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WEAK RELATIVE PSEUDOCOMPLEMENTS IN SEMILATTICES

Abstract. Weak relative pseudocomplementation on a meet semilattice S is a partial operation $*$ which associates with every pair (x, y) of elements, where $x \geq y$, an element z (the weak pseudocomplement of x relative to y) which is the greatest among elements u such that $y = u \wedge x$. The element z coincides with the pseudocomplement of x in the upper section $[y)$ and, if S is modular, with the pseudocomplement of x relative to y. A weakly relatively pseudomented semilattice is said to be extended, if it is equipped with a total binary operation extending $*$. We study congruence properties of the variety of such semilattices and review some of its subvarieties already described in the literature.

1. Introduction

A meet semilattice is said to be weakly relatively pseudocomplemented if, whenever $y \leq x$, there is a greatest element u such that $y = u \wedge x$. The concept goes back to [28], where the congruence lattice of a semilattice was shown to possess this property; the term, however, was introduced later in [35]. Weak relative pseudocomplements in congruence lattices (of various structures) are discussed also, for example, in [2, 16, 19, 41]; they are uncovered as well in lattices of closure operators [18, 31], certain subalgebra lattices of semigroups and groups [37, 38, 42], and in algebraic structures of constraint programming [5, 6]. Every meet-semidistributive algebraic (in particular, finite) lattice is weakly relatively pseudocomplemented [12, 17, 37]. Weak relative pseudocomplementation has been studied also in posets [15]. Sectionally pseudocomplemented semilattices introduced in [8, 10] (i.e., semilattices with pseudocomplemented principial filters) are just

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weakly relatively pseudocomplemented semilattices. See also [11] and references therein for information on sectionally pseudocomplemented semilattices and lattices.

Some explanations concerning the term 'sectionally pseudocomplemented', used in various senses in literature, could be helpful to the reader. It was initially introduced for a lower bounded meet semilattice in which all initial segments $[0, b]$ are pseudocomplemented (see [23, 24, 27, 32]). Further, semilattices with pseudocomplemented segments (mentioned in [25] as 'abschhnittspseudokomplementär') also were called in [26] sectionally pseudocomplemented. However, it is easily seen that an upper bounded meet semilattice is sectionally pseudocomplemented in the sense of the previous paragraph if and only if every segment, i.e., closed interval, in it is pseudocomplemented [12, 35].

On the other hand, the term 'weak relative pseudocomplementation' (interchangeably with 'weak implication') is used to name a certain operation in Nelson algebras [30, 36] (known also as quasi-pseudo-Boolean algebras [33]) and, more generally, in the so called weak Brouwerian semilattices with filter preserving operators [4, p. 355].

Let S be a weakly relatively pseudocomplemented semilattice. For elements $x, y \in S$ with $y \leq x$, we denote the element $\max\{u\colon\ y = u \wedge x\}$ by $x * y$. The operation $*$ introduced in this way is partial, and a natural question arises how to extend it to a total operation in a reasonable way and when it is possible to do (the extended operation may be viewed as an implication on the underlaying semilattice). Several answers to the question can be found in literature. In particular, the so called semi-Brouwerian semilattices [35] is a rather natural version of extended weakly relatively pseudocomplemented semilattices. In [35, 39], just the particular extended operation of semi-Brouwerian semilattices is called a weak relative pseudocomplementation.

Weakly relatively pseudocomplemented semilattices and their extensions is the subject of the present paper. In the subsequent section, weak relative pseudocomplementation is put into the context of some other kinds of relative complementation; in particular, it is shown there that in modular semilattices all weak relative pseudocomplements are relative pseudocomplements. The extension problem is addressed in Section 3, where the class (in fact, a variety) $\mathsf{EWR}^\wedge$ of all extended weakly relatively pseudocomplemented semilattices and its subvariety $\mathsf{dEWR}^\wedge$ consisting of distributive semilattices are introduced. These varieties are shown in Section 4 to have nice congruence properties. The subject of Section 5 is those $\mathsf{EWR}^\wedge$-algebras where all filters are congruence kernels. The class, in fact, the variety ABS of these algebras admits various other characteristics, which together motivate the name 'almost Brouwerian semilattice' for them. In particular, almost Brouwerian semilattices are just those extended weakly relatively pseudocomplemented semilattices that are congruence orderable; this latter property turns out to be equivalent to having equationally definable principal congruences. A few particular classes of algebras, which already have been studied in literature and turn out to be subvarieties of $\mathsf{EWR}^\wedge$, are reviewed in the final section. It also contains a detailed comparison of all subvarieties of $\mathsf{EWR}^\wedge$ considered in the paper.

2. Preliminaries

All semilattices we shall deal with in this paper will be meet semilattices.

Definition 1. Suppose that x, y are elements of a semilattice and $y \leq x$. The *weak pseudocomplement of x relative to y* is the element z defined by

$$z := \max\{u\colon\ u \wedge x = y\} = \max\{u \geq y\colon\ u \wedge x \leq y\}. \tag{1}$$

A semilattice is *weakly relatively pseudocomplemented* (or just *wr-pseudocomplemented*, for short) if all weak relative pseudocomplements (*wr-pseudocomplements*) in it exist. If $x * y$ stands for the weak pseudocomplement of x relative to y, then the (partial) operation $*$ in a wr-pseudocomplemented semilattice is called a *weak relative pseudocomplementation*, or just *wr-pseudocomplementation*.

The element z in (1) more explicitly is characterised by the condition

$$\text{if } y \leq x, \text{ then } y \leq u \text{ implies that } (u \leq z \text{ if and only if } u \wedge x \leq y).$$

The next lemma provides an axiomatic description of wr-pseudocomplementation.

Lemma 1. *Let S be a semilattice, and let $*$ be an additional partial operation on S such that $x * y$ is defined if and only if $y \leq x$. Then the following assertions are equivalent:*

(a) *$*$ is a wr-pseudocomplementation,*
(b) *$*$ satisfies the conditions*
  (b1) *if $y \leq x$ then $x \wedge (x * y) \leq y$,*
  (b2) *if y is the greatest lower bound of u and x, then $u \leq x * y$.*

Proof. Assume that $y \leq x$. The second defining condition in (1) with $z := x * y$ says that

$$y \leq x * y, \tag{i}$$

$$x \wedge (x * y) \leq y, \tag{ii}$$

$$\text{if } y \leq u \text{ and } u \wedge x \leq y, \text{ then } u \leq x * y. \tag{iii}$$

The inequality (i) actually follows from (iii) by $u := y$. Now, in virtue of the assumption, (b1) corresponds to (ii), and (b2), to (iii). ∎

In a wr-pseudocomplemented semilattice S, all elements of the kind $a * a$ are maximal; more exactly, $a * a = \max[a)$. It follows that the semilattice is up-directed: always $x, y \le (x \wedge y) * (x \wedge y)$, and then S actually must have the greatest element 1.

