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SOME SEMILATTICE DECOMPOSITIONS OF DIMONOIDS

Abstract. We show that the system of axioms of a dimonoid is independent and prove that every dimonoid with a commutative operation is a semilattice of archimedean subdimonoids, every dimonoid with a commutative periodic semigroup is a semilattice of unipotent subdimonoids, every dimonoid with a commutative operation is a semilattice of a-connected subdimonoids and every idempotent dimonoid is a semilattice of rectangular subdimonoids.

1. Introduction

The notion of a non-commutative Lie algebra (Leibniz algebra) appeared in the researches on a homology theory for Lie algebras [1]. It is well-known that for Lie algebras there is a notion of a universal enveloping associative algebra. Jean-Louis Loday [2] found a universal enveloping algebra for Leibniz algebras. Dialgebras play a role of such object, that is, vector spaces D with two bilinear associative operations $\prec$ and $\succ$ satisfying the following axioms:

$$(x \prec y) \prec z = x \prec (y \succ z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$(x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$. Dialgebras were investigated in different papers (see, for example, [2]–[7]). So, recently L. A. Bokut, Yuqun Chen and Cihua Liu [3] gave the composition-diamond lemma for dialgebras and obtained a Gröbner-Shirshov basis for dialgebras. Kolesnikov [4] has shown that any dialgebra can be obtained from some associative conformal algebra. The conformal algebras were introduced by Kac [8] as a formal language of the description of properties of algebraic structures occurring in mathematical physics. The notion of a variety of dialgebras was introduced in [4] with the help of the notion of an operad.

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A set D equipped with two binary associative operations $\prec$ and $\succ$ satisfying the axioms indicated above is called a dimonoid [2]. So, a dialgebra is a linear analogue of a dimonoid. At the present time dimonoids have became a standard tool in the theory of Leibniz algebras. One of the first results about dimonoids is the description of the free dimonoid generated by a given set [2]. With the help of properties of free dimonoids, free dialgebras were described and a homology of dialgebras was investigated [2]. In [9] K. Liu used the notion of a dimonoid to introduce the notion of a one-sided diring and studied basic properties of dirings. The notion of a diband of subdimonoids was introduced in [10]. This notion generalizes the notion of a band of semigroups [11] and is effective to describe structural properties of dimonoids. In terms of dibands of subdimonoids, in particular, it was proved that every commutative dimonoid is a semilattice of archimedean subdimonoids [10]. The semilattice decompositions of dimonoids also were given in [12]–[14]. In [15] the author constructed a free commutative dimonoid and described the least idempotent congruence on this dimonoid. The structure of an arbitrary diband of subdimonoids was described in [16]. In [17] it has been proved that the free dimonoid is a semilattice of s-simple subdimonoids each being a rectangular band of subdimonoids. Some new dialgebras were introduced in terms of dimonoids in [2]. It is also well-known that the notion of a dimonoid generalizes the notion of a digroup [6], [18]. Recently Phillips [18] gave a simple basis of independent axioms for the variety of digroups. Digroups play a prominent role in an important open problem from the theory of Leibniz algebras. Pirashvili [19] considered duplexes which are sets with two binary associative operations and described a free duplex. Dimonoids in the sense of Loday [2] are examples of duplexes. Moreover, it should be noted that algebras with two associative operations (so-called bisemigroups) were considered earlier in some other aspects in the paper of B. M. Schein [20].

Obviously, if the operations of a dimonoid coincide then it becomes a semigroup. Therefore studying dimonoids via semigroup techniques may constitute a research direction.

The purpose of this work is to obtain some semilattice decompositions of dimonoids. In section 2 we give necessary definitions and some properties of dimonoids (Lemmas 1–7 and Theorem 8). In section 3 we show that the system of axioms of a dimonoid is independent (Theorem 9) and give different examples of dimonoids (Propositions 10–14). In section 4 we prove that every dimonoid with a commutative operation is a semilattice of archimedean subdimonoids (Theorem 15), every dimonoid with a commutative periodic semigroup is a semilattice of unipotent subdimonoids (Theorem 16), every dimonoid with a commutative operation is a semilattice of a-connected subdimonoids (Theorem 17) and every idempotent dimonoid is a semilattice of rectangular subdimonoids (Theorem 18). Theorems 15, 16 and 18 extend, respectively, Theorem 2 from [10] about the decomposition of commutative dimonoids into semilattices of archimedean subdimonoids, Schwarz's theorem [21] about the decomposition of commutative periodic semigroups into semilattices of unipotent semigroups and McLean's decomposition [22] of bands into semilattices of rectangular bands. In this section we also construct examples of dimonoids with one and two idempotent operations.

