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MONOUNARY ALGEBRAS WITH SAME QUASIORDERS OR RETRACTS

Abstract. Let (A, f) be a monounary algebra. We describe all monounary algebras (A, g) having the same set of quasiorders, $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$. It is proved that if $\mathsf{Quord}(A, f)$ does not coincide with the set of all reflexive and transitive relations on the set A and (A, f) contains no cycle with more than two elements, then f is uniquely determined by means of $\mathsf{Quord}(A, f)$. In the opposite case, $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ if and only if $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$. Further, we show that, except the case when $\mathsf{Quord}(A, f)$ coincides with the set of all reflexive and transitive relations, if the monounary algebras (A, f) and (A, g) have the same quasiorders, then they have the same retracts. Next we characterize monounary algebras which are determined by their sets of retracts and connected monounary algebras which are determined by their sets of quasiorders.

1. Introduction

A quasiorder of an algebra is a binary relation on its support, which is reflexive, transitive and compatible with all fundamental operations of the algebra.

In many papers quasiorders of algebras are studied. The system of all quasiorders of an algebra is a complete algebraic lattice with respect to inclusion. Also, by [2], [12], every algebraic lattice is isomorphic to the quasiorder lattice of a suitable algebra.

Let us notice that quasiorders of an algebra $\mathcal{A}$ can be considered as a common generalization of partial orders which are compatible with all operations of $\mathcal{A}$ and its congruences.

We will deal with monounary algebras. The first goal of this paper is, for a given monounary algebra (A, f), a characterization of all monounary algebras (A, g) such that the algebras (A, f) and (A, g) have the same sets of quasiorders. This problem is in a close connection with the papers [3]–[6],

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where similar problems were investigated for congruences or endomorphisms, respectively.

Further, in the present paper we study an analogous question for sets of retracts of a given monounary algebra (A, f).

The notion of a retract is in mathematics commonly known: by a retract it is understood a homomorphic image of an algebra by an arbitrary retraction, i.e., by an idempotent endomorphism.

The investigation of homomorphisms and retracts of monounary algebras has been shown to be a useful tool for studying some questions concerning algebras of arbitrary type. Novotný [11], [10] remarks that constructions of homomorphism of general algebras can be reduced to constructions of homomorphisms of monounary algebras. Also, it is possible to apply constructions of retracts of monounary algebras for obtaining all retracts of any algebra.

We will substantially apply results of [5] and [6], in which all monounary algebras (A, f) satisfying the condition $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$ were described, for a given monounary algebra (A, f).

2. Preliminary and auxiliary results

The symbol $\mathbb{N}$ will be used for the set of all positive integers and $\mathbb{Z}$ for the set of all integers.

A monounary algebra is defined as a pair (A, f) where A is a nonempty set and $f : A \to A$ is a mapping (a unary operation on A).

Let (A, f) be a monounary algebra. Then (A, f) is called *connected* if for arbitrary $x, y \in A$ there are $n, m \in \mathbb{N} \cup \{0\}$ such that $f^n(x) = f^m(y)$. A maximal connected subalgebra of (A, f) is called a *connected component*.

For $x \in A$, the connected component containing x is denoted $K^f(x)$.

An element $x \in A$ is called *cyclic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$. If $n(x)$ is the least positive integer with this property, then the set $\{x, f^1(x), f^2(x), \dots, f^{n(x)-1}(x)\}$ is said to be a *cycle*.

A cycle of (A, f) is said to be *small* if it has at most two elements; otherwise it is *large*.

If $k \in \mathbb{N}$, then we denote by A^f_k the set of all elements of connected components possessing a k-element cycle, and by B^f_k the set of all cyclic elements of A^f_k.

For a cyclic element $x \in A$ and $k \in \mathbb{N}$ we denote by $C^f_0[x]$ the cycle containing x, and by induction we set

$$C^f_k[x] = \{y \in A \setminus C^f_{k-1}[x] : f(y) \in C^f_{k-1}[x]\}.$$

The notion of a *degree* $s^f(y)$ of an element $y \in A$ was defined by M. Novotný (cf., e.g., [9] or [3]) as follows. Let us denote by $Y^{(\infty)}$ the set of all elements $y \in A$ such that there exists a sequence $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ of elements belonging to A with the property $y_0 = y$ and $f(y_n) = y_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $Y^{(0)} = \{y \in A : f^{-1}(y) = \emptyset\}$. Now we define a set $Y^{(\lambda)} \subseteq A$ for an ordinal $\lambda \neq 0$ by induction. Assume that we have defined $Y^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$Y^{(\lambda)} = \Big\{y \in A \smallsetminus \bigcup_{\alpha<\lambda} Y^{(\alpha)} : f^{-1}(y) \subseteq \bigcup_{\alpha<\lambda} Y^{(\alpha)}\Big\}.$$

The sets $Y^{(\lambda)}$ are pairwise disjoint. For each $y \in A$, either $y \in Y^{(\infty)}$ or there is an ordinal λ with $y \in Y^{(\lambda)}$. In the former case we put $s^f(y) = \infty$, in the latter we set $s(y) = \lambda$. Suppose that if λ is an ordinal, then $\lambda < \infty$.

The set of all equivalence relations on the set A will be denoted $\mathsf{Equiv}(A)$. Further, the set of all reflexive and transitive relations on A will be denoted $\mathsf{Reftr}(A)$.

Let $\Delta = \{(a, a) : a \in A\}$ be the identity on the set A. Clearly, it is the smallest quasiorder and also the smallest congruence of the algebra (A, f).

For $a, b \in A$ let $\alpha^f(a, b)$ and $\theta^f(a, b)$ be the smallest quasiorder of (A, f) and the smallest congruence of (A, f), respectively, such that $(a, b) \in \alpha^f(a, b)$ and $(a, b) \in \theta^f(a, b)$.