The subsequent definition generalizes the concept of relative annihilator along the lines of [15] (but the restriction $b \le a$, which was assumed there, is now removed). We first associate an equivalence relation $a^\sharp$ on S with every element $a \in S$ by

$$(x, y) \in a^\sharp \text{ if and only if } a \wedge x = a \wedge y.$$

It is worth to note that $a \le b$ if and only if $b^\sharp \subseteq a^\sharp$. A *weak annihilator* $\langle x, y\rangle$ *of* x *relative to* y is the equivalence class of $a^\sharp$ containing y. Observe that $x * y = \max\langle x, y\rangle$ whenever $y \le x$. A semilattice in which all weak relative annihilators have the greatest element is a *semi-Brouwerian semilattice* in the sense of [35]. As noted on p. 426 in [35], if $z = \max\langle x, y\rangle$, then $\langle x, y\rangle = [x \wedge y, z]$; the element z is called there the weak pseudocomplementation of x relative to y regardless of whether the inequality $y \le x$ holds (cf. also [39]). We still shall use the term in the restricted sense of Definition 1. As $\langle x, y\rangle = \langle x, x \wedge y\rangle$, a semilattice is semi-Brouwerian if and only if it is wr-pseudocomplemented.

Recall that, in a semilattice with the least element, the pseudocomplement of an element a is the greatest element disjoint from a. An arbitrary semilattice S is said to be *sectionally pseudocomplemented* if every its upper section $[p)$ is pseudocomplemented. Therefore, if x and y are elements of S with $x \ge y$, then the pseudocomplement of x in the upper section $[y)$ is an element z such that

$$z = \max\{u \ge y\colon\ u \wedge_y x = y\},$$

where $\wedge_y$ stands for the local meet operation in $[y)$. As easily seen, an element z of a semilattice is the weak pseudocomplement of x relative to y if and only if it is the pseudocomplement of x in $[y)$ (Proposition 8 in [15] shows that this is not necessery in arbitrary posets). Moreover, if z indeed is the pseudocomplement of x in $[y)$ and $x \le y'$, then $z \wedge y'$ is the pseudocomplement of x in the segment $[y, y']$. We thus come to conclusions already mentioned in Introduction.

Proposition 2. [35, Proposition 2.3] *The classes of wr-pseudocomplemented semilattices, sectionally pseudocomplemented semilattices and upper bounded semilattices with pseudocomplemented segments coincide.*

Let us now compare weak relative pseudocomplementation with relative pseudocomplementation. Recall that z is the *pseudocomplement of x relative to y* if

$$z = \max\{u\colon\ u \wedge x \leq y\}. \tag{2}$$

The results below are particular cases of those obtained in Section 4 of [15] for arbitrary posets. We provide them with simpler proofs.

Lemma 3. *Let S be a semilattice, and let x, y be its elements such that $y \leq x$. Then an element z is the pseudocomplement of x relative to y if and only if it is the weak relative pseudocomplement and y satisfies the condition*

$$\text{if } u \wedge x \leq y, \text{ then } y = u' \wedge x \text{ for some } u' \geq u. \tag{3}$$

Proof. Assume that $x \geq y$ and that z satisfies (2). Then $y \leq z$, and (1) also holds. Furthermore, if $u \wedge x \leq y$ for some u, put $u' = z$: then $u \leq u'$ and $u' \wedge x \leq y$. As $y \leq z, x$, eventually $y = u' \wedge x$.

If, conversely, z satisfies (1), then $z \wedge x \leq y$. If, moreover, $u \wedge x \leq y$ for some u, then (3) provides an element u' such that $u' \wedge x = y$ and $u \leq u'$. But $u' \leq z$ by the choice of z; so, $u \leq z$. Therefore, z satisfies also (2). ∎

An element p of a semilattice S is said to be *modular* if

$$x \wedge y \leq p \leq y \text{ implies that } p = x' \wedge y \text{ for some } x' \geq x,$$

and *distributive* if

$$x \wedge y \leq p \text{ implies that } p = x' \wedge y' \text{ for some } x' \geq x \text{ and } y' \geq y.$$

For example, the element y in (3) is modular (recall the supposition $y \leq x$). S itself is modular (distributive) if all its elements are modular (distributive). The subsequent corollary extends to semilattices Theorem 3 of [41] proved for lattices with pseudocomplemented segments.

Corollary 4. *A semilattice is relatively pseudocomplemented if and only if it is weakly relatively pseudocomplemented and modular.*

Proof. In virtue of the preceding lemma, it only remains to note that a semilattice in which the pseudocomplement of x relative to b exists whenever $x \geq b$ is still relatively pseudocomplemented: for arbitrary x and y, $u \wedge x \leq y$ if and only if $u \wedge x \leq x \wedge y$, and then the pseudocomplement of x relative to $x \wedge y$ is also its pseudocomplement relative to y. ∎

As every relatively pseudocomplemented semilattice is known to be distributive (see, e.g., Theorem 3.3 in [35]), we come to the following semilattice analogue of [41, Theorem 2].

Theorem 5. *A modular wr-pseudocomplemented semilattice is distributive.*

3. Extending wr-pseoudocomplementation

We shall deal in this section with algebras of kind $(A, \wedge, \to, 1)$, where $(A, \wedge, 1)$ is a semilattice with the largest element 1 and $\to$ is a (total) binary operation on A. Let us call such algebras *arrow semilatices.* Occasionally, we shall consider also operations $\rightsquigarrow$ and $\leftrightsquigarrow$ on an arrow semilattice defined by

$$(4) \qquad x \rightsquigarrow y := x \to (x \wedge y), \quad x \leftrightsquigarrow y := (x \rightsquigarrow y) \wedge (y \rightsquigarrow x).$$

Given an arrow semilattice A, we denote the derived algebra $(A, \wedge, \rightsquigarrow, 1)$, called the *shadow* of A, by $A^{\rightsquigarrow}$.

Definition 2. We call a wr-pseudocomplemented semilattice *extended* if it is equipped with a binary operation $\to$ satisfying the condition

$$(5) \qquad x \to y = x * y \text{ whenever } y \leq x,$$

and consider such semilattices as arrow semilattices. Equivalently, an extended wr-pseudocomplemented semilattice is an arrow semilatice such that $x \to y$ is the wr-pseudocomplement of x relative to y whenever $y \leq x$. Let us denote the class of all extended wr-pseudocomplemented semilattices (*EWR-semilattices*, for short) by $\mathsf{EWR}^{\wedge}$.

For example, it follows from Lemma 3 that every semilattice with relative pseudocomplementation, or Brouwerian semilattice, is an EWR-semilattice. Semi-Brouwerian semilattices (see the previous section) also may, and will, be regarded as extended wr-pseudocomplemented semilattices, where $a \to b$ stands for $\max\langle a, b\rangle$. Then

$$(6) \qquad x \to y = x * (x \wedge y).$$

This relationship, which is essentially the equation (2.6) in [35], may be used as an alternative definition of the operation $\to$ in terms of meet and wr-pseudocomplementation; just in this way a total binary extension of sectional pseudocomplementation in a meet semilattice was introduced in [10].

Being wr-pseudocomplemented, each EWR-semilattice incorporates a semi-Brouwerian semilattice in a sense. The following proposition states this in precise terms.

