \alpha_i \circ \widetilde{\pi}_{i,A_i} : (\Pi_{k\in I}F_k)(A_i) \to A_i$. Referring to 1.7, we define an algebra of type $\Pi_{k\in I}F_k$ by $\bigotimes_{j\in I}\mathcal{A}_j := \Pi_{j\in I}(A_j, \delta_j)$.

Theorem 4.3. *Let $(F_i)_{i\in I}$ be a family of* **Set***-endofunctors and $(\mathcal{A}_i)_{i\in I}$ a family of algebras such that, for each $i \in I$, $\mathcal{A}_i = (A_i, \alpha_i)$ is an F_i-algebra. We assume that*

(i) *if* $\Pi_{k\in I}F_k$ *is trivial, then* F_i *is trivial for each* $i \in I$,
(ii) *if* $(\Pi_{k\in I}F_k)(\emptyset) = \emptyset$, *then* $F_i(\emptyset) = \emptyset$ *for all* $i \in I$.

For each $i \in I$, *the projection* $\pi_i : \Pi_{j\in I}A_j \to A_i$ *is a weak homomorphism from* $\bigotimes_{j\in I}\mathcal{A}_j$ *to* $\mathcal{A}_i$.

Proof. We adopt the notations of 4.2. Let $i \in I$. By definition, π_i is a homomorphism from $\bigotimes_{j\in I}\mathcal{A}_j$ to (A_i, δ_i). According to 3.3(v), it suffices to prove ${\mathcal{A}_i}^{F_i}{\equiv}^{\Pi_{k\in I}F_k}(A_i, \delta_i)$.

Let L be an arbitrary set, $S \subseteq A_i^L$. Consider the restricted projections $(\pi_l|_S : S \to A_i)_{l\in L}$ and the projection $\widetilde{\pi}_{i,S} : (\Pi_{k\in I}F_k)(S) \to F_i(S)$. For each $l \in L$, it holds $F_i(\pi_l|_S) \circ \widetilde{\pi}_{i,S} = \widetilde{\pi}_{i,A_i} \circ (\Pi_{k\in I}F_k)(\pi_l|_S)$, as for all $a \in (\Pi_{k\in I}F_k)(S)$, we observe

$$\begin{aligned}(F_i(\pi_l|_S))(\widetilde{\pi}_{i,S}(a)) &= (F_i(\pi_l|_S))(a(i))\\ &= \widetilde{\pi}_{i,A_i}((F_k(\pi_l|_S)(a(k)))_{k\in I})\\ &= \widetilde{\pi}_{i,A_i}((\Pi_{k\in I}F_k)(\pi_l|_S)(a)).\end{aligned}$$

With these preliminary considerations, we are going to to show

$$S \in \mathrm{Sub}_{F_i}(\mathcal{A}_i^L) \iff S \in \mathrm{Sub}_{\Pi_{k\in I}F_k}((A_i, \delta_i)^L).$$

"$\Longrightarrow$": If $\sigma : F_i(S) \to S$ is a structure mapping so that (S, σ) is a subalgebra of $\mathcal{A}_i^L$, then we define $\rho := \sigma \circ \widetilde{\pi}_{i,S} : (\Pi_{k\in I}F_k)(S) \to S$. Immediately, for each $l \in L$, we infer

$$\begin{aligned}\pi_l|_S \circ \rho &= \pi_l|_S \circ \sigma \circ \widetilde{\pi}_{i,S}\\ &= \alpha_i \circ F_i(\pi_l|_S) \circ \widetilde{\pi}_{i,S}\\ &= \alpha_i \circ \widetilde{\pi}_{i,A_i} \circ (\Pi_{k\in I}F_k)(\pi_l|_S)\\ &= \delta_i \circ (\Pi_{k\in I}F_k)(\pi_l|_S),\end{aligned}$$

wherefore, by 2.3, (S, ρ) is a subalgebra of $(A_i, \delta_i)^L$.
"$\Longleftarrow$": Conversely, if there exists a structure mapping $\rho : (\Pi_{k\in I}F_k)(S) \to S$ such that (S, ρ) is a subalgebra of $(A_i, \delta_i)^L$, then define $\sigma : F_i(S) \to S$ by $\sigma^\bullet := \{(\widetilde{\pi}_{i,S}(s), \rho(s)) \mid s \in (\Pi_{k\in I}F_k)(S)\}$. On account of (i), (ii) and 4.1(ii), $\widetilde{\pi}_{i,S}$ is surjective. Moreover, for all $(a_1, a_2) \in \ker \widetilde{\pi}_{i,S}$ and $l \in L$, we obtain

