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THE UNCOUNTABLE COFINALITY OF THE AUTOMORPHISM GROUP OF THE COUNTABLE UNIVERSAL DISTRIBUTIVE LATTICE

Dedicated to Rüdiger Göbel on the occasion of his 70th birthday.

Abstract. We show that the automorphism group of the countable universal distributive lattice has strong uncountable cofinality, and we adapt the method to deduce the strong uncountable cofinality of the automorphism group of the countable universal generalized boolean algebra.

1. Introduction

In [6] a detailed analysis was given of the automorphism group of the countable universal homogeneous distributive lattice $\mathbb{D}$. In particular, its normal subgroups were determined, and the small index property was verified. The question of whether $\mathrm{Aut}(\mathbb{D})$ has uncountable cofinality was however left open, and it is our purpose in this paper to establish this. The same methods are used to demonstrate the uncountable cofinality of the automorphism group of the countable universal homogeneous generalized boolean algebra.

The background to this problem is explained in [7], but we recap here on the main points. A group G is said to have *uncountable cofinality* if it cannot be written as the union of a countable chain of proper subgroups. This notion has been studied by a number of authors, originally Koppelberg and Tits [13], who, in response to a question of Serre, proved the uncountable cofinality of the direct power of infinitely many copies of a finite perfect group. Macpherson and Neumann [14] established the same result for the

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symmetric group on a countably infinite set, and building on these methods, uncountable cofinality has been established in a number of other cases, for instance the automorphism group $A(\mathbb{Q})$ of the rational numbers as an ordered set [10], for other (partially or totally) ordered sets [5, 7], and for the homeomorphism groups of certain topological spaces [4]. For a survey of this field, see Thomas [15]. A rather stronger condition is that G have *strong uncountable cofinality*, and this means that G cannot be written as the union of an ascending chain of proper subsets $(U_n : n \in \omega)$ each closed under formation of inverses, and such that for each i and j, there is k such that $U_iU_j \subseteq U_k$. It was shown in [5] that strong uncountable cofinality is equivalent to uncountable cofinality together with a property introduced in [2] called 'Bergman's property': for any generating set E for G which contains the identity and is closed under inverses, there is $n \in \mathbb{N}$ such that $G = E^n$. This property has also found considerable recent interest, cf., e.g., [2, 4, 5, 12]. We work here exclusively with strong uncountable cofinality, thereby establishing for the present automorphism groups both uncountable cofinality and the Bergman property.

2. The uncountable cofinality of the automorphism group of the countable universal homogeneous distributive lattice

In this section we shall establish the uncountable cofinality of $\mathrm{Aut}(\mathbb{D})$. This follows a combination of methods used in other cases, principally those of the rationals [10] and the countable atomless boolean algebra [4]. For these we require the following definitions.

First we recall what $\mathbb{D}$ is. It is known that the class of finite distributive lattices is an amalgamation class, so it follows by the general Fraïssé theory (see [11] for instance) that there is a unique countable universal homogeneous distributive lattice, which we denote by $\mathbb{D}$. This has no greatest or least element, all the maximal chains are isomorphic to $\mathbb{Q}$, and any 'interval' $[a, b] = \{x \in \mathbb{D} : a \leq x \leq b\}$ is itself a lattice, which is isomorphic to the countable atomless boolean algebra. Modifying this a little, we may also consider the class of finite 'generalized boolean algebras', which are finite distributive lattices with a least element 0. This is an amalgamation class under the class of maps which are required to fix 0 (as well as the lattice operations $\wedge$ and $\vee$), and the resulting structure is a universal homogeneous generalized boolean algebra $\mathbb{B}$. For this structure, there is a least but no greatest element, all maximal chains are isomorphic to the rational interval $[0, 1)$, and again any interval is isomorphic to the countable atomless boolean algebra. Both $\mathbb{D}$ and $\mathbb{B}$ are 'relatively complemented', which means that for any $a \leq x \leq b$ there is $y \in [a, b]$ such that $x \wedge y = a$ and $x \vee y = b$. We shall quote various results proved in [6] about $\mathbb{D}$ and $\mathbb{B}$.