It is easy to see that the following lemma is valid:

Lemma 2.1. *Let (A, f) be a monounary algebra. If $a, b \in A$, a and b do not belong to the same connected component, then $\alpha^f(a, b) = \Delta \cup \{(f^i(a), f^i(b)) : i \in \mathbb{N} \cup \{0\}\}$.*

From the paper of Berman [1] concerning congruences it follows that if $n \in \mathbb{N}$, then θ is a congruence relation of an n-element cycle (C, f) if and only if there is $d \in \mathbb{N}$ such that d divides n and $[x]_\theta = \{f^k(x) : k \equiv 0 \mod d\}$ for each $x \in C$.

The congruence with this property will be denoted θ_d. It is easy to verify that for each $x \in C$, θ_d is the smallest congruence of the cycle (C, f), containing the pair $(x, f^d(x))$; moreover, θ_d does not depend on the choice of x. In [8] the following assertion was proved:

Lemma 2.2. *Let (A, f) be an n-element cycle, $n \in \mathbb{N}$. Then* $\mathsf{Quord}(A, f) = \mathsf{Con}(A, f) = \{\theta_d : d$ divides $n\}$.

Now we are going to describe several properties of a monounary algebra (A, f), which can be characterized by means of quasiorders (congruences) of (A, f), without using the operation explicitly; about such properties it will be said that they can be *determined by means of quasiorders* (*determined by means of congruences*, respectively).

We will suppose that (A, f) is a monounary algebra, $\operatorname{card} A > 2$.

Lemma 2.3. ([5], 4.10.1) $\mathsf{Con}(A, f) = \mathsf{Equiv}(A)$ *if and only if either* f *is the identity function or* f *is a constant function on* A.

Lemma 2.4. $\mathsf{Quord}(A, f) = \mathsf{Reftr}(A)$ *if and only if either* f *is the identity function or* f *is a constant function on* A.

Proof. Suppose that $\mathsf{Quord}(A, f) = \mathsf{Reftr}(A)$. Then $\alpha^f(x, y) = \Delta \cup \{(x, y)\}$ for each $x, y \in A$. If there are $a, b \in A$, $a \neq b$ with $f(a) \neq f(b)$, then $(f(a), f(b)) \in \alpha^f(a, b) \setminus \Delta$, from which it follows $f(a) = a$, $f(b) = b$, thus f is the identity function on A. Otherwise f is a constant function on A.

The opposite implication is trivial. ∎

Corollary 2.5. $\mathsf{Con}(A, f) = \mathsf{Equiv}(A)$ *if and only if* $\mathsf{Quord}(A, f) = \mathsf{Reftr}(A)$.

According to [5], 2.1 and 2.6 we have the following assertion:

Lemma 2.6.

(i) *Let* $x \in A$. *The property that* $K^f(x)$ *possesses no cycle can be determined by means of congruences.*
(ii) *If there exists a connected component of* (A, f) *possessing no cycle, then* f *can be determined by means of congruences.*

Corollary 2.7.

(i) *Let* $x \in A$. *The property that* $K^f(x)$ *possesses no cycle can be determined by means of quasiorders.*
(ii) *If there exists a connected component of* (A, f) *possessing no cycle, then* f *can be determined by means of quasiorders.*

Lemma 2.8. *Let* $a, b \in A$, $a \neq b$. *Then* $\{a, b\}$ *is a cycle if and only if* $\alpha^f(a, b) = \Delta \cup \{(a, b), (b, a)\}$.

Proof. If $\{a, b\}$ is a two-element cycle, then the assertion is valid. Suppose that $\alpha^f(a, b) = \Delta \cup \{(a, b), (b, a)\}$. Clearly, $(f(a), f(b)) \in \alpha^f(a, b)$. Since $f(a) = f(b)$ implies $\alpha^f(a, b) = \Delta \cup \{(a, b)\}$, a contradiction, we get $f(a) \neq f(b)$. This yields $(f(a), f(b)) \in \{(a, b), (b, a)\}$. If $f(a) = a, f(b) = b$, then $\alpha^f(a, b) = \Delta \cup \{(a, b)\}$, which is a contradiction as well. Therefore $f(a) = b, f(b) = a$, i.e., $\{a, b\}$ is a cycle. ∎

Lemma 2.9. *Let* x, y, v *be distinct elements of* A. *Then the property that* $f(x) = y$, $f(y) = f(v) = v$ *can be determined by means of quasiorders.*

Proof. Due to [5], 3.3, the following two conditions are equivalent:

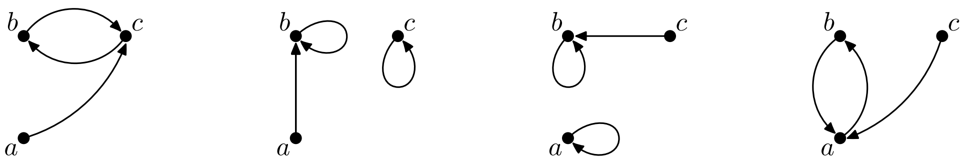
- (a) $f(x) = y$, $f(y) = f(v) = v$, or (b) $f(x) = v$, $f(y) = f(v) = y$, or (c) $f(x) = x$, $f(y) = v$, $f(v) = y$,
- $\theta^f(y, v) = \Delta \cup \{y, v\}^2$, $\theta^f(x, y) = \theta^f(x, v) = \Delta \cup \{x, y, v\}^2$.

The second condition is expressed by means of congruences, hence by means of congruences we can deduce that one of the cases (a), (b), (c) occurs. We will show that the case (a) can be distinguished from (b) and from (c), when we apply quasiorders.