Proposition 6. *An arrow semilattice is an EWR-semilattice if and only if its shadow is a semi-Brouwerian semilattice.*

Proof. Assume that A is an arrow semilattice. If A is in $\mathsf{EWR}^{\wedge}$, then (5) implies that the operations $\rightsquigarrow$ and $*$ are interrelated as follows:

$$(7) \qquad x \rightsquigarrow y = x * (x \wedge y).$$

Hence, the semilattice $A^{\rightsquigarrow}$ belongs to $\mathsf{EWR}^{\wedge}$ and is even semi-Brouwerian (compare (7) with (6)). Conversely, if the semilattice $A^{\rightsquigarrow}$ is semi-Brouwerian, then (7) holds and implies (5), i.e., A is an EWR-semilattice. ■

The subsequent characteristic of $\mathsf{EWR}^{\wedge}$ follows immediately from Lemma 1 and (5) (we consider any inequality $t_1 \leq t_2$ as an equation $t_1 = t_1 \wedge t_2$).

Proposition 7. *The class $\mathsf{EWR}^{\wedge}$ is a variety determined by the semilattice axioms and identities*

$(\to_1)$: $x \wedge (x \to (x \wedge y)) \leq y,$
$(\to_2)$: $x \leq y \to (x \wedge y).$

For further reference, we list some elementary properties of the operations $\to$ and $\rightsquigarrow$ in extended wr-pseudocomplemented semilattices.

Lemma 8. *The following holds in every EWR-semilattice.*

$(\to_3)$: $x \to x = 1,$
$(\to_4)$: $1 \to x = x,$
$(\to_5)$: $x \wedge (x \to (x \wedge y)) = x \wedge y,$
$(\to_6)$: $x \rightsquigarrow x = 1,$
$(\to_7)$: $1 \rightsquigarrow x = x,$
$(\to_8)$: $x \rightsquigarrow 1 = 1,$
$(\to_9)$: $x \leq y \rightsquigarrow x,$
$(\to_{10})$: $x \wedge (x \rightsquigarrow y) = x \wedge y,$
$(\to_{11})$: $x \leq y$ *iff* $x \rightsquigarrow y = 1.$

Proof. It follows from $(\to_2)$ that $1 \leq x \to x$ and $x \leq 1 \to x$, and from $(\to_1)$, that $1 \to x \leq x$. Therefore $(\to_3)$, $(\to_4)$ and, furthermore, $(\to_6)$ and $(\to_7)$ hold. $(\to_5)$ and $(\to_{10})$ are equivalent to the conjunction of $(\to_1)$ and the inequality $x \wedge y \leq x \to (x \wedge y)$, a consequence of $(\to_2)$. The identity $(\to_8)$ follows from $(\to_3)$, and $(\to_{11})$ follows from $(\to_3)$ and $(\to_1)$. The identity $(\to_9)$ is another version of $(\to_2)$. ■

We now move to distributive EWR-semilattices. The next proposition immediately follows from Theorem 5 above and (in view of Proposition 6) Corollary 3.4 in [35], which states that a semi-Brouwerian semilattice is Brouwerian if and only if it is distributive. We give it a short direct proof.

Proposition 9. *The following conditions on an EWR-semilattice A are equivalent:* (i) *A is modular,* (ii) *A is distributive,* (iii) *$A^{\rightsquigarrow}$ is a Brouwerian semilattice.*

Proof. (i) and (ii) are equivalent by Theorem 5. As every Brouwerian semilattice is distributive, (iii) implies (ii). It remains to recall Lemma 3 and the fact that the pseudocomplement of x relative to arbitrary y coincides with its pseudocomplement relative to $x \wedge y$; the equality (7) then shows that (in a modular semilattice) the operation $\rightsquigarrow$ is relative pseudocomplementation. ■

Let $\mathsf{dEWR}^{\wedge}$ stand for the class of all distributive EWR-semilattices. Given such a semilattice $A := (A, \wedge, \to, 1)$, we denote by A^{+} the common expansion $(A, \wedge, \rightsquigarrow, \to, 1)$ of A and the Brouwerian semilattice $A^{\rightsquigarrow}$. Notice that A is term equivalent to A^{+}. The class BS^{+} of all these expansions is a variety definable by the set of equations consisting of (i) $\mathsf{EWR}^{\wedge}$ axioms for $(A, \wedge, \to, 1)$, (ii) Brouwerian semilattice axioms for $(A, \wedge, \rightsquigarrow, 1)$, and (iii) the first identity from (4) (due to which it immediately follows that $\mathsf{dEWR}^{\wedge}$ is likewise an equational class).

Actually, it is enough to include in (ii) only any identity that causes a semi-Brouwerian semilattice $A^{\rightsquigarrow}$ to be Brouwerian. An example of such an identity, provided by Proposition 2.1 in [35], is $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$ (see also subsection 6.1 below). Summing up, we come to the following conclusion.

Corollary 10. *The class $\mathsf{dEWR}^{\wedge}$ is a subvariety of $\mathsf{EWR}^{\wedge}$ determined by the identity*

$$(\to_{12}):\quad x \to (x \wedge y \wedge z) = (x \to (x \wedge y)) \wedge (x \to (x \wedge z)).$$

4. Some congruence properties of $\mathsf{EWR}^{\wedge}$

In this section we turn to congruence properties of the variety $\mathsf{EWR}^{\wedge}$. The reader is referred to the monographs [7], [9] or [11] for information on general congruence properties of algebras and varieties mentioned in the subseqent theorem. We only remind that an algebra is arithmetical (at 1) if and only if it is congruence distributive and permutable (at 1); permutable at 1 algebras are known also as subtractive. As usual, the notation $\Theta(a, b)$ stands for the principal congruence generated by the pair (a, b), i.e., the intersection of all congruences θ such that $a\,\theta\,b$. The lattice of congruences of an algebra A is denoted by $C(A)$.

Theorem 11. *Every EWR-semilattice A has the following properties:*

(a) *it is arithmetical at* 1,
(b) *it is weakly regular at* 1 *(*1*-regular).*

Hence,

(c) *A is congruence distributive,*
(d) *the congruence kernels $1/\theta$ of A form a lattice $N(A)$ (with intersection as its meet),*
(e) *the mapping $\theta \mapsto 1/\theta$ is a lattice isomorphism $C(A) \to N(A)$.*

Furthermore,

(f) *if A is distributive, then it is arithmetical.*

Proof. (a) By [9, Theorem 8.3.2(iii)]. The identities ($\to_6$), ($\to_8$) and ($\to_4$) show that $t(x,y) := y \rightsquigarrow x$ is the corresponding witness term: $t(x,x) = 1$, $t(1,x) = 1$, $t(x,1) = x$.

(b) By [9, Theorem 6.4.3(iii)]. The identities ($\to_6$) and ($\to_{11}$) show that $t(x,y) := x \leftrightsquigarrow y$ is the witness term: $t(x,x) = 1$, and if $t(x,y) = 1$, then $x = y$.