2. Preliminaries

A nonempty set D with two binary operations $\prec$ and $\succ$ satisfying the following five axioms:

$(D1)$ $$(x \prec y) \prec z = x \prec (y \prec z),$$

$(D2)$ $$(x \prec y) \prec z = x \prec (y \succ z),$$

$(D3)$ $$(x \succ y) \prec z = x \succ (y \prec z),$$

$(D4)$ $$(x \prec y) \succ z = x \succ (y \succ z),$$

$(D5)$ $$(x \succ y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$, is called a dimonoid (see [2, p. 11]).

A map f from a dimonoid D_1 to a dimonoid D_2 is a homomorphism, if $(x \prec y)f = xf \prec yf$, $(x \succ y)f = xf \succ yf$ for all $x, y \in D_1$. A subset T of a dimonoid $(D, \prec, \succ)$ is called a subdimonoid, if for any $a, b \in D$, a, $b \in T$ implies $a \prec b$, $a \succ b \in T$.

As usual N denotes the set of positive integers.

Let $(D, \prec, \succ)$ be a dimonoid, $a \in D$, $n \in N$. Denote the degree n of an element a concerning the operation $\prec$ (respectively, $\succ$) by a^n (respectively, by na).

Lemma 1. ([10], Lemma 1) *Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\prec$. For all $b, c \in D$, $m \in N$, $m > 1$,*

$$(b \prec c)^m = b^m \succ c^m = (b \succ c)^m.$$

Lemma 2. ([10], Lemma 4) *Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\succ$. For all $b \in D$, $m \in N$,*

$$2\, b^m = (2m)\, b.$$

A commutative idempotent semigroup is called a semilattice. A commutative semigroup S is separative, if for any $s, t \in S$, $s^2 = st = t^2$ implies $s = t$. A semigroup S is called globally idempotent, if $S^2 = S$.

Lemma 3. *The operations of a dimonoid $(D, \prec, \succ)$ coincide, if one of the following conditions holds:*

(i) $(D, \prec)$ *is a semilattice;*
(ii) $(D, \prec)$ *is a left cancellative (cancellative) semigroup;*
(iii) $(D, \prec)$ *is a commutative separative semigroup;*
(iv) $(D, \prec)$ *is a commutative globally idempotent semigroup.*

Proof. (i) For all $x, y, z \in D$ we have

$$(x \succ y) \prec z = z \prec (x \succ y) = (z \prec x) \prec y = x \prec (y \prec z) = x \succ (y \prec z)$$

according to the commutativity of the operation $\prec$ and the axioms $(D1)$, $(D2)$, $(D3)$ of a dimonoid. Substituting $z = y$ in the last equality and using the idempotent property of the operation $\prec$, we obtain $x \prec y = x \succ y$.

(ii) By the axioms $(D1), (D2)$ of a dimonoid we have

$$(x \prec y) \prec z = x \prec (y \prec z) = x \prec (y \succ z)$$

for all $x, y, z \in D$. Hence, using the left cancellability, we obtain $y \prec z = y \succ z$ for all $y, z \in D$.

Analogously, the case with a cancellative semigroup can be proved.

(iii) Let x, y be arbitrary elements of D. Assume $a = x \prec y, b = x \succ y$. Then

$$a^2 = (x \prec y) \prec (x \prec y) = (x \prec y)^2,$$
$$a \prec b = (x \prec y) \prec (x \succ y) = (x \prec y)^2,$$
$$b^2 = (x \succ y) \prec (x \succ y) = (x \succ y)^2 = (x \prec y)^2$$

according to the axioms $(D1), (D2)$ of a dimonoid and Lemma 1. As the commutative semigroup $(D, \prec)$ is separative, then $a^2 = a \prec b = b^2$ implies $a = b$.