If (c) is valid, then $\{y, v\}$ is a two-element cycle and due to 2.8, $\alpha^f(y, v)$ is a symmetric relation; if (a) holds, then $\alpha^f(y, v) = \Delta \cup \{(y, v)\}$, which fails to be symmetric. In the case (a), $\alpha^f(x, y) = \Delta \cup \{(x, y), (y, v), (x, v)\}$, and in (b), $\alpha^f(x, y) = \Delta \cup \{(x, y), (v, y)\}$. ■

Lemma 2.10. ([5], 3.12) *Let $a, b, c \in A$ be distinct. The following conditions are equivalent:*

- (a) $f(a) = f(b) = c, f(c) = b$, *or* (b) $f(a) = f(b) = b, f(c) = c$, *or* (c) $f(a) = a, f(b) = f(c) = b$, *or* (d) $f(a) = b, f(b) = f(c) = a$,
- $\theta^f(a, b) = \Delta \cup \{a, b\}^2$, $\theta^f(b, c) = \Delta \cup \{b, c\}^2$, $\theta^f(a, c) = \Delta \cup \{a, b, c\}^2$.



(a) (b) (c) (d)

The next lemma deals with the case that no triple with the property of 2.9 exists.

Lemma 2.11. *Assume that for each $x, y, v \in A$ such that $x \neq y$ and $f(x) = y$, $f(y) = f(v) = v$, there holds $y = v$. Let a, b, c be distinct elements of A. Then the property $f(a) = f(b) = b, f(c) = c$ can be determined by means of quasiorders.*

Proof. Since in the cases (a) and (d) in Lemma 2.10 the algebra (A, f) possesses a two-element cycle and a two-element cycle can be determined by means of quasiorders due to 2.8, we must distinguish the case (b) from the case (c). If (b) holds, then $\alpha^f(a, c) = \Delta \cup \{(a, c), (b, c)\}$, otherwise $\alpha^f(a, c) = \Delta \cup \{(a, c), (a, b)\}$. ■

Further, we will use the following results:

Lemma 2.12. ([5], 3.1) *A large cycle (as a set of elements) can be determined by means of congruences.*

Lemma 2.13. ([5], 1.4, 1.5) *Suppose that each connected component of (A, f) possesses a small cycle. Let $u, w \in A$ be such distinct elements, that either $f(u) = f(w) = w$ or $f(u) = w, f(w) = u$. If $t \in A$, then $f(t)$ can be determined by means of congruences.*

Lemma 2.14. *Suppose that* $\mathsf{Quord}(A,f) \neq \mathsf{Reftr}(A)$.

(i) *The property that each connected component of* (A,f) *possesses a small cycle can be determined by means of quasiorders.*
(ii) *If each connected component of* (A,f) *possesses a small cycle, then* f *can be determined by means of quasiorders.*

Proof. By 2.7, the case when there is a connected component of (A,f) containing no cycle, can be determined by means of quasiorders. Since each large cycle can be determined by means of congruences in view of 2.12, we obtain that (i) is valid.

Suppose that each connected component of (A,f) has a small cycle. First assume that there is a two-element cycle in (A,f), which in view of 2.8 can be determined by means of quasiorders. Then $f(t)$ can be determined by means of congruences for each $t \in A$ by 2.13, hence the assertion is valid. Now let (A,f) possess no two-element cycle and let there exist distinct elements $x,y,v \in A$ with $f(x) = y$, $f(y) = f(v) = v$. Due to 2.9, this case can be determined by means of quasiorders. Then the assertion follows from 2.13 (take $u = y, w = v$). Finally, suppose that no two-element cycle and no x,y,v having the above property exist in (A,f). Then 2.4 and the assumption $\mathsf{Quord}(A,f) \neq \mathsf{Reftr}(A)$ imply that there are distinct elements $a,b,c \in A$ such that $f(a) = f(b) = b, f(c) = c$; this is determined by means of quasiorders in view of 2.11, too. Again 2.13 for $u = a, w = b$ implies that the assertion holds. ∎

3. Same quasiorders

In the first part of this section we will deal with the assumption

- (A,f) is a monounary algebra such that each of its components contains a cycle,
- there exists at least one large cycle in (A,f).

In view of 2.7 and by 2.12, this property can be determined by means of quasiorders.

At the end of the section we summarize all obtained results for the general case into our main theorem concerning the same quasiorders.

First let us recall some results of [5] and [6] which will be applied in this section (they are denoted by the numbers of the corresponding assertions in [5] and [6]). The notations introduced here will be used also in the following section without other quotation.

(6.4) The sets A_1^f and A_2^f can be determined by means of congruences. If $x \in A_1^f \cup A_2^f$, then $f(x)$ can be described by means of congruences.

(6.5) Let x belong to a large cycle, $n \in \mathbb{N} \cup \{0\}$. The set $C_n^f[x]$ can be determined by means of congruences.

Corollary 3.1. *Let $k \in \mathbb{N}$. The sets A_k^f and B_k^f can be determined by means of congruences. Also, connected components can be determined by means of congruences.*

Proof. If $k = 1, 2$, then the assertion is valid due to **(6.4)**. Let $k > 2$. By 2.12, B_k^f is determined by means of congruences, and then according to **(6.5)**, the corresponding components can be determined by means of congruences as well. ■

(6.7.1) Let the assumption of the previous assertion hold. If $y \in C_n^f[x]$ and $n > 1$, then $f(y)$ can be described by means of congruences.

(6.8) Let x belong to a large cycle, $y \in C_1^f[x]$, $z \in C_0^f[x]$. The equality $f(y) = f(z)$ can be determined by means of congruences.

The set of pairs (y, z) from **(6.8)** is denoted P^f. It can be determined by means of congruences.

(6.9) Let x, v, z belong to the same large cycle C with k elements. Next suppose that $n \in \mathbb{N}$, $y \in C_{n+1}^f[x]$. Then the pair (z, v) with $f^{n+1}(z) = f^{n+1}(y)$, $v = f(z)$ can be determined by means of congruences.

If (z, v) are as in **(6.9)**, then we say that this pair is *determined by the surroundings of the cycle C*. By $M^f(x)$ we denote the system of all pairs which are determined by the surroundings of $C = C_0^f[x]$. Then the set $M^f(x)$ can be determined by means of congruences.