By a *coterminal $\mathbb{Z}$-chain* in $\mathbb{D}$ (or more generally in any partially ordered set) we understand a family $\{a_i : i \in \mathbb{Z}\}$ indexed by the integers, such that $a_i < a_{i+1}$ for each i, and for every $x \in \mathbb{D}$, there are i and j such that $a_i \leq x \leq a_j$. A *moiety* is a subset of $\mathbb{D}$ of the form $\bigcup_{n\in\mathbb{Z}}(a_{2n}, a_{2n+1})$, for some coterminal $\mathbb{Z}$-chain $\{a_i : i \in \mathbb{Z}\}$. The sets (a_{2n}, a_{2n+1}) are called 'components' of the moiety. The notion of moiety was originally introduced by Neumann [3] in the proof of the small index property for the symmetric group on ω, signifying a set which is 'half' of the whole, and versions of the same idea have appeared in many other similar proofs of the small index property or uncountable cofinality.

For any permutation group G, we denote the setwise and pointwise stabilizers of a subset A of the set on which G acts by $G_{\{A\}}$ and G_A respectively.

We need the following result, which we give without proof. This is similar to Lemmas 2.2 and 2.3 in [6], adapted for our current purposes.

Lemma 2.1. *For $i = 1, 2$, let L_i be a relatively complemented distributive lattice, and $L'_i \subseteq L_i$ a sublattice such that the smallest relatively complemented sublattice of L_i containing L'_i is L_i itself. Then each isomorphism from L'_1 to L'_2 extends uniquely to an isomorphism of L_1 to L_2.*

Theorem 2.2. *The automorphism group of the countable universal homogeneous distributive lattice $\mathbb{D}$ has strong uncountable cofinality.*

Proof. Let $U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots \subseteq \mathrm{Aut}(\mathbb{D}) = G$ have union equal to G and be such that $U_n^{-1} = U_n$ and $(\forall i, j)(\exists k)U_iU_j \subseteq U_k$. We aim to show that $U_n = G$ for some n. As in [7], this is accomplished in a series of steps. By [9] Thm.II.4.20, the countable atomless boolean algebra $\mathbb{B}$ is generated by some maximal chain. Now choose a coterminal $\mathbb{Z}$-chain $(a_n)_{n\in\mathbb{Z}}$ in $\mathbb{D}$. Then each $[a_n, a_{n+1}]$ is isomorphic to $\mathbb{B}$, so we let C_n be a maximal chain of $[a_n, a_{n+1}]$ which generates $[a_n, a_{n+1}]$ as a boolean algebra. We remark that $C = \bigcup_{n\in\mathbb{Z}} C_n$ generates $\mathbb{D}$ as a relatively complemented lattice, as follows by the argument given in the proof of Lemma 2.3 in [6], and we fix this C in what follows.

(1) Any isomorphism f from C to C extends uniquely to an automorphism of $\mathbb{D}$.

This is immediate by Lemma 2.1 from the fact that C generates $\mathbb{D}$ as a relatively complemented lattice.

(2) Let (a_n), (a'_n) be coterminal $\mathbb{Z}$-chains in C. Then any isomorphism f from $C \cup \bigcup_{n\in\mathbb{Z}}[a_{2n}, a_{2n+1}]$ to $C \cup \bigcup_{n\in\mathbb{Z}}[a'_{2n}, a'_{2n+1}]$ extends uniquely to an automorphism of $\mathbb{D}$.

This is proved by the same method as (1).

(3) There is m_0 such that $G_{\{C\}} \subseteq U_{m_0}$.

To prove this we consider the intersections of the sequence (U_n) with the setwise stabilizer $G_{\{C\}}$ of C in G. Since $C \cong \mathbb{Q}$, and by (1), $G_{\{C\}} \cong A(\mathbb{Q})$. By the strong uncountable cofinality of $A(\mathbb{Q})$ [5] we deduce that there is m_0 such that $U_{m_0} \cap G_{\{C\}} = G_{\{C\}}$, which gives the result.

(4) There are a moiety $M = \bigcup_{n\in\mathbb{Z}}(a_{2n}, a_{2n+1})$ such that each a_n lies in C, and an integer $m_1 \geq m_0$, such that every automorphism fixing $C \setminus M$ pointwise agrees with some member of U_{m_1} on M.