(iv) Let $x, y \in D$ and $y = y_1 \prec y_2$, $y_1, y_2 \in D$. Then

$$x \prec y = x \prec (y_1 \prec y_2) = (y_2 \prec x) \prec y_1 = y_2 \prec (x \succ y_1) = (x \succ y_1) \prec y_2$$
$$= x \succ (y_1 \prec y_2) = x \succ y$$

according to the commutativity of the operation $\prec$ and the axioms $(D1)$, $(D2)$, $(D3)$ of a dimonoid. ∎

Lemma 4. *Let $(D, \prec, \succ)$ be an arbitrary dimonoid. For all $x, y, t \in D$, $n \in N$*

(i) $(x \prec y)^n \succ t = n(x \succ y) \succ t = n(x \prec y) \succ t$;
(ii) $t \prec n(x \succ y) = t \prec (x \prec y)^n = t \prec (x \succ y)^n$.

Proof. We prove (i) using an induction on n. For $n = 1$ we have

$$(x \prec y) \succ t = (x \succ y) \succ t$$

according to the axioms $(D4),(D5)$ of a dimonoid. Let $(x \prec y)^k \succ t = k(x \succ y) \succ t$ for $n = k$. Then for $n = k+1$ we obtain

$$\begin{aligned}(x \prec y)^{k+1} \succ t &= ((x \prec y) \prec (x \prec y)^k) \succ t = ((x \prec y) \succ (x \prec y)^k) \succ t \\ &= ((x \succ y) \succ (x \prec y)^k) \succ t = (x \succ y) \succ ((x \prec y)^k \succ t) \\ &= (x \succ y) \succ k(x \succ y) \succ t = (k+1)(x \succ y) \succ t\end{aligned}$$

according to the axioms $(D4)$, $(D5)$ of a dimonoid and the supposition. Thus, $(x \prec y)^n \succ t = n(x \succ y) \succ t$ for all $n \in N$.

Now we show that $(x \prec y)^n \succ t = n(x \prec y) \succ t$ for all $x, y, t \in D$, $n \in N$. For $n = 1$, obviously, the equality is correct. Let $(x \prec y)^k \succ t = k(x \prec y) \succ t$ for $n = k$. Then for $n = k+1$ we obtain

$$\begin{aligned}(x \prec y)^{k+1} \succ t &= ((x \prec y) \prec (x \prec y)^k) \succ t = (x \prec y) \succ ((x \prec y)^k \succ t) \\ &= (x \prec y) \succ k(x \prec y) \succ t = (k+1)(x \prec y) \succ t\end{aligned}$$

according to the axioms $(D4),(D5)$ of a dimonoid and the supposition. Thus, $(x \prec y)^n \succ t = n(x \prec y) \succ t$ for all $n \in N$.

Dually, the equalities (ii) can be proved. ■

Lemma 5. *Let $(D, \prec, \succ)$ be an arbitrary dimonoid. For all $x \in D$, $n \in N$*

(i) $x^n \succ x = (n+1)x$;
(ii) $x \prec nx = x^{n+1}$.

Proof. We prove (i) using an induction on n. For $n = 1$ we have $x \succ x = 2x$. Let $x^k \succ x = (k+1)x$ for $n = k$. Then for $n = k+1$ we obtain

$$x^{k+1} \succ x = (x \prec x^k) \succ x = x \succ (x^k \succ x) = x \succ (k+1)x = (k+2)x$$

according to the axiom $(D4)$ of a dimonoid and the supposition. Thus, $x^n \succ x = (n+1)x$ for all $n \in N$.

Dually, the equality (ii) can be proved. ■

Let S be a semigroup and $a \in S$. The elements $x, y \in S$ are called a-connected, if there exist $n, m \in N$ such that $(xa)^n \in yaS$ and $(ya)^m \in xaS$. The semigroup S is a-connected, if x, y are a-connected for all $x, y \in S$ [23].

Note that if $(xa)^n \in yaS$ and $(ya)^m \in xaS$, then $(xa)^p \in yaS$ and $(ya)^p \in xaS$, where $p = \max\{n, m\}, n, m, p \in N$ [23].