The further properties are well-known consequences of (a) and (b). For (c), see (b), (a) and Theorem 8.2.8 of [9]. For (d), see (a) and Propositions 1.4 and 1.5 in [40]. The surjective mapping indicated in (e) is a homomorphism ([40, Proposition 1.2]). The congruence kernels of the algebra A are its normal ideals in the general terminology assumed in that paper. By (a) and (b), $\mathsf{EWR}^{\wedge}$ is an ideal-determined variety ([40, p. 206]; see also [9, Theorem 10.1.13]), i.e., every ideal of A is the kernel of exactly one congruence. So, the mapping $\theta \mapsto 1/\theta$ is also injective and, eventually, an isomorphism.

As to (f), by Proposition 9, $A^{\rightsquigarrow}$ is a Brouwerian semilattice. It is known well that every Brouwerian semilattice is arithmetical (see Theorem 4.3.1 in [11]). Since $C(A) \subseteq C(A^{\rightsquigarrow})$, the algebra A itself also is arithmetical. ∎

It follows from the proof of (b) that $x \leftrightsquigarrow y$ is the so called *Gödel equivalence term* for $\mathsf{EWR}^{\wedge}$ (see [9, Definition 9.4.1] or p. 336 in [4]). The subsequent lemma, suggested by Theorem 3.6 in [35], gives a characteristic property of the term.

Lemma 12. *Suppose that θ is a congruence of an EWR-semilattice A. Then*

$$x \,\theta\, y \text{ if and only if } (x \leftrightsquigarrow y) \in 1/\theta.$$

Proof. The kernel $1/\theta$ of θ is a filter of A. If $x \,\theta\, y$, then $(x \rightsquigarrow y) \,\theta\, (y \rightsquigarrow y) = 1$ by ($\to_6$), and $(x \rightsquigarrow y) \in 1/\theta$. Likewise, $(y \rightsquigarrow x) \in 1/\theta$. Therefore, $(x \leftrightsquigarrow y) \in 1/\theta$. If, conversely, $(x \leftrightsquigarrow y) \in 1/\theta$, then $(x \rightsquigarrow y) \,\theta\, 1$, wherefrom $x \wedge y = (x \wedge (x \rightsquigarrow y)) \,\theta\, x$ (see ($\to_{10}$)). Likewise, $(x \wedge y) \,\theta\, y$, and then $x \,\theta\, y$. ∎

An arrow semilattice A is said to be *strongly 1-regular* (see [29, p. 483]), if, for all $a, b \in A$, there are $c, d \in A$ such that $\Theta(a,b) = \Theta(d,1)$ and $\Theta(a,1) \vee \Theta(b,1) = \Theta(c,1)$. Therefore, A is strongly 1-regular if and only if every compact (i.e., finitely generated) congruence of A is of the form $\Theta(a,1)$. Clearly, every strongly 1-regular algebra is 1-regular. The next result generalises a part of Theorem 4.1 in [29].

Theorem 13. *Every EWR-semilattice is strongly 1-regular: for all a, b,*

(a) $\Theta(a,b) = \Theta(a \leftrightsquigarrow b, 1)$,

(b) $\Theta(a,1) \vee \Theta(b,1) = \Theta(a \wedge b, 1)$.

Proof. By Lemma 12, $a\,\theta\,b$ if and only if $(a \leftrightsquigarrow b)\,\theta\,1$ for every congruence θ; this immediately implies (a). For (b), abridge the notations $\Theta(a \wedge b, 1)$, $\Theta(a, 1)$, $\Theta(b, 1)$ to θ, θ_1, θ_2, respectively. Then $a = (a \wedge 1)\,\theta\,(a \wedge (a \wedge b)) = (a \wedge b)\,\theta\,1$. Thus, $\Theta(a, 1) \subseteq \Theta(a \wedge b, 1)$, and similarly $\Theta(b, 1) \subseteq \Theta(a \wedge b, 1)$. On the other hand, $(a \wedge b)\ (\theta_1 \vee \theta_2)\ (1 \wedge 1) = 1$, wherefrom $\Theta(a \wedge b, 1) \subseteq \Theta(a, 1) \vee \Theta(b, 1)$. ∎

The set $\mathit{Cp}(A)$ of principal congruences of any algebra A is a lower bounded join subsemilattice of $C(A)$ (the identity relation $\Theta(1, 1)$ is its least element). The theorem shows that, for EWR-semilattices, the mapping $a \mapsto \Theta(a, 1)$ is a surjective semilattice antihomomorphism $A \to \mathit{Cp}(A)$. Following the general definition in [22, p. 600], [4, p. 356], we say that an arrow semilattice A is *congruence orderable* if this mapping is injective, and *Fregean* if A is congruence orderable and 1-regular.

Proposition 14. *An EWR-semilattice A is congruence orderable if and only if the mapping $a \mapsto \Theta(a, 1)$ is an anti-isomorphism between the semilattices A and $\mathit{Cp}(A)$:*

$$a \leq b \text{ if and only if } \Theta(b, 1) \subseteq \Theta(a, 1). \tag{8}$$

Theorem 11(b) implies that a congruence orderable EWR-semilattice is always Fregean. Due to Theorem 13, the subsequent proposition is, in fact, a particular case of Theorem 4.3(1,2) of [29]: for the generic operations & and $\triangle$ mentioned in it, we may substitute $\wedge$ and $\leftrightsquigarrow$ respectively. (In [29], 1-regularity is not required in the definition of Fregean algebra.)

Proposition 15. *Suppose that A is a congruence orderable EWR-semilattice. Then $A^{\rightsquigarrow}$ is a Brouwerian semilattice if and only if $C(A) = C(A^{\rightsquigarrow})$.*

The reader is referred to [3, 4] for information on equationally definable principal congruences.

Theorem 16. *Consider the following conditions on an arrow semilattice A:*

(a) *$A^{\rightsquigarrow}$ is a Brouwerian semilattice, and every congruence of $A^{\rightsquigarrow}$ is a congruence of A, i.e., $C(A) = C(A^{\rightsquigarrow})$,*
(b) *A has equationally definable principal congruences: if $\theta = \Theta(a, b)$, then*

$$x\,\theta\,y \text{ if and only if } (a \leftrightsquigarrow b) \wedge x = (a \leftrightsquigarrow b) \wedge y,$$

(c) *for every a, $\Theta(a, 1) = a^{\sharp}$,*
(d) *A is congruence orderable.*

If A is in $\mathsf{EWR}^{\wedge}$, then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). *If A is also distributive, then* (d) *implies* (a).

Remark: we shall see in the next section that actually a congruence orderable EWR-semilattice is always distributive (Theorem 21 and Corollary 20), so that all conditions (a)–(d) are, in fact, equivalent in every EWR-semilattice.

Proof. Assume that $A \in \mathsf{EWR}^{\wedge}$.

(a) $\to$ (b). If the semilattice $A^{\rightsquigarrow}$ is indeed Brouwerian, then it has equationally definable principal congruences ([3], p. 199 and Example 15 of Section 1). If A has the same congruences, then equations (4) allow us to conclude that (b) holds.

(b) $\to$ (c). By virtue of ($\to_7$) and ($\to_8$), condition (b) implies that $(x, y) \in \Theta(a, 1)$ if and only if $x\, a^\sharp\, y$.