Let us remind that in the following four lemmas we will suppose that (A, f) is a monounary algebra such that each its component contains a cycle and that there exists at least one large cycle. Further, let g be a unary operation on A such that $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$.

Let $a, b \in A$, $a \neq b$. We will distinguish several cases and prove in each of them, that $\alpha^f(a, b) = \alpha^g(a, b)$.

Lemma 3.2. *If a, b belong to the same cycle, then $\alpha^f(a, b) = \alpha^g(a, b)$.*

Proof. According to 2.2 the assumption yields $\alpha^f(a, b) = \theta^f(a, b) = \theta^g(a, b) = \alpha^g(a, b)$. ■

Lemma 3.3. *If a, b belong to distinct cycles, then $\alpha^f(a, b) = \alpha^g(a, b)$.*

Proof. Suppose that $a \in C$, $b \in D$, where C and D are m-element and n-element cycles, respectively. Let $d = \gcd(m, n)$. Then

$$\theta^f(a, b) = \Delta \cup \{(f^i(x), f^j(y)) : \{x, y\} \subseteq \{a, b\}, d \text{ divides } i - j\}.$$

Next, Lemma 2.1 implies $\alpha^f(a,b) = \Delta \cup \{(f^i(a), f^j(b)) : d \text{ divides } i-j\}$, thus we obtain

$$\alpha^f(a,b) = \Delta \cup \big(\theta^f(a,b) \smallsetminus (D \times A \ \cup \ C \times C)\big).$$

Since $\theta^f(a,b) = \theta^g(a,b)$, we get $\alpha^f(a,b) = \alpha^g(a,b)$. ∎

Let us remark, that the case of Lemma 3.3 is included in the next lemma, nevertheless, having proved 3.3 separately, it is more transparent.

Lemma 3.4. *Let a, b belong to distinct connected components. Then $\alpha^f(a,b) = \alpha^g(a,b)$.*

Proof. There are cyclic elements $x, y \in A$ such that $a \in C_k[x]$, $b \in C_n[y]$, where $k, n \in \mathbb{N} \cup \{0\}$. Without loss of generality suppose that $n \geq k$. In view of **(6.7.1)**, $f^i(a) = g^i(a)$ for each $0 \leq i < k$, $f^i(b) = g^i(b)$ for each $0 \leq i < n$. Then

$$\begin{aligned}\alpha^f(a,b) &= \Delta \cup \{(a,b), (f(a), f(b)), \dots, (f^{k-1}(a), f^{k-1}(b))\} \\ &\quad \cup \alpha^f(f^{k-1}(a), f^{k-1}(b)) \\ &= \Delta \cup \{(a,b), (g(a), g(b)), \dots, (g^{k-1}(a), g^{k-1}(b))\} \\ &\quad \cup \alpha^f(g^{k-1}(a), g^{k-1}(b)).\end{aligned}$$

Denote $a' = g^{k-1}(a), b' = g^{k-1}(b)$. We have

$$\alpha^f(a',b') = \Delta \cup \big(\theta^f(a',b') \smallsetminus (K^f(y) \times A \ \cup \ K^f(x) \times K^f(x))\big),$$

from which, according to 3.1, it follows that $\alpha^f(a',b') = \alpha^g(a',b')$. Therefore $\alpha^f(a,b) = \alpha^g(a,b)$. ∎

Lemma 3.5. *Let a, b belong to the same connected component. Then $\alpha^f(a,b) = \alpha^g(a,b)$.*

Proof. There is a cyclic $x \in A$ such that $a \in C_k[x]$, $b \in C_n[x]$, where $k, n \in \mathbb{N} \cup \{0\}$. Let $n \geq k$. If $n = k = 0$, then a, b are cyclic and we can use 3.2.

First assume that $0 = k < n$. By **(6.7.1)**, $f^i(b) = g^i(b)$ for each $0 \leq i < n$. This yields

$$\alpha^f(a,b) = \Delta \cup \big(\theta^f(a,b) \smallsetminus \{f^i(b) : 0 \leq i \leq n-1\} \times A\big) =$$
$$\Delta \cup \big(\theta^g(a,b) \smallsetminus \{g^i(b) : 0 \leq i \leq n-1\} \times A\big) = \alpha^g(a,b).$$

Now suppose that $0 < k \leq n$. As in the previous proof, $f^i(a) = g^i(a)$ for each $0 \leq i < k$, $f^i(b) = g^i(b)$ for each $0 \leq i < n$. Also analogously, if we denote $a' = g^{k-1}(a), b' = g^{k-1}(b)$, then to finish the proof it suffices to show that $\alpha^f(a',b') = \alpha^g(a',b')$. Thus let $a' \neq b'$. Then

$$\alpha^f(a',b') = \Delta \cup \big(\theta^f(a',b') \smallsetminus \{f^i(b') : 0 \leq i \leq n-k\} \times A\big).$$

Since for $0 \leq i \leq n-k$,

$$f^i(b') = f^i(g^{k-1}(b)) = f^i(f^{k-1}(b)) = f^{i+k-1}(b) = g^{i+k-1}(b) = g^i(b'),$$

we obtain that $\alpha^f(a', b') = \alpha^g(a', b')$. ∎

Before formulating the main result on the same quasiorders, we will divide monounary algebras into four pairwise disjoint types:

A monounary algebra (A, f) with more than two elements is of exactly one of the following types:

(T1) f is the identity function or f is a constant function on A,
(T2) (A, f) possesses a connected component without a cycle,
(T3) (A, f) is neither of type (T1) nor of (T2) and all its cycles are small,
(T4) each connected component of (A, f) possesses a cycle and at least one of them is large.