Recall that a semigroup S is called archimedean, if for any $a, b \in S$ there exists $n \in N$ such that b^n belongs to the principial two-sided ideal $J(a)$ generated by a. If $(D, \prec, \succ)$ is a dimonoid, then we denote the semigroup $(D, \prec)$ (respectively, $(D, \succ)$) with an identity by $D^1_{\prec}$ (respectively, by $D^1_{\succ}$).

Lemma 6. *Let $(D, \prec, \succ)$ be a dimonoid and let $a \in D$ be an arbitrary fixed element. Then*

(i) *If* $(D, \prec)$ *is a* a*-connected semigroup, then* $(D, \succ)$ *is a* a*-connected semigroup;*

(ii) $(D, \prec)$ *is an archimedean semigroup if and only if* $(D, \succ)$ *is an archimedean semigroup.*

Proof. (i) Let $(D, \prec)$ be a a-connected semigroup, $x, y \in D$. Then there exists $n \in N$ such that $(x \prec a)^n \in y \prec a \prec D$ and $(y \prec a)^n \in x \prec a \prec D$. Hence

$$(x \prec a)^n = y \prec a \prec t_1, \tag{1}$$

$$(y \prec a)^n = x \prec a \prec t_2 \tag{2}$$

for some $t_1, t_2 \in D$. Assume $t_3 = t_1 \succ x \succ a$, $t_4 = t_2 \succ y \succ a$. Multiply the equalities (1) and (2) by $x \succ a$ and, respectively, by $y \succ a$:

$$\begin{aligned}(x \prec a)^n \succ (x \succ a) &= n(x \succ a) \succ (x \succ a) = (n+1)(x \succ a)\\ &= (y \prec a \prec t_1) \succ (x \succ a) = ((y \prec a) \succ t_1) \succ (x \succ a)\\ &= y \succ a \succ t_1 \succ x \succ a = y \succ a \succ t_3,\end{aligned}$$

$$\begin{aligned}(y \prec a)^n \succ (y \succ a) &= n(y \succ a) \succ (y \succ a) = (n+1)(y \succ a)\\ &= (x \prec a \prec t_2) \succ (y \succ a) = ((x \prec a) \succ t_2) \succ (y \succ a)\\ &= x \succ a \succ t_2 \succ y \succ a = x \succ a \succ t_4\end{aligned}$$

according to Lemma 4(i) and the axioms $(D4), (D5)$ of a dimonoid. Thus, $(n+1)(x \succ a) \in y \succ a \succ D$, $(n+1)(y \succ a) \in x \succ a \succ D$. Consequently, $(D, \succ)$ is a a-connected semigroup.

(ii) Let $(D, \prec)$ be an archimedean semigroup. Then for all $a, b \in D$ there exist $x, y \in D^1_{\prec}$, $n \in N$ such that $x \prec a \prec y = b^n$. Multiply both parts of the last equality by b concerning the operation $\succ$:

$$\begin{aligned}(x \prec a \prec y) \succ b &= ((x \prec a) \prec y) \succ b = (x \prec a) \succ (y \succ b)\\ &= x \succ a \succ (y \succ b) = b^n \succ b = (n+1)b\end{aligned}$$

according to the axioms $(D4), (D5)$ of a dimonoid and Lemma 5(i). Analogously, using the axioms $(D1), (D2)$ of a dimonoid and Lemma 5(ii), we can prove the sufficiency. ■

A dimonoid $(D, \prec, \succ)$ will be called an idempotent dimonoid or a diband, if $x \prec x = x = x \succ x$ for all $x \in D$.

Lemma 7. *Let* $(D, \prec, \succ)$ *be an idempotent dimonoid. Then* $(D, \prec)$ *is a rectangular band if and only if* $(D, \succ)$ *is a rectangular band.*

Proof. If $(D, \prec)$ is a rectangular band, $a, b \in D$, then $a \prec b \prec a = a$. From the last equality we have

$$\begin{aligned}(a \prec b \prec a) \succ a &= (a \prec (b \succ a)) \succ a \\ &= a \succ ((b \succ a) \succ a) = a \succ b \succ a = a \succ a = a\end{aligned}$$

according to the axioms $(D2), (D4), (D5)$ of a dimonoid and the idempotent property of the operation $\succ$. Hence $(D, \succ)$ is a rectangular band.