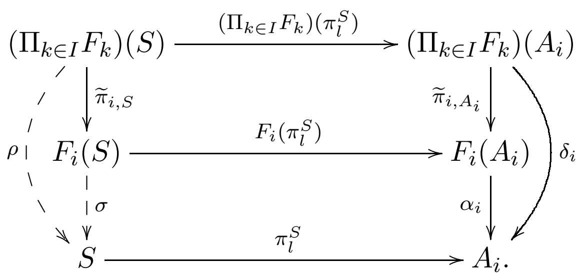
$$\begin{aligned}(\pi_l|_S \circ \rho)(a_1) &= (\delta_i \circ (\Pi_{k\in I}F_k)(\pi_l|_S))(a_1)\\ &= (\alpha_i \circ \widetilde{\pi}_{i,A_i} \circ (\Pi_{k\in I}F_k)(\pi_l|_S))(a_1)\\ &= (\alpha_i \circ F_i(\pi_j|_S) \circ \widetilde{\pi}_{i,S})(a_1)\\ &= (\alpha_i \circ F_i(\pi_j|_S) \circ \widetilde{\pi}_{i,S})(a_2)\\ &= (\alpha_i \circ \widetilde{\pi}_{i,A_i} \circ (\Pi_{k\in I}F_k)(\pi_l|_S))(a_2)\\ &= (\delta_i \circ (\Pi_{k\in I}F_k)(\pi_l|_S))(a_2)\\ &= (\pi_l|_S \circ \rho)(a_2).\end{aligned}$$

Thus, we conclude $\ker \widetilde{\pi}_{i,S} \subseteq \ker \rho$ and, hence, σ is well-defined. For each $l \in L$, we deduce

$$\begin{aligned} \pi_l|_S \circ \sigma \circ \widetilde{\pi}_{i,S} &= \pi_l|_S \circ \rho = \delta_i \circ (\Pi_{k\in I} F_k)(\pi_l|_S) \\ &= \alpha_i \circ \widetilde{\pi}_{i,A_i} \circ (\Pi_{k\in I} F_k)(\pi_l|_S) \\ &= \alpha_i \circ F_i(\pi_l|_S) \circ \widetilde{\pi}_{i,S}. \end{aligned}$$

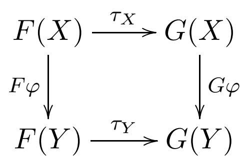
According to the surjectivity of $\widetilde{\pi}_{i,S}$, it follows $\pi_l|_S \circ \sigma = \alpha_i \circ F_i(\pi_l|_S)$. By 2.3, (S, σ) is a subalgebra of $\mathcal{A}_i^L$. ■

The proof of 4.3 is illustrated by the diagram



The second construction we want to present here requires that the considered functors are naturally transformable to the identity functor. Of course, this assumption is motivated by the universal algebraic case, as we will see in 4.7(ii).

Definition 4.4. ([8]) Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors. A *natural transformation* τ *from* F *to* G associates to every $\mathcal{C}$-object X a $\mathcal{D}$-morphism $\tau_X : F(X) \to G(X)$ such that the following condition is satisfied: For every $\mathcal{C}$-morphism $\varphi : X \to Y$, it holds $G\varphi \circ \tau_X = \tau_Y \circ F\varphi$, i.e. the diagram



commutes.