Theorem 3.6. *Let (A, f) be a monounary algebra with* card $A > 2$ *and let g be a unary operation on A.*

(i) *For $m \in \{1, 2, 3, 4\}$, the property that (A, f) is of type* (Tm) *can be determined by means of quasiorders.*
(ii) *If (A, f) is of type* (T2) *or* (T3)*, then f can be determined by means of quasiorders.*
(iii) *If (A, f) is of type* (T1), (T2) *or* (T4)*, then* $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ *if and only if* $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$.

Proof. The condition (i) for $m = 1$ follows from 2.4, for $m = 2$ from 2.7 and for $m = 3$ from 2.14. Since an algebra is of type (T4) if and only if it is of none of types (T1), (T2), (T3), the condition is satisfied also for $m = 4$. According to 2.7 and 2.14 we get (ii).

Let us show (iii). Since $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ implies $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$, it suffices to prove only the converse implication. Suppose that $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$. First let (A, f) be of type (T1). Then (i) yields that (A, g) be of type (T1), too. We get by 2.4 that (A, f) is of type (T1) if and only if $\mathsf{Quord}(A, f) = \mathsf{Reftr}(A)$, and analogously for (A, g). Therefore $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$. If (A, f) is of type (T2), then 2.6 yields that $g = f$, thus the assertion holds trivially. If (A, f) is of type (T4), then we obtain that $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ is valid according to 3.4 and 3.5. ∎

4. Connection between quasiorders and retracts

In this section we describe a connection between quasiorders and retracts. Namely, we show that if (A, f) and (A, g) are monounary algebras with $\mathsf{Quord}(A, f) \neq \mathsf{Reftr}(A)$ and such that (A, f) and (A, g) have the same quasiorders, then they have the same retracts.

Further, we characterize the monounary algebras which are uniquely determined by means of their sets of retracts. Also, a characterization of connected monounary algebras which are uniquely determined by means of their sets of quasiorders is found.

In what follows, the theorem characterizing retracts of connected monounary algebras will be frequently applied.

Theorem 4.1. ([7], 1.2) *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following condition is satisfied:*

If $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s^f(y) \leq s^f(z)$.

Suppose that (A, f) is a monounary algebra, K_i, $i \in I$ is the system of all its connected components and $\operatorname{card} I > 1$. For $i \in I$ denote by f_i the operation f reduced to the set K_i. If $B \subseteq A$ is a retract of (A, f), it is easy to show that for each $i \in I$ such that $B \cap K_i \neq \emptyset$, the set $B \cap K_i$ is a retract of (K_i, f_i). Further, for each $i \in I$ such that $B \cap K_i = \emptyset$ there is $j \in I$ with $B \cap K_j \neq \emptyset$ and there exists a homomorphism $\varphi_{ij} : K_i \to K_j$. Conversely, if $B \subseteq A$ satisfies these two conditions, then B is a retract of (A, f).

4.1. Same quasiorders imply same retracts

For a monounary algebra (A, f) we will denote by $\mathsf{R}(A, f)$ the set of all retracts of (A, f).

Lemma 4.2. *Assume that (A, f) and (A, g) are monounary algebras such that $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g) \neq \mathsf{Reftr}(A)$. If (A, f) is connected, then $\mathsf{R}(A, f) = \mathsf{R}(A, g)$.*

Proof. If A has 2 elements and $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g) \neq \mathsf{Reftr}(A)$, then (A, f) and (A, g) form a cycle, thus $g = f$ and $\mathsf{R}(A, f) = \mathsf{R}(A, g)$.

If (A, f) contains no cycle or (A, f) possesses a small cycle (i.e., (A, f) is of type (T2) or (T3)), then 3.6 implies $g = f$, thus the assertion holds trivially. Further, we can suppose that (A, f) contains a large cycle C. Let $x \in C$. According to **(6.5)**, if $k \in \mathbb{N} \cup \{0\}$, then the set $C_k^f[x]$ can be determined by means of congruences. By **(6.7.1)**, if $y \in C_k^f[x]$ and $k > 1$, then $f(y)$ can be described by means of congruences, i.e., $g(y) = f(y)$.

Suppose that $B \in \mathsf{R}(A, f)$. Obviously, $C \subseteq B$. In view of the characterization of retracts (cf. 4.1) we have

$$\forall y \in f^{-1}(B)\ \exists z \in B : f(y) = f(z),\ s^f(y) \leq s^f(z).$$

To prove that $B \in \mathsf{R}(A, g)$, take $y \in g^{-1}(B)$. Denote $t = g(y) \in B$. If $t \in C$, then there is $z \in C$ such that $g(y) = g(z)$. In this case $s^g(y) \leq \infty = s^g(z)$.

Now let $t \in C_k^f(x)$, $k \in \mathbb{N}$. Then $y \in C_{k+1}^f(x)$, which yields $f(y) = g(y) = t$, i.e, $y \in f^{-1}(B)$. From this it follows that there exists $z \in B$ such that

$f(y) = f(z)$, $s^f(y) \leq s^f(z)$. By **(6.5)** and **(6.7.1)** we obtain that

$$\{v \in A : \exists m \in \mathbb{N} \text{ with } f^m(v) = t\} = \{v \in A : \exists m \in \mathbb{N} \text{ with } g^m(v) = t\}$$

and $f(u) = g(u)$ for each $u \in \{v \in A : \exists m \in \mathbb{N}$ with $f^m(v) = t\}$. This yields $s^f(u) = s^g(u)$ for each $u \in \{v \in A : \exists m \in \mathbb{N}$ with $f^m(v) = t\}$. Hence $g(y) = f(y) = f(z) = g(z)$ and $s^g(y) = s^f(y) = s^f(z) = s^g(z)$. Therefore $B \in \mathsf{R}(A, g)$.

Similarly, $\mathsf{R}(A, g) \subseteq \mathsf{R}(A, f)$, thus we have shown that $\mathsf{R}(A, f) = \mathsf{R}(A, g)$. ■

Theorem 4.3. *Suppose that* (A, f) *and* (A, g) *are monounary algebras such that* $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g) \neq \mathsf{Reftr}(A)$. *Then* $\mathsf{R}(A, f) = \mathsf{R}(A, g)$.