Conversely, from the equality $a \succ b \succ a = a$ we obtain

$$\begin{aligned}a \prec (a \succ b \succ a) &= a \prec (a \succ (b \succ a)) = (a \prec a) \prec (b \succ a) \\ &= a \prec (b \succ a) = a \prec b \prec a = a \prec a = a\end{aligned}$$

according to the axioms $(D2)$, $(D5)$ of a dimonoid and the idempotent property of the operation $\prec$. Hence $(D, \prec)$ is a rectangular band. ∎

The notion of a diband of subdimonoids was introduced in [10] and investigated in [16]. Recall this definition.

If $\varphi : S \to T$ is a homomorphism of dimonoids, then the corresponding congruence on S will be denoted by Δ_φ.

Let S be an arbitrary dimonoid, J be some idempotent dimonoid. Let

$$\alpha : S \to J : x \mapsto x\alpha,$$

be a homomorphism. Then every class of the congruence Δ_α is a subdimonoid of the dimonoid S, and the dimonoid S itself is a union of such dimonoids S_ξ, $\xi \in J$ that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \,|(x;t) \in \Delta_\alpha\},$$
$$S_\xi \prec S_\varepsilon \subseteq S_{\xi \prec \varepsilon}, \quad S_\xi \succ S_\varepsilon \subseteq S_{\xi \succ \varepsilon},$$
$$\xi \neq \varepsilon \Rightarrow S_\xi \bigcap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a diband of subdimonoids (or S is a diband J of subdimonoids S_ξ $(\xi \in J)$). If J is a band (=idempotent semigroup), then we say that S is a band J of subdimonoids S_ξ $(\xi \in J)$. If J is a commutative band (=semilattice), then we say that S is a semilattice J of subdimonoids S_ξ $(\xi \in J)$.

Let S be a diband J of subdimonoids S_ξ, $\xi \in J$. Note that if the operations of S coincide, then S is a band of semigroups [11].

If ρ is a congruence on the dimonoid $(D, \prec, \succ)$ such that $(D, \prec, \succ)/\rho$ is an idempotent dimonoid, then we say that ρ is an idempotent congruence.

Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\prec$, $a, b \in D$. We say that a $\prec$-divide b and write $a_\prec | b$, if there exists such element x from $D^1_\prec$ that $a \prec x = b$. A dimonoid $(D, \prec, \succ)$ will be called commutative, if semigroups $(D, \prec)$ and $(D, \succ)$ are commutative.

Define a relation η on the dimonoid $(D, \prec, \succ)$ with a commutative operation $\prec$ by

$$a\eta b \text{ if and only if there exist positive integers}$$
$$m, n,\ m \neq 1, n \neq 1 \text{ such that } a_{\prec}|b^m,\ b_{\prec}|a^n.$$

Theorem 8. ([10], Theorem 1) *The relation η on the dimonoid $(D, \prec, \succ)$ with a commutative operation $\prec$ is the least idempotent congruence, and $(D, \prec, \succ)/\eta$ is a commutative idempotent dimonoid which is a semilattice.*

3. Independence of axioms and examples of dimonoids

In this section we show that the system of axioms of a dimonoid is independent and give different examples of dimonoids.

The following theorem proves the independence of axioms of a dimonoid.

Theorem 9. *The system of axioms $(D1)$, $(D2)$, $(D3)$, $(D4)$, $(D5)$ of a dimonoid is independent.*

Proof. Let N be the set of positive integers. Define the operations $\prec$ and $\succ$ on N by

$$x \prec y = 2y, \quad x \succ y = y$$

for all $x, y \in N$. The model $(N, \prec, \succ)$ satisfies the axioms $(D2)$–$(D5)$, but not $(D1)$. Indeed,

$$(x \prec y) \prec z = 2z = x \prec (y \succ z),$$
$$(x \succ y) \prec z = 2z = x \succ (y \prec z),$$
$$(x \prec y) \succ z = z = x \succ (y \succ z),$$
$$x \succ (y \succ z) = z = (x \succ y) \succ z,$$
$$x \prec (y \prec z) = 4z \neq 2z = (x \prec y) \prec z$$

for all $x, y, z \in N$.