This is done by a standard diagonalization argument as in earlier proofs. We start with any moiety of the form $\bigcup_{n\in\mathbb{Z}}(a'_{2n}, a'_{2n+1})$ where each a'_n lies in C, and express it as the disjoint union of an infinite sequence of moieties $(M_i : i \geq m_0)$. We shall show that M in the statement of (4) may be taken as some M_i with $i = m_1 \geq m_0$. If not, then for each $i \geq m_0$ there is $g_i \in G_{C\setminus M_i}$ which does not agree with any member of U_i on M_i. Let g be the map on $\bigcup_{i\geq m_0} M_i$ obtained by 'patching' all these g_is, that is, which agrees with g_i on M_i and which fixes all members of $C\setminus\bigcup_{n\in\mathbb{Z}}(a'_{2n}, a'_{2n+1})$. By (2), g extends to an automorphism of $\mathbb{D}$, also written as g. Since $G = \bigcup_{i\in\omega} U_i$, $g \in U_i$ for some $i \geq m_0$. But now g and g_i agree on M_i, which is a contradiction.

In what follows we fix this choice of M and (a_n).

(5) There is $m_2 \geq m_1$ such that $G_{C\setminus M} \subseteq U_{m_2}$.

For this we choose $h \in G_{C\setminus M}$ which fixes each component of M setwise but acts non-trivially there. Let $h \in U_m$. Since $[a_{2n}, a_{2n+1}]$ is isomorphic to the countable atomless boolean algebra, and also using (2) again, such h exists. Now Anderson showed in [1] that the automorphism group of the countable atomless boolean algebra is simple, and furthermore, that for any non-identity elements f and f', there are f_i such that

$$f' = f^{f_1}(f^{-1})^{f_2} f^{f_3}(f^{-1})^{f_4} f^{f_5}(f^{-1})^{f_6}$$

(where superscripts indicate conjugation). Thus on each component of M, we may write any $g \in G_{C\setminus M}$ as a product of 6 conjugates of h and its inverse of this form, and by (2), we may find $f_i \in G_{C\setminus M}$ such that $g = h^{f_1}(h^{-1})^{f_2}h^{f_3}$ $(h^{-1})^{f_4}h^{f_5}(h^{-1})^{f_6}$. By (4), the f_i agree with members f'_i of U_{m_1} on M. Hence g agrees with $h^{f'_1}(h^{-1})^{f'_2}h^{f'_3}(h^{-1})^{f'_4}h^{f'_5}(h^{-1})^{f'_6}$ on $\mathbb{D}$. Since $h \in U_m$ and each f'_i lies in U_{m_1}, using $(\forall j,k)(\exists l)U_jU_k \subseteq U_l$, we find the desired $m_2 \geq m, m_1$.

(6) There is $m_3 \geq m_2$ such that for every moiety M' of the form $\bigcup_{n\in\mathbb{Z}}(a_{i_{2n}}, a_{i_{2n+1}})$, where i_n are integers such that $i_n < i_{n+1}$ for all n, $G_{C\setminus M'} \subseteq U_{m_3}$.

To see the truth of this we let m_3 be such that $U_{m_0}U_{m_2}U_{m_0} \subseteq U_{m_3}$. Now we observe that there is $g \in G_{\{C\}}$ such that $ga_n = a_{i_n}$ for each n. Let $h \in G_{C\setminus M'}$. Then $g^{-1}hg \in G_{C\setminus M}$, so by (5) lies in U_{m_2}. Hence $h \in gU_{m_2}g^{-1}$. But by (3), $g \in U_{m_0}$, so $h \in U_{m_0}U_{m_2}U_{m_0} \subseteq U_{m_3}$.

(7) There is $m_4 \geq m_3$ such that for every choice of (i_n) as in (6), every automorphism fixing each a_{i_n} lies in U_{m_4}.

Let m_4 be such that $U_{m_3}U_{m_3} \subseteq U_{m_4}$, and let $g \in G$ fix each a_{i_n}. Then g may be written in the form $g_2 g_1$ where g_1 fixes all members of $C \cap \bigcup_{n\in\mathbb{Z}}[a_{i_{2n}}, a_{i_{2n+1}}]$ and g_2 fixes all members of $C \cap \bigcup_{n\in\mathbb{Z}}[a_{i_{2n+1}}, a_{i_{2n+2}}]$. By (6), $g_1, g_2 \in U_{m_3}$, and hence $g \in U_{m_4}$.