Proof. If $g = f$, then the assertion holds trivially, thus assume that $g \neq f$. Due to 3.6, each connected component of (A, f) possesses a cycle. Let K_i, $i \in I$ be the system of all connected components of (A, f). By 3.1, K_i, $i \in I$ is the system of all connected components of (A, g), too. For $i \in I$ denote by f_i, g_i the operation f, g, respectively, reduced to the set K_i. Then the assumption yields that if $i \in I$, then $\mathsf{Quord}(K_i, f_i) = \mathsf{Quord}(K_i, g_i)$, hence $\mathsf{R}(K_i, f_i) = \mathsf{R}(K_i, g_i)$ according to 4.2. Analogously as above, it suffices to prove that $\mathsf{R}(A, f) \subseteq \mathsf{R}(A, g)$.

Let $B \in \mathsf{R}(A, f)$, $i \in I$. First assume that $B \cap K_i \neq \emptyset$. Then $B \cap K_i$ is a retract of (K_i, f_i), thus $B \cap K_i$ is a retract of (K_i, g_i). Now let $B \cap K_i = \emptyset$. Then there is $j \in I$ with $B \cap K_j \neq \emptyset$ and there is a homomorphism φ_{ij} of (K_i, f_i) to (K_j, f_j). A homomorphism from (K_i, f_i) to (K_j, f_j) exists if and only if for the cycles C_i and C_j of (K_i, f_i) and (K_j, f_j), respectively, we have

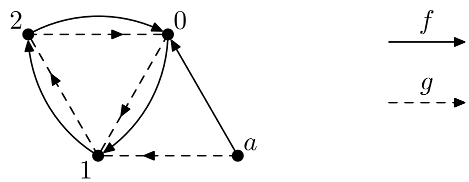
$$\operatorname{card} C_j \text{ divides } \operatorname{card} C_i.$$

This condition is equivalent with the fact that there exists a homomorphism ψ_{ij} of (K_i, g_i) to (K_j, g_j). Thus we have shown that B is a retract of (A, g). ■

The following example shows that the converse implication from Theorem 4.3 fails to hold:

Example 4.4. Let $A = \{0, 1, 2, a\}$, $f(0) = g(0) = 1 = g(a)$, $f(1) = g(1) = 2$, $f(2) = g(2) = 0 = f(a)$. Then

$$\alpha^f(a, 2) = \Delta \cup \{(a, 2)\} \neq \Delta \cup (A \times \{0, 1, 2\}) = \alpha^g(a, 2),$$
$$\mathsf{Quord}(A, f) \neq \mathsf{Quord}(A, g),$$
$$\mathsf{R}(A, f) = \{A, \{0, 1, 2\}\} = \mathsf{R}(A, g).$$



4.2. Uniqueness of an operation by means of retracts

Let (A, f) be a monounary algebra and $x, y \in A$ be arbitrary elements. We say that a retract $M \in \mathsf{R}(A, f)$ *separates* the elements x, y (or x, y are *separable by* M) if $x \in M$ and $y \notin M$, or $x \notin M$ and $y \in M$.

We say that (A, f) has the *retract separation property* if every pair of distinct elements is separable by some retract of (A, f).

Let us notice that each retract is a subalgebra, thus for $M \in \mathsf{R}(A, f)$, if $a \in M$ then $f^n(a) \in M$, for each $n \in \mathbb{N} \cup \{0\}$.

Theorem 4.5. *Let (A, f) be a monounary algebra containing no two element cycle as a component. The following conditions are equivalent:*

(1) $\mathsf{R}(A, f) = \mathsf{R}(A, g)$ *implies $g = f$ for each unary operation g on A;*
(2) *(A, f) has the retract separation property;*
(3) (a) *(A, f) has no cycle with more than one element,*
 (b) *if $z \in A$, $f(z) \neq z$ and $x_1 \in f^{-1}(z)$, then there exists $x_2 \in f^{-1}(z)$, $x_2 \neq x_1$ such that $s^f(x_1) \leq s^f(x_2)$.*

Proof. (1) $\implies$ (2) Suppose that there exist $a, b \in A$ which are not separable by any retract from $\mathsf{R}(A, f)$. We define an operation g on A as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \notin f^{-1}(a) \cup f^{-1}(b) \cup \{a, b\}, \\ a & \text{if } x \in f^{-1}(b), \\ b & \text{if } x \in f^{-1}(a), \\ f(a) & \text{if } x = b, \\ f(b) & \text{if } x = a. \end{cases}$$

Note that the elements a, b "interchanged their positions" in the algebra (A, g). The mapping $\varphi : A \to A$ defined by $\varphi(a) = b$, $\varphi(b) = a$ and $\varphi(x) = x$ for $x \notin \{a, b\}$ is an isomorphism between (A, f) and (A, g). Since M is a retract of (A, f) if and only if $\varphi(M)$ is a retract of (A, g) and since a, b are not separable, we obtain that $\mathsf{R}(A, f) = \mathsf{R}(A, g)$. According to the assumptions there is no two element cycle as a component of (A, f), thus $g \neq f$.

(2) $\implies$ (1) Let (A, f), (A, g) be such algebras that $\mathsf{R}(A, f) = \mathsf{R}(A, g)$ and suppose that (A, f) has the retract separation property. Under this assumption we show that $f = g$.

If there is an element $a \in A$ with $f(a) = a$, then the one-element subset $\{a\}$ is a retract, thus $g(a) = a$ and vice versa, the same holds for the operation g. Hence $f(a) = a$ if and only if $g(a) = a$.