Assume $x \prec y = x$, $x \succ y = 2x$ for all $x, y \in N$. Similarly to the preceding case we can show that the model $(N, \prec, \succ)$ satisfies the axioms $(D1)$–$(D4)$, but not $(D5)$.

Let $x \prec y = x + y$, $x \succ y = y$ for all $x, y \in N$. In this case the model $(N, \prec, \succ)$ satisfies the axioms $(D1)$, $(D3)$–$(D5)$, but not $(D2)$. Indeed,

$$x \prec (y \prec z) = x + y + z = (x \prec y) \prec z,$$
$$(x \succ y) \prec z = y + z = x \succ (y \prec z),$$
$$(x \prec y) \succ z = z = x \succ (y \succ z),$$
$$x \succ (y \succ z) = z = (x \succ y) \succ z,$$
$$(x \prec y) \prec z = x + y + z \neq x + z = x \prec (y \succ z)$$

for all $x, y, z \in N$.

Assume $x \prec y = x$, $x \succ y = x + y$ for all $x, y \in N$. Similarly to the preceding case we can show that the model $(N, \prec, \succ)$ satisfies the axioms $(D1)$–$(D3)$, $(D5)$, but not $(D4)$.

Finally we construct the last model. Let X be an arbitrary nonempty set, $|X| > 1$ and let X^* be the set of finite nonempty words in the alphabet X. We denote the first (respectively, the last) letter of a word $w \in X^*$ by $w^{(0)}$ (respectively, by $w^{(1)}$). Define the operations $\prec$ and $\succ$ on X^* by

$$w \prec u = w^{(0)}, \quad w \succ u = u^{(1)}$$

for all $w, u \in X^*$. The model $(X^*, \prec, \succ)$ satisfies the axioms $(D1)$, $(D2)$, $(D4)$, $(D5)$, but not $(D3)$. Indeed,

$$\begin{aligned} w \prec (u \prec \omega) &= w^{(0)} = (w \prec u) \prec \omega, \\ (w \prec u) \prec \omega &= w^{(0)} = w \prec (u \succ \omega), \\ (w \prec u) \succ \omega &= \omega^{(1)} = w \succ (u \succ \omega), \\ w \succ (u \succ \omega) &= \omega^{(1)} = (w \succ u) \succ \omega \end{aligned}$$

for all $w, u, \omega \in X^*$. As $|X| > 1$, then there exists $u \in X^*$ such that $u^{(1)} \neq u^{(0)}$. Then

$$(w \succ u) \prec \omega = u^{(1)} \neq u^{(0)} = w \succ (u \prec \omega)$$

for all $w, \omega \in X^*$. ∎

Now we give examples of dimonoids.

a) Let S be a semigroup and let f be its idempotent endomorphism. Define the operations $\prec$ and $\succ$ on S by

$$x \prec y = x(yf), \ x \succ y = (xf)y$$

for all $x, y \in S$.

Proposition 10. ([10], Proposition 1) *$(S, \prec, \succ)$ is a dimonoid.*

b) Let S and T be semigroups, $\theta : T \to S$ be a homomorphism. Define the operations $\prec$ and $\succ$ on $S \times T$ by

$$(s, t) \prec (p, g) = (s, tg), (s, t) \succ (p, g) = ((t\theta)p, tg)$$

for all $(s, t), (p, g) \in S \times T$.

Proposition 11. ([10], Proposition 2) *$(S \times T, \prec, \succ)$ is a dimonoid.*

c) Let $2N$ be the set of even positive integers and $2N$-1 be the set of odd positive integers. Fix $t, t_1, t_2 \in 2N$-1 and define the operations $\prec$ and $\succ$ on N by

$$x \prec y = \begin{cases} x + y + t_1, \ x, y \in 2N, \\ \quad t \quad \text{otherwise}, \end{cases}$$

$$x \succ y = \begin{cases} x + y + t_2, \ x, y \in 2N, \\ \quad t \quad \text{otherwise} \end{cases}$$

for all $x, y \in N$.