(c) $\to$ (d). By Proposition 14, as (c) implies (8).

(d) $\to$ (a). If A is distributive, then $A^{\rightsquigarrow}$ is a Brouwerian semilattice (Proposition 9), and we may apply Proposition 15. ∎

5. Congruences and filters of $\mathsf{EWR}^{\wedge}$-semilattices

Every congruence kernel of an $\mathsf{EWR}^{\wedge}$-algebra A is a filter of its underlying semilattice, i.e., $N(A) \subseteq F(A)$, where $F(A)$ stands for the lattice of filters of A. We are now going to find out when the converse holds.

Given a filter F of A, we denote by $F^\sharp$ the equivalence relation $\{(x, y)\colon\ a \wedge x = a \wedge y$ for some $a \in F\}$. It is the smallest semilattice congruence of which F is the kernel; we call such relations *filter-induced.* The transformation $F \mapsto F^\sharp$ is injective, for always $1/(F^\sharp) = F$. A *filter congruence* of A is a congruence of A that is induced in this way by some filter.

Notice that, in a particular case when F is the principal filter $[a)$, $F^\sharp$ coincides with the relation $a^\sharp$ introduced in Section 2: $b \geq a$ and $x\, b^\sharp\, y$ imply that $x\, a^\sharp\, y$.

Lemma 17. *Every congruence of an* EWR*-semilattice is a filter congruence; more exactly,* $\theta = (1/\theta)^\sharp$.

Proof. Let F stand for the kernel of θ. If $x\,\theta\, y$, then $a := (x \leftrightsquigarrow y) \in F$ (Lemma 12). By ($\to_9$) and ($\to_{10}$), then $a \wedge x = x \wedge y = a \wedge y$, and $x\, F^\sharp\, y$. Conversely, if $x\, F^\sharp\, y$, then $a \wedge x = a \wedge y$ for some a with $a\,\theta\, 1$. It follows that $(a \wedge x)\,\theta\, x$, $(a \wedge y)\,\theta\, y$ and $x\,\theta\, y$. Eventually, $\theta = F^\sharp$. ∎

Therefore, for every $F \in N(A)$,

$$x\, F^\sharp\, y \text{ if and only if } (x \leftrightsquigarrow y) \in F.$$

Lemma 18. *Suppose that F is a filter of an* EWR*-semilattice A. The following assertions are equivalent:*

(a) *F is a congruence kernel,*
(b) *$F^\sharp$ is a congruence,*

(c) *for every* $x \in F$ *and all* $y, z \in A$,

(9) $$(y \to z)\, F^\sharp\, ((x \wedge y) \to (x \wedge z)).$$

Proof. (a) $\leftrightarrow$ (b). It follows from the previous lemma and equality $F = 1/F^\sharp$ that if $\theta \in C(A)$, then

(10) $$F = 1/\theta \text{ if and only if } \theta = F^\sharp.$$

(b) $\to$ (c). If $F^\sharp$ is a congruence and $x \in F$, then $1\, F^\sharp\, x$, wherefrom $y\, F^\sharp\, (x \wedge y)$ and $z\, F^\sharp\, (x \wedge z)$. Now (9) follows.

(c) $\to$ (b). The equivalence $F^\sharp$ is always compatible with $\wedge$. Further, if $x\, F^\sharp\, y$ and $u\, F^\sharp\, v$, then $a \wedge x = a \wedge y$ and $b \wedge u = b \wedge v$ for appropriate $a, b \in F$. As $c := a \wedge b \in F$, we now use (9):

$$(x \to u)\, F^\sharp\, ((c \wedge x) \to (c \wedge u)) = ((c \wedge y) \to (c \wedge v))\, F^\sharp\, (y \to v).$$

Hence, $(x \to u)\, F^\sharp\, (y \to v)$, and $F^\sharp$ is compatible also with $\to$. ∎

Theorem 19. *The following conditions on an* EWR*-semilattice* A *are equivalent:*

(a) *every filter of* A *is a congruence kernel, i.e.,* $F(A) = N(A)$,
(b) *every principal filter is a congruence kernel,*
(c) *every filter-induced equivalence on* A *is a congruence,*
(d) *the mapping* $F \mapsto F^\sharp$ *is a lattice isomorphism* $F(A) \to C(A)$,
(e) A *satisfies the identity*

$(\to_{13})$: $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$.

Proof. Equivalence of (a) and (c) follows from the preceding lemma, (b) is a part of (a), and (e) implies (a): due to $(\to_{13})$, the condition (c) in the lemma is fulfilled for every filter F. Further, if this condition is fulfilled for all principal filters, then $(\to_{13})$ holds; so (b) implies (e). At last, (a) and (d) are equivalent by virtue of Theorem 11(e) and (10). ∎

Definition 3. An $\mathsf{EWR}^\wedge$-algebra satisfying $(\to_{13})$ will be called an *almost Brouwerian semilattice.* We let ABS stand for the variety of all such semilattices.

If A is an almost Brouwerian semilattice, then, in virtue of Theorems 19(d) and 11(c), its lattice of filters is distributive. It is well-known (see, e.g., Lemma 5.1(iii) of Chapter II in [20]) that this conclusion extends to the semilattice A itself.

Corollary 20. *Every almost Brouwerian semilattice is distributive.*

Of course, every Brouwerian semilattice is almost Brouwerian. We now can relate the properties of filters listed in Theorem 19 with the congruence properties from Theorem 16.

Theorem 21. *An* EWR*-semilattice is almost Brouwerian if and only if it is congruence orderable.*

Proof. Assume that A is an almost Brouwerian semilattice. According to Proposition 9, then $A^{\rightsquigarrow}$ is a Brouwerian semilattice, so that the mapping $F \mapsto F^{\sharp}$ is an isomorphism from $F(A^{\rightsquigarrow})$ onto $C(A^{\rightsquigarrow})$ (see, e.g., Corollary 4.3.4 in [11]). But the algebras A and $A^{\rightsquigarrow}$ have the same filters, and, by Theorem 19(d), also the same congruences. By Theorem 16, then A is congruence orderable.

Now assume that A is a congruence orderable EWR-semilattice. Then $[a) = 1/\Theta(a,1)$: by (8), $b \in [a)$ if and only if $\Theta(b,1) \subseteq \Theta(a,1)$, which implies that $(b,1) \in \Theta(a,1)$, and the converse implication holds in virtue of the definition of principal congruence. Therefore, $[a)$ is always a congruence kernel, and A is almost Brouwerian by Theorem 19(b,e). ∎

An operation on a Brouwerian semilattice B is said to be *compatible* if it preserves all congruences of B ([22, p. 609]; cf. [29, p. 497]). Corollary 4.1 of [22] asserts that a strongly 1-regular congruence orderable variety is term equivalent to a variety of Brouwerian algebras with (additional) compatible operations. Theorem 19(a,c) and Proposition 9 together with the discussion subsequent to the latter yield the following specification of this result.

Corollary 22. *The variety* ABS *is term equivalent to the variety* BS^{+} *of Brouwerian semilattices* $(A, \wedge, \rightsquigarrow, \rightarrow, 1)$ *with a compatible operation* $\rightarrow$.