Definition 4.5. Let $(F_i)_{i\in I}$ be a family of **Set**-endofunctors, $(\mathcal{A}_i)_{i\in I}$ a family of algebras and $\tau = (\tau_i)_{i\in I}$ a family of natural transformations such that, for each $i \in I$, $\mathcal{A}_i = (A_i, \alpha_i)$ is an F_i-algebra, and τ_i is a natural transformation from F_i to the identity functor Id : **Set** $\to$ **Set**. Consider the canonical injections $(\iota_{j,A_i} : F_j(A_i) \to (\Sigma_{k\in I} F_k)(A_i))_{i,j\in I}$. For each $i \in I$, there exists a unique map $\delta_i : (\Sigma_{k\in I} F_k)(A_i) \to A_i$ satisfying $\alpha_i = \delta_i \circ \iota_{i,A_i}$ and $\tau_{j,A_i} = \delta_i \circ \iota_{j,A_i}$ for all $j \in I \setminus \{i\}$. Referring to 1.7, we define an algebra of type $\Sigma_{k\in I} F_k$ by $\bigoplus_{j\in I}^{\tau} \mathcal{A}_j := \Pi_{j\in I}(A_j, \delta_j)$.

Theorem 4.6. *Let $(F_i)_{i\in I}$ be a family of* **Set**-*endofunctors, $(\mathcal{A}_i)_{i\in I}$ a family of algebras and $\tau = (\tau_i)_{i\in I}$ a family of natural transformations such that, for each $i \in I$, $\mathcal{A}_i = (A_i, \alpha_i)$ is an F_i-algebra, and τ_i is a natural transformation from F_i to the identity functor* $\mathrm{Id} : \mathbf{Set} \to \mathbf{Set}$. *Then, for each $i \in I$, the projection $\pi_i : \Pi_{j\in I} A_j \to A_i$ is a weak homomorphism from $\bigoplus^{\tau}_{j\in I} \mathcal{A}_j$ to $\mathcal{A}_i$.*

Proof. We adopt the notations of 4.5. Let $i \in I$. By definition, π_i is a homomorphism from $\bigoplus_{j\in I} \mathcal{A}_j$ to (A_i, δ_i). With regard to 3.3(v), it suffices to prove ${\mathcal{A}_i}^{F_i} \equiv^{\Sigma_{k\in I} F_k} (A_i, \delta_i)$.

Let L be an arbitrary set and $S \subseteq A_i^L$. Furthermore, consider the restricted projections $(\pi_l|_S : S \to A_i)_{l\in L}$ and the canonical injections $(\iota_{j,S} : F_j(S) \to (\Sigma_{k\in I} F_k)(S))_{j\in I}$. For all $j \in I$, $l \in L$, it is easy to infer $(\Sigma_{k\in I} F_k)(\pi_l|_S) \circ \iota_{j,S} = \iota_{j,A_i} \circ F_j(\pi_l|_S)$. Similarly to the proof of 4.3, it only remains to verify

$$S \in \mathrm{Sub}_{F_i}(\mathcal{A}_i^L) \iff S \in \mathrm{Sub}_{\Sigma_{k\in I} F_k}((A_i, \delta_i)^L).$$

"$\Longrightarrow$": Let $\sigma : F_i(S) \to S$ be a structure map such that (S, σ) is a subalgebra of $\mathcal{A}_i^L$. Consider the unique map $\rho : (\Sigma_{k\in I} F_k)(S) \to S$ satisfying $\sigma = \rho \circ \iota_{i,S}$ and $\tau_{j,S} = \rho \circ \iota_{j,S}$ for all $j \in I \setminus \{i\}$. For each $l \in L$, we have

$$\begin{aligned} \pi_l|_S \circ \rho \circ \iota_{i,S} &= \pi_l|_S \circ \sigma = \alpha_i \circ F_i(\pi_l|_S) \\ &= \delta_i \circ \iota_{i,A_i} \circ F_i(\pi_l|_S) = \delta_i \circ (\Sigma_{k\in I} F_k)(\pi_l|_S) \circ \iota_{i,S} \end{aligned}$$

and, for all $j \in I \setminus \{i\}$,

$$\begin{aligned} \pi_l|_S \circ \rho \circ \iota_{j,S} &= \pi_l|_S \circ \tau_{j,S} = \tau_{j,A_i} \circ F_j(\pi_l|_S) \\ &= \delta_i \circ \iota_{j,A_i} \circ F_j(\pi_l|_S) = \delta_i \circ (\Sigma_{k\in I} F_k)(\pi_l|_S) \circ \iota_{j,S}. \end{aligned}$$