To conclude the proof, choose any $g \in G$. We can find a sequence of integers $(i_n)_{n\in\mathbb{Z}}$ such that for every n, both $a_{i_{n+1}}$ and $g(a_{i_{n+1}})$ are greater than both a_{i_n} and $g(a_{i_n})$. Then for each n, $a_{i_{2n+1}}$ and $g(a_{i_{2n+1}})$ lie between $a_{i_{2n}}$ and $a_{i_{2n+2}}$. Hence there is an automorphism h fixing each $a_{i_{2n}}$ and taking $g(a_{i_{2n+1}})$ to $a_{i_{2n+1}}$. Now hg fixes each $a_{i_{2n+1}}$, so by (7), $g = h^{-1}hg \in U_{m_4}U_{m_4} \subseteq U_{m_5}$, for some (fixed) $m_5 \geq m_4$. ∎

3. The uncountable cofinality of the automorphism group of the countable universal homogeneous generalized boolean algebra

We can use the same method to show the uncountable cofinality of the automorphism group of the closely related generalized boolean algebra. This is a combination of the proof given in the previous section, and the strong uncountable cofinality of the automorphism group of the countable atomless boolean algebra, established in [4] (remembering that this automorphism group is isomorphic to the group of homeomorphisms to itself of Cantor space).

Theorem 3.1. *The automorphism group of the countable universal homogeneous generalized boolean algebra* $\mathbb{B}$ *has strong uncountable cofinality.*

Proof. We first remark that in [6], some of the lemmas for $\mathbb{D}$ carry over to $\mathbb{B}$ with small modifications. For instance, Lemma 2.6 there says that three earlier results carry over with $\mathbb{Z}$-chains replaced by ω-chains. The corresponding versions of (1) and (2) in the proof of Theorem 2.2 are thus as follows:

(1) For a suitable choice of maximal chain C of $\mathbb{B}$, any isomorphism from C to C extends uniquely to an automorphism of $\mathbb{B}$.

As before, this C may be chosen to be a maximal chain generating $\mathbb{B}$, and we let this be fixed for the rest of the proof.

(2) If (a_n), (a'_n) are cofinal ω-sequences in C such that $a_0 = a'_0 = 0$, then any isomorphism from $C \cup \bigcup_{n\in\omega}[a_{2n}, a_{2n+1}]$ to $C \cup \bigcup_{n\in\omega}[a'_{2n}, a'_{2n+1}]$ (or from $C \cup \bigcup_{n\in\omega}[a_{2n+1}, a_{2n+2}]$ to $C \cup \bigcup_{n\in\omega}[a'_{2n+1}, a'_{2n+2}]$) extends uniquely to an automorphism of $\mathbb{B}$.

We continue with versions of the other steps in the proof of Theorem 2.2. The proofs of (3), (4), (5), (6), and (7) are as before.

(3) There is m_0 such that $G_{\{C\}} \subseteq U_{m_0}$.

(4) There are a moiety $M = \bigcup_{n\in\omega}(a_{2n+1}, a_{2n+2})$ such that each a_n lies in C, and $m_1 \geq m_0$, such that every automorphism fixing $C \setminus M$ pointwise agrees with some member of U_{m_1} on M.

This time, by 'moiety' we mean a subset of $\mathbb{B}$ of the form $\bigcup_{n\in\omega}(a_{2n+1}, a_{2n+2})$ where (a_n) is a cofinal ω-sequence.

From now on, M and the sequence (a_n) are fixed.

(5) There is $m_2 \geq m_1$ such that $G_{C\setminus M} \subseteq U_{m_2}$.

We also need a version of this for 'moieties' which may include the bottom level.

(5′) There is $m_3 \geq m_2$ such that $G_{C\setminus M'} \subseteq U_{m_3}$, where $M' = \bigcup_{n\in\omega}(a_{2n}, a_{2n+1})$.