6. Some other subvarieties of $\mathsf{EWR}^{\wedge}$

6.1. Semi-Brouwerian semilattices

The extension rule (6) implies that every wr-pseudocomplemented semilattice can uniquely be extended to a semi-Brouwerian semilattice. In fact, an EWR-semilattice is semi-Brouwerian if and only if it satisfies the identity

$(\rightarrow_{14})$: $x \rightarrow y = x \rightarrow (x \wedge y)$,

a counterpart of (6). Therefore, the class SBS of all semi-Brouwerian semilattices is a subvariety of $\mathsf{EWR}^{\wedge}$. That SBS is a variety, was discovered already in [35]: it follows from Proposition 2.1 in [35] and the discussion subsequent to it that the equations $(\rightarrow_{14})$ and

$(\rightarrow_{15})$: $y \leq x \rightarrow y$,
$(\rightarrow_{16})$: $x \wedge (x \rightarrow y) = x \wedge y$

are the characteristic axioms for $\rightarrow$ in an arbitrary semi-Brouwewrian semilattice (see also [39, Theorem 3]). One more equational description of SBS can be obtained from Theorem 3 of [10]. Note that, due to ($\to_{15}$), the identity ($\to_{16}$) can be weakened to

($\to_{17}$): $x \wedge (x \to y) \leq y$.

A comparison of ($\to_{15}$) and ($\to_{17}$) with ($\to_2$) and ($\to_1$) immediately gives us another proof for Proposition 6.

A semi-Brouwerian semilattice A is Brouwerian if and only it is distributive (Corollary 3.4 in [35]). According to Propositions 2.1 and 3.5 of [35], this condition is in its turn obeyed if any of the following equations is fulfilled in A:

$$x \to (y \wedge z) = (x \to y) \wedge (x \to z), \quad x \to (y \to z) = (x \wedge y) \to z.$$

The paper [39] contains a long list of identities that hold in semi-Brouwerian lattices, as well as various identities of Brouwerian lattices that generally do not hold in semi-Brouwerian lattices. Among the latter there are some well-known pure implicational formulas, for example,

($\to_{18}$): $x \leq (x \to y) \to y$.

Regretfully, the term 'semi-Brouwerian semilattice' has been used also in various other senses. Another class of algebras interesting in the context of the pesent paper and having the same name is reviewed in subsection 6.4.

6.2. Semilattices with sectional join-pseudocomplementation

Sectionally pseudocomplemented lattices are discussed also in [8]. However, instead of (6), another relation

$$x \to y := (x \vee y) * y \tag{11}$$

(in notation of the present paper) was used in [8] to turn the partial operation $*$ into a total one. In the further papers [12, 13], as well as in the monograph [11] sectionally pseudocomplemented lattices are treated as extended by means of (11). The class of such lattices is a subvariety of EWR-lattices, which is defined by one additional axiom

($\to_{19}$): $x \to y = (x \vee y) \to y$.

In [14], the definition (11) was extended to sectionally pseudocomplemented posets as follows:

$$x \to y := \max\{z * y\colon\ x, y \leq z\}; \tag{12}$$

however, the maxima at the right need not always exist. A sectionally pseudocomplemented poset extended by means of (12) was called *sectionally j-pseudocomplemented* ('j' refers to the join operation in (11); a semi-Brouwerian semilattice could likewise be called sectionally m-pseudocomplemented with 'm' for meet in (6)).

Proposition 23. *The class* $\mathsf{SjP}^{\wedge}$ *of all sectionally j-pseudocomplemented semilattices is a subvariety of* $\mathsf{EWR}^{\wedge}$ *specified by equations* $(\to_{16})$, $(\to_{18})$ *and*

$$(\to_{20}):\ \ y \to z \leq (x \wedge y) \to z.$$

Proof. It follows from [14, Corollary 4(b)] that the operation $\to$ in a SjP-semilattice can be characterised by the equations $(\to_2)$, $(\to_4)$, $(\to_{16})$, $(\to_{18})$, $(\to_{20})$ and $(x \wedge y) \to y = 1$. The latter one is a consequence of $(\to_{20})$ and $(\to_3)$. On the other hand, $(\to_1)$ is an easy concequence of $(\to_{16})$. ∎

Observe that the three $\mathsf{SjP}^{\wedge}$-axioms are fulfilled in every Brouwerian semilattice.

6.3. Sectionally pseudocomplemented semilattices as total algebras

Theorem 3 of [21] states that a meet semilattice with the greatest element is sectionally pseudocomplemented if and only if it admits a (total) binary operation $\to$ subject to axioms $(\to_3)$, $(\to_{16})$ and

$$(\to_{21}):\ \ x \wedge ((x \wedge y) \to z) = x \wedge (y \to (x \wedge z)).$$

Its proof implies that the operation $\to$ obeys also (5). Against this background, sectionally pseudocomplemented semilattices are treated in [21] as arrow semilattices satisfying the mentioned axioms (but see the last paragraph of this subsection). The class of all these algebras is thus a subvariety of $\mathsf{EWR}^{\wedge}$. To avoid any confusion, we continue to use the term 'sectionally pseudocomplemented semilattice' in its initial sense of Section 2, and refer to the extended wr-semilattices from the subvariety as to $\mathsf{SPS}_{\mathrm{HK}}$-algebras ('HK' for the authors' names).

The identity $(\to_{21})$ coincides with the identity R3s from Section 3 of [35], which holds in every semi-Brouwerian algebra. Therefore, SBS is a subvariety of $\mathsf{SPS}_{\mathrm{HK}}$. (It was also stated in the proof of Theorem 3 in [21] that all axioms of $\mathsf{SPS}_{\mathrm{HK}}$ are fulfilled in a sectionally pseudocomplemented semilattice extended by $(\to_6)$.) As the following example of an EWR-semilattice borrowed from [34] shows, the subvariety is proper.

Example 1. Let $\mathbf{A}_2$ be the meet semilattice $\{0, 1\}$ with $0 \leq 1$ and the operation $\to$ defined by $x \to y = 1$ iff $x = y$. It belongs to $\mathsf{SPS}_{\mathrm{HK}}$, but does not satisfy $(\to_{14})$ (take $x = 0$, $y = 1$).

The same example refutes Theorem 4 of [21], which states that the operation $\to$ in a distributive $\mathsf{SPS}_{\mathrm{HK}}$-algebra is relative pseudocomplementation (the identity $(\to_{14})$ is implicitly used in its proof). Consequently, Theorem 11 of [21], which rests on that theorem (and asserts that an $\mathsf{SPS}_{\mathrm{HK}}$-algebra in which the natural correspondence $\theta \mapsto 1/\theta$ between congruences and filters is one-to-one, is a Brouwerian semilattice), is likewise wrong (cf. Theorem 19 above). Furthermore, an unjustified use of ($\to_{14}$) appears also in the proof of Theorem 15 in [21].

Based on [21] are Sections 5.1 and 5.2 in [11]. The above comments concern respectively Theorems 5.1.4, 5.2.7 and 5.2.10 therein.