Therefore, it holds $\pi_l|_S \circ \rho = \delta_i \circ (\Sigma_{k\in I} F_k)(\pi_l|_S)$. By 2.3, (S, ρ) is a subalgebra of $(A_i, \delta_i)^L$.

"$\Longleftarrow$": Conversely, if there exists a structure map $\rho : (\Sigma_{k\in I} F_k)(S) \to S$ such that (S, ρ) is a subalgebra of $(A_i, \delta_i)^L$, then we define $\sigma := \rho \circ \iota_{i,S} : F_i(S) \to S$. For each $l \in L$, it follows

$$\begin{aligned} \pi_l|_S \circ \sigma &= \pi_l|_S \circ \rho \circ \iota_{i,S} = \delta_i \circ (\Sigma_{k\in I} F_k)(\pi_l|_S) \circ \iota_{i,S} \\ &= \delta_i \circ \iota_{i,A_i} \circ F_i(\pi_l|_S) = \alpha_i \circ F_i(\pi_l|_S). \end{aligned}$$

According to 2.9, (S, σ) is a subalgebra of $(\mathcal{A}_i)^L$. ∎

The Theorems 4.3 and 4.6 are instances of situations from which the notion of a weak homomorphism arises. The following examples illustrate that, in general, $\bigotimes_{j\in I} \mathcal{A}_j$ and $\bigoplus^{\tau}_{j\in I} \mathcal{A}_j$ are not categorical products of the algebras $(\mathcal{A}_i)_{i\in I}$. In fact, these structures fail to satisfy the corresponding universal property.

Examples 4.7. Let $\underline{\Omega_i} = (\Omega_i, \mathrm{ar}_i)$ $(i \in I)$ be a family of algebraic types as in 2.2(i). Suppose that, for each $i \in I$, we have $\Omega_i \neq \emptyset$ and $\mathrm{ar}_i[\Omega_i] \subseteq \mathbb{N}_{\geq 1}$. Consider the functors $F_i := F_{\underline{\Omega}_i}$ $(i \in I)$ from 2.2(i) and let $(\mathcal{A}_i)_{i \in I}$ be a family of algebras such that, for each $i \in I$, $\mathcal{A}_i = (A_i, \alpha_i)$ is an F_i-algebra.

(i) According to 4.3, for each $i \in I$, the projection π_i is a weak homomorphism from $\bigotimes_{j \in I} \mathcal{A}_j$ to $\mathcal{A}_i$.

(ii) Let $\sigma_i : \Omega_i \to \mathbb{N}_{\geq 1}$ $(i \in I)$ be a family of maps such that $\sigma_i(\omega) \in \{1, \ldots, \mathrm{ar}_i(\omega)\}$ for all $\omega \in \Omega_i$ and $i \in I$. Then, for each $i \in I$, a natural transformation τ_i from F_i to Id can be defined as follows: For any set X, put

$$\tau_{i,X} : F_{\underline{\Omega}_i}(X) \to X : (\omega, (x_1, \ldots, x_{\mathrm{ar}_i(\omega)})) \mapsto x_{\sigma_i(\omega)}.$$

Theorem 4.6 states that, for each $i \in I$, π_i is a weak homomorphism from $\bigoplus_{j \in I}^{\tau} \mathcal{A}_j$ to $\mathcal{A}_i$.