Since $[a_0, a_1]$ is isomorphic to the countable atomless boolean algebra, by [4], $\mathrm{Aut}([a_0, a_1])$ has strong uncountable cofinality. By considering the intersections of the sequence (U_n) with $G_{C\setminus[a_0,a_1]}$ (which by (2) is isomorphic to $\mathrm{Aut}([a_0, a_1])$) we find m' such that $G_{C\setminus[a_0,a_1]} \subseteq U_{m'}$. Choose $m_3 \geq m_2$ such that $U_{m_0}U_{m_2}U_{m_0}U_{m'} \subseteq U_{m_3}$.

Now let $g \in G_{C\setminus M'}$. Then we may write g as g_2g_1 where g_1 fixes all members of $C \setminus [a_0, a_1]$ and g_2 fixes all members of $C \setminus \bigcup_{n\geq 1}[a_{2n}, a_{2n+1}]$. Let $h \in G_{\{C\}}$ map a_n to a_{n+1} for all $n \geq 1$. Then $h^{-1}g_2h$ fixes all members of $C \setminus M$, so by (5), lies in U_{m_2}. Hence $g = h(h^{-1}g_2h)h^{-1}g_1 \in U_{m_0}U_{m_2}U_{m_0}U_{m'} \subseteq U_{m_3}$.

(6) There is $m_4 \geq m_3$ such that for every moiety M' of the form $\bigcup_{n\in\omega}(a_{i_{2n+1}}, a_{i_{2n+2}})$, where i_n are natural numbers such that $i_n < i_{n+1}$ for all n, $G_{C\setminus M'} \subseteq U_{m_3}$.

(7) There is $m_5 \geq m_4$ such that for every choice of (i_n) as in (6), every automorphism fixing each a_{i_n} lies in U_{m_5}.

The proof is concluded as in Theorem 2.2. ∎

We remark in conclusion that in [7], we established the strong uncountable cofinality for various uncountable chains. It is tempting to ask whether these results extend to the distributive lattices generated by these chains, in the style of the present paper. It would be necessary to derive results about simplicity similar to those in [1], in a strong version providing explicitly for expressions using finitely many conjugates, such as we used in step (5) of the proof of Theorem 2.2.

References

[1] R. D. Anderson, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. 80 (1958), 955–963.

[2] G. M. Bergman, *Generating infinite symmetric groups*, Bull. London Math. Soc. 38 (2006), 429–440.

[3] J. D. Dixon, P. M. Neumann, S. Thomas, *Subgroups of small index in infinite permutation groups*, Bull. London Math. Soc. 18 (1986), 580–586.

[4] M. Droste, R. Göbel, *Uncountable cofinalities of permutation groups*, J. London Math. Soc. 71 (2005), 335–344.

[5] M. Droste, W. C. Holland, *Generating automorphism groups of chains*, Forum Math. 17 (2005), 699–710.

[6] M. Droste, D. Macpherson, *The automorphism group of the universal distributive lattice*, Algebra Universalis 43 (2000), 295–306.

[7] M. Droste, J. K. Truss, *Uncountable cofinalities of automorphism groups of linear and partial orders*, Algebra Universalis 62 (2009), 75–90.

[8] A. M. W. Glass, *Ordered Permutation Groups*, London Math. Soc. Lecture Notes No. 55 (1981), Cambridge University Press.

[9] G. A. Grätzer, *General Lattice Theory*, 2nd edition, Birkhäuser Verlag, Basel, 2003.

[10] C. Gourion, *À propos du groupe des automorphismes de* $(\mathbb{Q}, \leq)$, C. R. Acad. Sci. Paris 315 (1992), 1329–1331.

[11] W. Hodges, *Model Theory*, Cambridge University Press, Cambridge, 1993.

[12] A. S. Kechris, C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. London Math. Soc. 94 (2006), 302–350.

[13] S. Koppelberg, J. Tits, *Une propriété des produits directs infinis de groupes finis isomorphes*, C. R. Acad. Sci. Paris 279 (1974), 583–585.

[14] H. D. Macpherson, P. M. Neumann, *Subgroups of infinite permutation groups*, J. London Math. Soc. 42 (1990), 64–84.

[15] S. Thomas, *Cofinalities of infinite permutation groups*, in: Advances in Algebra and Model Theory, edited by M. Droste and R. Göbel, Gordon and Breach, pages 101–120, 1997.

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