There is also some nonconformity in terminology: the algebra $\mathbf{A}_2$ is an example of an $\mathsf{SPS}_{\mathrm{HK}}$-algebra which happens to be a lattice, but nevertheless is not a sectionally pseudocomplemented lattice in the sense of the preceding subsection, for it does not satisfy ($\to_{19}$) (again, take $x = 0$ and $y = 1$). Conversely, there are such lattices which, considered as semilattices, are not sectionally pseudocomplemented semilattices in the sense of this section.

6.4. Semi-Brouwerian semilattices: another version

In [34], a semi-Heyting algebra is defined to be a bounded lattice with an additional operation $\to$ fulfilling ($\to_3$), ($\to_{16}$) and ($\to_{13}$). It is also proposed there to call a semi-Brouwerian semilattice any meet semilattice with 1 and operation $\to$ which is subject to these three axioms. To avoid the evident terminological conflict, we choose here the symbol $\mathsf{SBS}_{\mathrm{S}}$ for the variety of these algebras.

As noted in Section 13 of [34], most results obtained in that paper for semi-Heyting algebras hold true, when appropriately modified (if needed) also for $\mathsf{SBS}_{\mathrm{S}}$-algebras. In particular, every $\mathsf{SBS}_{\mathrm{S}}$-algebra has pseudocomplemented segments and is, hence, an EWR-semilattice. (Actually, it was proved already in [10, Lemma 3] that an arrow semilattice satisfying ($\to_3$), ($\to_{16}$) ($\to_{13}$) is sectionally pseudocomplemented.) Furthermore, ($\to_{13}$) can be replaced by two identities

$$x \wedge (y \to z) = x \wedge ((x \wedge y) \to z), \quad x \wedge (y \to z) = x \wedge (y \to (x \wedge z)).$$

It immediately follows that $\mathsf{SBS}_{\mathrm{S}}$ is a subvariety of $\mathsf{SPS}_{\mathrm{HK}}$.

Really, the variety $\mathsf{SBS}_{\mathrm{S}}$ coincides with ABS: the axiom ($\to_3$) of $\mathsf{SBS}_{\mathrm{S}}$ holds already in $\mathsf{EWR}^{\wedge}$, while ($\to_{16}$) follows from ($\to_5$) in virtue of ($\to_{13}$) (with $y = x$); on the other hand, ($\to_1$) and ($\to_2$), the $\mathsf{EWR}^{\wedge}$-axioms, follow by substituting $x \wedge y$ for y in ($\to_{16}$) and, respectively, x for y and $x \wedge y$ for z in ($\to_{13}$). It should be noted in this context that several results proved above have been obtained in [34] in another way for $\mathsf{SBS}_{\mathrm{S}}$-algebras. Moreover, Theorem 7.5 of [34] provides a description of subdirectly irreducible semi-Heyting algebras. We transfer it to almost Brouwerian semilattices in the following form.

Proposition 24. *A non-trivial almost Brouwerian semilattice is subdirectly irreducible if and only if there is the greatest element in it below* 1.

6.5. Interrelations between the subvarieties

We already have noticed several inclusions between the varieties of arrow semilattices discussed in the last four sections. Let us sum up the relevant information.

Variety	Axioms
$\mathsf{EWR}^\wedge$	$(\to_1) + (\to_2)$
$\mathsf{dEWR}^\wedge$	$\mathsf{EWR}^\wedge + (\to_{12})$
ABS	$\mathsf{EWR}^\wedge + (\to_{13})$
SBS	$\mathsf{EWR}^\wedge + (\to_{14})$
	$(\to_{14}) + (\to_{15}) + (\to_{16})$
$\mathsf{SjP}^\wedge$	$\mathsf{EWR}^\wedge + (\to_{16}) + (\to_{18}) + (\to_{20})$
$\mathsf{SBS}_{\mathrm{HK}}$	$(\to_3) + (\to_{16}) + (\to_{21})$
$\mathsf{SBS}_{\mathrm{S}}$	$(\to_3) + (\to_{13}) + (\to_{16})$

Proposition 25. *The poset of the considered varieties of arrow semilattices consists of the following chains (BS is the variety of Brouwerian semilattices):*

$$\mathsf{BS} \subseteq \mathsf{ABS}\ (= \mathsf{SBS}_{\mathrm{S}}) \subseteq \mathsf{dEWR}^\wedge \subseteq \mathsf{EWR}^\wedge,$$
$$\mathsf{BS} \subseteq \mathsf{SBS} \subseteq \mathsf{SBS}_{\mathrm{HK}} \subseteq \mathsf{EWR}^\wedge,$$
$$\mathsf{BS} \subseteq \mathsf{ABS} \subseteq \mathsf{SBS}_{\mathrm{HK}} \subseteq \mathsf{EWR}^\wedge,$$
$$\mathsf{BS} \subseteq \mathsf{SjP}^\wedge \subseteq \mathsf{EWR}^\wedge$$

(see Section 5 (page 662), Corollary 20, subsections 6.1, 6.3, 6.4, 6.2).

A number of subvarieties of semi-Heyting and $\mathsf{SBS}_{\mathrm{S}}$-algebras have been considered in [34]. See also [1].

We are now going to examine relations between these varieties more carefully. To proceed, we need more examples of EWR-semilattices.

Example 2. Let $A := \{0, a, 1\}$ be a three-element chain with $0 < a < 1$. Clearly, it is a distributive semilattice. An operation $\to$ on A is an extended wr-pseudocomplementation if and only if it fits in with the partial table on the left:

$*$	0	a	1
0	1		
a	0	1	
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	0	1	0
1	0	a	1

$\to$	0	a	1
0	1	0	a
a	0	1	1
1	0	a	1

Therefore, any completion of it leads us to an algebra in $\mathsf{dEWR}^\wedge$. Let as denote by $\mathbf{A}_3^1$ the algebra in which $\to$ has the table in the middle, and by $\mathbf{A}_3^2$, that on the right.

Example 3. Let A be the non-distributive five-element semilattice (pentagon) with two maximal chains $0 < a < c < 1$ and $0 < b < 1$. An operation $\to$ on A is an extended wr-pseudocomplementation if and only if it fits in with the partial table on the left:

$*$	0	a	b	c	1
0	1				
a	b	1			
b	c		1		
c	b	a		1	
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	c	a	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	0	1
b	c	a	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

Calculations by means of (12) lead us to the table in the middle. We have thus obtained the single $\mathsf{SjP}^\wedge$-algebra based on the pentagon, which we denote by $\mathbf{A}_5^1$. It is also an $\mathsf{SPS}_{\mathrm{HK}}$-algebra. Example 2 of [21] shows another $\mathsf{SBS}_{\mathrm{HK}}$-algebra based on the same pentagon, which differs from $\mathbf{A}_5^1$ only in that $b \to a = c$. The algebra $\mathbf{A}_5^2$ with the table on the right does not satisfy $(\to_{16})$ (put $x = a, y = c$) and, hence, does not belong to these two classes.