By the way of contradiction suppose that there exists an element $a \in A$ with $a \neq f(a) \neq g(a) \neq a$. According to the assumption there is a retract $M \in \mathsf{R}(A, f) = \mathsf{R}(A, g)$ separating $f(a)$ and $g(a)$. Without loss of generality we may assume that $f(a) \in M$ and $g(a) \notin M$. Since $f^n(a) \in M$ for $n \geq 1$ we have $g(a) \neq f^n(a)$ for $n \geq 1$.

Obviously, $g(a) \notin \bigcup_{n\in\mathbb{N}} f^{-n}(a)$. Otherwise there exists $m \in \mathbb{N}$ with $f^m(g(a)) = a$ and thus any retract $K \in \mathsf{R}(A, f)$ containing $g(a)$ also contains a. Conversely, any retract $K \in \mathsf{R}(A, g)$ contains together with the element a also $g(a)$. Since $\mathsf{R}(A, f) = \mathsf{R}(A, g)$, we obtain that the elements a and $g(a)$ are not separable, which is a contradiction.

Finally, suppose that a and $g(a)$ are incomparable with respect to the operation f, i.e., there is no $n \in \mathbb{N} \cup \{0\}$ satisfying $f^n(g(a)) = a$ or $f^n(a) = g(a)$. Denote

$$M_1 = M \cup \bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(a).$$

Now using 4.1, it can be verified that M_1 is a retract of (A, f). Clearly, $a \in M_1$. Since $g(a) \notin M$ and $g(a) \notin \bigcup_{n\in\mathbb{N}\cup\{0\}} f^{-n}(a)$, we get that $g(a) \notin M_1$. This yields a contradiction, because $M_1 \in \mathsf{R}(A, g)$.

(2) $\implies$ (3) Let (2) hold. Clearly (a) is valid, since distinct elements of a cycle are not separable. Suppose that $z \in A$ is such that $f(z) \neq z$ and $x_1 \in f^{-1}(z)$. There exists a retract M separating z and x_1. Since $f(x_1) = z$ and M is a subalgebra of (A, f), we obtain $x_1 \notin M$, $z \in M$. We have $x_1 \in f^{-1}(M)$, thus according to 4.1, there exists $x_2 \in f^{-1}(z) \cap M$ with $s^f(x_1) \leq s^f(x_2)$. Since $x_1 \notin M$ and $x_2 \in M$, we get $x_1 \neq x_2$.

(3) $\implies$ (2) Let (3) be valid and let a, b be distinct elements of A. First assume that they belong to the same connected component, which yields that there exist $m, n \in \mathbb{N} \cup \{0\}$ with $f^m(a) = f^n(b)$. We can suppose that for any $m', n' \in \mathbb{N} \cup \{0\}$ such that $m' \leq m$ or $n' \leq n$, $f^{m'}(a) \neq f^{n'}(b)$ holds. Let $n \leq m$.

Suppose that $n > 0$. Denote $a' = f^{m-1}(a)$, $b' = f^{n-1}(b)$. Then $f(a') = f(b')$. Without loss of generality, $s^f(a') \leq s^f(b')$. Denote

$$M = A \smallsetminus \bigcup_{k\in\mathbb{N}\cup\{0\}} f^{-k}(a').$$

According to 4.1, M is a retract and $a \notin M$, $b \in M$, thus a, b are separable.

Let $n = 0$. Then $m > 0$. If $f(b) = b$, then $\{b\}$ is a retract not containing a. Suppose that $f(b) \neq b$ and put $a' = f^{m-1}(a)$. We have $f(a') = b$, hence according to (b) there is $x \in f^{-1}(b)$, $x \neq a'$ such that $s^f(a') \leq s^f(x)$. This implies that there exists a retract M such that $a' \notin M$, $b \in M$. As above, M separates a, b.

Now assume that a, b belong to distinct components. If $f(a) = a$, then $\{a\}$ is a retract separating a, b. Thus let $f(a) \neq a$. Denote K the connected component containing the element a. According to the previous consideration, the elements $a, f(a)$ can be separated by a retract M of the (connected) monounary algebra (K, f). Obviously, $a \notin M$. Now put

$$M_1 = (A \smallsetminus K) \cup M.$$

Clearly, M_1 is a retract of (A, f) with $a \notin M_1$, $b \in M_1$, hence M_1 separates a, b. ■

4.3. Uniqueness of an operation by means of quasiorders and congruences

In this subsection we will characterize all connected monounary algebras (A, f) such that for each unary operation g on A, $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ implies $g = f$.

In view of Theorem 3.6, for monounary algebras possessing no large cycles, $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ implies $g = f$. Note that for monounary algebras possessing no large cycles, the analogous implication for congruences does not hold in general.

Let us remark that we do not deal with the case when (A, f) fails to be connected, because in such characterization we would obtain too complicated conditions.

Theorem 4.6. ([6], 6.10) *Suppose that (A, f) is a connected monounary algebra with a large cycle $C = C_0^f[x]$, $x \in A$. Let g be a unary operation on the set A. The following conditions are equivalent:*

(1) $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$;
(2) (a) *C is a cycle of (A, g) and* $\mathsf{Con}(C, f) = \mathsf{Con}(C, g)$,
  (b) $M^g(x) = M^f(x)$,
  (c) $g(u) = f(u)$ *for each* $u \in \bigcup_{m>1} C_m^f[x]$,
  (d) $P^g = P^f$.

Note that the notions of P^f and M^f were defined above, between Corollary 3.1 and Lemma 3.2.

If $n \in \mathbb{N}$, then we denote by $\mathbb{Z}_n$ the set of all integers modulo n. We write $\mathbb{Z}_n = \{0_n, 1_n, \dots, (n-1)_n\}$ and the operations are counted modulo n.

In what follows, let $\mathsf{n} = p_1^{j_1} p_2^{j_2} \dots p_k^{j_k}$, $k, j_1, \dots, j_k \in \mathbb{N}$, where $p_1, \dots, p_k$ are distinct primes.