(iii) Assume that $|I| \geq 2$, $F_i = \mathrm{Id}$ and $\mathcal{A}_i = (A, \alpha)$ for all $i \in I$, where A is a set with $|A| \geq 2$ and $\alpha : A \to A$ a map with $|\alpha[A]| \geq 2$. In accordance with 4.2, we get the Id^I-algebra $\bigotimes_{j \in I} \mathcal{A}_j = (A^I, \gamma)$ with $\gamma : (A^I)^I \to A^I$ defined by $\gamma(a)(i) := \alpha(a(i)(i))$ for all $a \in (A^I)^I$, $i \in I$. Furthermore, the diagonal map $\delta : A \to A^I$, given by $\delta(a)(i) := a$ for all $a \in A$, $i \in I$, is the unique map satisfying $\pi_i \circ \delta = \mathrm{id}_A$ for each $i \in I$. By 3.2(iii), id_A is a weak homomorphism from (A, α) to itself. However, since $\delta[A] = \{a \in A^I \mid \forall i, j \in I : a(i) = a(j)\}$ is not closed in $\bigotimes_{j \in I} \mathcal{A}_j$, δ is not a weak homomorphism from (A, α) to $\bigotimes_{j \in I} \mathcal{A}_j$.

(iv) Suppose that $|I| \geq 2$, $F_i = \mathrm{Id}$, $\tau_{i,X} = \mathrm{id}_X$ and $\mathcal{A}_i = (A, \alpha)$ for all $i \in I$, where A is a set with $|A| \geq 2$ and $\alpha : A \to A$ a constant map with $\alpha(a) = \alpha^*$ for all $a \in A$. With regard to 4.5, one can compute the $(I \times \mathrm{Id})$-algebra $\bigoplus_{j \in I}^{\tau} \mathcal{A}_j = (A^I, \gamma)$ with $\gamma : I \times A^I \to A^I$ defined by

$$\gamma(i, a)(j) = \begin{cases} \alpha^* & \text{if } i = j, \\ a(j) & \text{otherwise} \end{cases}$$

for all $a \in A^I$, $i, j \in I$. As in (iii), $\delta[A]$ is not closed in $\bigoplus_{j \in I}^{\tau} \mathcal{A}_j$. Thus, δ is not a weak homomorphism from $\mathcal{A}$ to $\bigoplus_{j \in I}^{\tau} \mathcal{A}_j$.

Concluding remarks

This work provides basic tools for further research concerning weak homomorphisms and some related category theoretic aspects. For instance, from [1] we know that factorization structures as $(\mathbf{E}^{\mathcal{F}}, \mathbf{M}^{\mathcal{F}})$ are very useful tools for the investigation of certain reflexive subcategories. With this in mind, future considerations could address the classification of $\mathbf{E}^{\mathcal{F}}$-*reflexive subcategories* of $\mathbf{Set}^{\mathcal{F}}$ (see [1]). Another interesting question is whether and under which conditions it is possible to generalize the given results by replacing **Set** by an arbitrary category $\mathcal{C}$ with a suitable factorization structure for morphisms and considering those $\mathcal{C}$-endofunctors which are compatible with this factorization structure in some sense.

Several times in this paper, we needed to assume the Axiom of Choice. This was to be expected, since the definition of a weak homomorphism is formulated in terms of infinite direct products of algebras. However, it is possible to replace the assumption of the Axiom of Choice by requiring an additional property of the functors under consideration. An endofunctor F of **Set** *strongly preserves epimorphisms* if the following condition is fulfilled: Whenever $\varphi : A \to B$ is a surjective map, I an arbitrary set and $R \subseteq B^I$, then the map $F\psi$ is surjective, where ψ is defined by

$$\psi : (\varphi^I)^{-1}[R] \to R : a \to \varphi^I(a).$$

If we assume that the investigated functors strongly preserve epimorphisms, we do not need the Axiom of Choice to prove the obtained results. But then the Axiom of Choice is significant for the cardinality of this restricted class of functors. Indeed, it is not difficult to show that the Axiom of Choice is equivalent to each of the following two statements:

(a) Every **Set**-endofunctor strongly preserves epimorphisms.
(b) The identity functor $\mathrm{Id} : \mathbf{Set} \to \mathbf{Set}$ strongly preserves epimorphisms.

So, for practical reasons, we decided to assume the Axiom of Choice in this work. But as we have seen, this is not the only way to deal with the outlined problem.

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