Theorem 26. *All inclusions indicated in Proposition* 25 *are proper.*

Proof. (a) $\mathsf{BS} \neq \mathsf{ABS}$. The arrow semilattice $\mathbf{A}_2$ from Example 1 is almost Brouwerian but not Brouwerian, for the implication $x \leq y \Rightarrow x \to y = 1$ fails in it for $x = 0$, $y = 1$.

(b) $\mathsf{ABS} \neq \mathsf{dEWR}^\wedge$. The EWR-semilattice $\mathbf{A}_3^1$ is distributive. However, it does not belong to ABS, for the equality $a \to 1 = 0$ contradicts to $(\to_{16})$, one of the $\mathsf{SPS}_{\mathrm{S}}$-axioms.

(c) $\mathsf{dEWR}^\wedge \neq \mathsf{EWR}^\wedge$. The pentagon-based EWR-semilattices from Example 3 are not distributive.

(d) $\mathsf{BS} \neq \mathsf{SBS}$. As noted on page 664, the Brouwerian inequality $(\to_{18})$ need not hold in a semi-Brouwerian semilattice.

(e) $\mathsf{SBS} \neq \mathsf{SBS}_{\mathrm{HK}}$. See Example 1.

(f) $\mathsf{SBS}_{\mathrm{HK}} \neq \mathsf{EWR}^\wedge$. The algebra $\mathbf{A}_3^1$ is not an $\mathsf{SBS}_{\mathrm{HK}}$-algebra: it falsifies $(\to_{21})$ with $x = y = a$, $z = 1$.

(g) $\mathsf{ABS} \neq \mathsf{SBS}_{\mathrm{HK}}$. The $\mathsf{SBS}_{\mathrm{HK}}$-algebra $\mathbf{A}_5^1$ does not belong to ABS, for all ABS-algebras are distributive.

(h) $\mathsf{BS} \neq \mathsf{SjP}^\wedge$. As the $\mathsf{SjP}^\wedge$-algebra $\mathbf{A}_5^1$ is not distributive, it does not belong to BS.

(i) $\mathsf{SjP}^\wedge \neq \mathsf{EWR}^\wedge$. The $\mathsf{SjP}^\wedge$-axiom $(\to_{20})$ fails in the EWR-semilattice $\mathbf{A}_3^1$: put $x = a$ and $y = z = 1$. ∎

Proposition 27. *The chains in Proposition* 25 *are maximal: no other inclusion holds between the varieties.*

Proof. (j) $\mathsf{SBS} \not\subseteq \mathsf{dEWR}^\wedge$ by (d) and Proposition 9. By Proposition 25, then also $\mathsf{SPS}_{\mathrm{HK}} \not\subseteq \mathsf{dEWR}^\wedge$and $\mathsf{SBS} \not\subseteq \mathsf{ABS}$.

(k) $\mathsf{dEWR}^\wedge \not\subseteq \mathsf{SBS}_{\mathrm{HK}}$: see the argument in (f) and observe that $\mathbf{A}_3^1$ is distributive. By Proposition 25, then also $\mathsf{dEWR}^\wedge \not\subseteq \mathsf{SBS}$.

(l) $\mathsf{ABS} \not\subseteq \mathsf{SBS}$: if an almost Brouwerian semilattice is semi-Brouwerian, then, being distributive, it should be Brouwerian. By (a), there are non-Brouwerian ABS-algebras.

(m) $\mathsf{ABS} \not\subseteq \mathsf{SjP}^\wedge$: the arrow semilattice $\mathbf{A}_2$ from Example 1 is almost Brouwerian, while the values $x = 0$, $y = 1$ do not satisfy (12). By Proposition 25, then also $\mathsf{dEWR}^\wedge \not\subseteq \mathsf{SjP}^\wedge$.

(n) $\mathsf{SjP}^\wedge \not\subseteq \mathsf{dEWR}^\wedge$: the $\mathsf{SjP}^\wedge$-algebra $\mathbf{A}_5^1$ is not distributive. By Proposition 25, then also $\mathsf{SjP}^\wedge\not\subseteq \mathsf{ABS}$.

(o) $\mathsf{SBS} \not\subseteq \mathsf{SjP}^\wedge$: see the argument in (d) and recall that $(\rightarrow_{18})$ is one of the axioms of $\mathsf{SjP}^\wedge$. By Proposition 25, then also $\mathsf{SPS}_{\mathrm{HK}} \not\subseteq \mathsf{SjP}^\wedge$.

(p) $\mathsf{SjP}^\wedge \not\subseteq \mathsf{SPS}_{\mathrm{HK}}$: the $\mathsf{SjP}^\wedge$-algebra $\mathbf{A}_5^1$ falsifies $(\rightarrow_{21})$ with $x = b$, $y = 1$ and $z = a$. By Proposition 25, then also $\mathsf{SjP}^\wedge \not\subseteq \mathsf{SBS}$. ∎

Proposition 28. *We have*

$$\mathsf{dEWR}^\wedge \cap \mathsf{SBS} = \mathsf{ABS} \cap \mathsf{SBS} = \mathsf{BS} = \mathsf{SBS} \cap \mathsf{SjP}^\wedge.$$

In contrary,

$$\mathsf{SPS}_{\mathrm{HK}}\cap \mathsf{SjP}^\wedge \neq \mathsf{BS} \quad \text{and} \quad \mathsf{dEWR}^\wedge \cap \mathsf{SBS}_{\mathrm{HK}} \neq \mathsf{ABS}.$$

Thus, BS is the only variety in Proposition 25 that is an intersection of others.

Proof. We already know that a semi-Brouwerian semilattice that is distributive or satisfies $(\rightarrow_{18})$ is Brouwerian; recall that the latter inequality is an $\mathsf{SjP}^\wedge$-axiom. Further, as explained in Example 3, $\mathbf{A}_5^1$ is both an $\mathsf{SBS}_{\mathrm{HK}}$-algebra and $\mathsf{SjP}^\wedge$-algebra; since it is not distributive, it does not belong to BS. The distributive $\mathsf{EWR}^\wedge$-algebra $\mathbf{A}_3^2$ belongs to $\mathsf{SPS}_{\mathrm{HK}}$, but not to ABS: values $x = a, y = 0, z = 1$ do not satisfy $(\rightarrow_{13})$. ∎

Proposition 29. *We have*

$$\mathsf{ABS} \cup \mathsf{SBS} \neq \mathsf{SBS}_{\mathrm{HK}}, \text{ and } \mathsf{dEWR}^\wedge \cup \mathsf{SPS}_{\mathrm{HK}} \cup \mathsf{SjP}^\wedge \neq \mathsf{EWR}^\wedge.$$

Thus, none of the varieties in Proposition 25 is an union of others.

Proof. It was stated at the end of the previous proof that $\mathbf{A}_3^2$ belongs to $\mathsf{SPS}_{\mathrm{HK}}$ and do not belong to ABS. The algebra falsifies $(\rightarrow_{15})$ with $x = 0$ and $y = a$; therefore it does not belong also to SBS. As explained in Example 3, the EWR-semilattice $\mathbf{A}_5^2$ is not distributive and belongs neither to $\mathsf{SPS}_{\mathrm{HK}}$ nor $\mathsf{SjP}^\wedge$. ∎

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