Theorem 4.7. *Let (A, f) be a connected monounary algebra possessing a large cycle $C = \mathbb{Z}_\mathsf{n}$ (with the successor operation $x \mapsto x + 1_\mathsf{n}$ counted modulo n on C). The following conditions are equivalent:*

(1) $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$ *implies $g = f$ for each unary operation g on A,*
(2) *there are $m \in \mathbb{N}$ and $x^{(1)}, \dots, x^{(m)} \in \mathbb{Z}$ (i.e., $x_\mathsf{n}^{(1)}, \dots, x_\mathsf{n}^{(m)} \in C$), such that*

(a) $(x_{\mathsf{n}}^{(t)}, x_{\mathsf{n}}^{(t)} + 1_{\mathsf{n}})$ *is determined by means of the surroundings of* C *for each* $1 \le t \le m$,

(b) $\{0 \le r < p_i^{j_i} : r \equiv x^{(t)} \mod p_i^{j_i}, 1 \le t \le m\} = \{0, 1, \ldots, p_i^{j_i} - 1\}$ *for each* $1 \le i \le k$.

Proof. Assume that (1) holds. Let $x_{\mathsf{n}}^{(1)}, \ldots, x_{\mathsf{n}}^{(m)} \in C$ be the set of all $y \in C$ such that the pair $(y, y + 1_{\mathsf{n}})$ is determined by the surroundings of the cycle C.

Suppose that (b) fails to hold. This yields that there is $1 \le i \le k$ for which the above sets are not equal, i.e., there is $0 \le q < p_i^{j_i}$ such that q is not a reminder of any $x^{(t)}$, $1 \le t \le m$, after dividing by $p_i^{j_i}$.

We are going to define an operation g on A. Let $y_{\mathsf{n}} \in C$. Take $z \in \mathbb{Z}$ such that

$$z \equiv y + 1 \mod p_\iota^{j_\iota} \text{ for each } 1 \le \iota \le k,\ \iota \ne i,$$
$$z \equiv q \mod p_i^{j_i}.$$

Since the modules in the system of congruences are relatively prime, this system has a unique solution modulo n. Then we put $g(y_{\mathsf{n}}) = z_{\mathsf{n}}$. According to [6], Theorem 5.4, $\mathsf{Con}(C, f) = \mathsf{Con}(C, g)$.

For $y \in C_1[x]$ there is $y' \in C_0[x]$ such that $(y, y') \in P^f$; we set $g(y) = g(y')$. Finally, we set $g(u) = f(u)$ for each $u \in \bigcup_{m>1} C_m^f[x]$. This implies that the condition (2) of Theorem 4.6 is fulfilled, hence $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$. Since $g \ne f$, we obtain a contradiction to (1).

Conversely, assume that (2) is valid and let g be a unary operation on A with $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$. In order to prove $f = g$ it is sufficient to show that $g(y_{\mathsf{n}}) = f(y_{\mathsf{n}})$ for each $y_{\mathsf{n}} \in C$, because then (c) and (d) of Theorem 4.6 imply that $g = f$.

Let $y_{\mathsf{n}} \in C$. From (2)(b) it follows that to each $1 \le \iota \le k$ there exists $1 \le t_\iota \le m$ having the property $y \equiv x^{(t_\iota)} \mod p_\iota^{j_\iota}$. By (2)(a) we get $(x_{\mathsf{n}}^{(t_\iota)}, f(x_{\mathsf{n}}^{(t_\iota)})) = (x_{\mathsf{n}}^{(t_\iota)}, x_{\mathsf{n}}^{(t_\iota)} + 1_{\mathsf{n}}) \in M^f(y_{\mathsf{n}}) = M^g(y_{\mathsf{n}})$, which implies $g(x_{\mathsf{n}}^{(t_\iota)}) = x_{\mathsf{n}}^{(t_\iota)} + 1_{\mathsf{n}} = f(x_{\mathsf{n}}^{(t_\iota)})$. We will apply the Berman's result which was stated below Lemma 2.1. Since $y \equiv x^{(t_\iota)} \mod p_\iota^{j_\iota}$, we obtain that the pair $(y_{\mathsf{n}}, x_{\mathsf{n}}^{(t_\iota)}) \in \theta_{p_\iota^{j_\iota}} \in \mathsf{Con}(C, f)$. Hence $\theta_{p_\iota^{j_\iota}} \in \mathsf{Con}(C, g)$, which yields $(g(y_{\mathsf{n}}), g(x_{\mathsf{n}}^{(t_\iota)})) \in \theta_{p_\iota^{j_\iota}}$.

Let $z \in \mathbb{Z}$ be such that $z_{\mathsf{n}} = g(y_{\mathsf{n}})$. Now, from $(g(y_{\mathsf{n}}), g(x_{\mathsf{n}}^{(t_\iota)})) \in \theta_{p_\iota^{j_\iota}}$, we get the following system of congruences:

$$z \equiv x^{(t_\iota)} + 1 \equiv y + 1 \mod p_\iota^{j_\iota}, \text{ for each } 1 \le \iota \le k.$$

One of the solutions of this system is $z = y + 1$. Since the solution is unique modulo n, we obtain $g(y_{\mathsf{n}}) = z_{\mathsf{n}} = y_{\mathsf{n}} + 1_{\mathsf{n}} = f(y_{\mathsf{n}})$. Therefore $g = f$. ∎

According to Theorem 3.6, under the assumption of the previous theorem we have $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ if and only if $\mathsf{Con}(A, f) = \mathsf{Con}(A, g)$, which implies:

Corollary 4.8. *Let (A, f) be a connected monounary algebra possessing a large cycle $C = \mathbb{Z}_{\mathsf{n}}$. Then the condition $\mathsf{Quord}(A, f) = \mathsf{Quord}(A, g)$ implies $g = f$ for each unary operation g on A, is equivalent to the condition* (2) *of* Theorem 4.7.

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