Proposition 12. $(N, \prec, \succ)$ *is a commutative dimonoid.*

Proof. It is immediate to check that $(N, \prec, \succ)$ is a dimonoid. It is clear that the operations $\prec$ and $\succ$ are commutative. ■

d) Let A be an alphabet, $F[A]$ be the free commutative semigroup over A, G be a set of non-ordered pairs (p, q), $p, q \in A$. Define the operations $\prec$ and $\succ$ on the set $F[A] \bigcup G$ by

$$a_1...a_m \prec b_1...b_n = a_1...a_m b_1...b_n,$$

$$a_1...a_m \succ b_1...b_n = \begin{cases} a_1...a_m b_1...b_n, \ mn > 1, \\ (a_1, b_1), \ m = n = 1, \end{cases}$$

$$a_1...a_m \prec (p, q) = a_1...a_m \succ (p, q) = a_1...a_m pq,$$

$$(p, q) \prec a_1...a_m = (p, q) \succ a_1...a_m = pq a_1...a_m,$$

$$(p, q) \prec (r, s) = (p, q) \succ (r, s) = pqrs$$

for all $a_1...a_m$, $b_1...b_n \in F[A]$, $(p, q), (r, s) \in G$.

Proposition 13. ([15], Theorem 3) $(F[A] \bigcup G, \prec, \succ)$ *is the free commutative dimonoid.*

e) Let X be an arbitrary nonempty set. Considering the disjoint union

$$D(X) = \coprod_{n \geq 1} \underbrace{(X^n \bigcup ... \bigcup X^n)}_{n \text{ copies}}$$

and denoting by $x_1...\breve{x}_i...x_n$ an element in the i-th summand, define the operations $\prec$ and $\succ$ on $D(X)$ by

$$(x_1...\breve{x}_i...x_k) \prec (x_{k+1}...\breve{x}_j...x_l) = x_1...\breve{x}_i...x_l,$$

$$(x_1...\breve{x}_i...x_k) \succ (x_{k+1}...\breve{x}_j...x_l) = x_1...\breve{x}_j...x_l$$

for all $x_1...\breve{x}_i...x_k, \ x_{k+1}...\breve{x}_j...x_l \in D(X)$.

Proposition 14. ([2], Corollary 1.8) $(D(X), \prec, \succ)$ *is the free dimonoid on the set X.*

Other examples of dimonoids can be found in [2], [10], [15]–[17].

4. Decompositions

In this section we prove that every dimonoid with a commutative operation is a semilattice of archimedean subdimonoids (Theorem 15), every dimonoid with a commutative periodic semigroup is a semilattice of unipotent subdimonoids (Theorem 16), every dimonoid with a commutative operation is a semilattice of a-connected subdimonoids (Theorem 17) and every idempotent dimonoid is a semilattice of rectangular subdimonoids (Theorem 18).

We say that a dimonoid is archimedean, if its both semigroups are archimedean (see section 2).

Theorem 15. *Every dimonoid $(D, \prec, \succ)$ with a commutative operation $\prec$ is a semilattice Y of archimedean subdimonoids D_i, $i \in Y$.*

Proof. Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\prec$. By Theorem 8 $(D, \prec, \succ)/\eta$ is a semilattice. From the theorem by Tamura and Kimura [24] it follows that every class A of the congruence η is an archimedean semigroup concerning the operation $\prec$. Hence according to Lemma 6(ii) A is an archimedean semigroup concerning the operation $\succ$. Thus, A is an archimedean subdimonoid of $(D, \prec, \succ)$. ∎

This theorem extends Theorem 2 from [10] about the decomposition of commutative dimonoids into semilattices of archimedean subdimonoids and the theorem by Tamura and Kimura [24] about the decomposition of commutative semigroups into semilattices of archimedean semigroups.

Recall that a semigroup S is called a periodic semigroup, if every element of S has a finite order, that is, if for every element a of S the subsemigroup $\langle a \rangle = \{a, a^2, ..., a^n, ...\}$ generated by a contains a finite number of different elements.

A dimonoid $(D, \prec, \succ)$ will be called unipotent, if it contains exactly one element $x \in D$ such that $x \prec x = x \succ x = x$. If ρ is a congruence on the dimonoid $(D, \prec, \succ)$ such that the operations of $(D, \prec, \succ)/\rho$ coincide and it is a semilattice, then we say that ρ is a semilattice congruence.

Theorem 16. *Every dimonoid $(D, \prec, \succ)$ with a commutative periodic semigroup $(D, \prec)$ is a semilattice L of unipotent subdimonoids D_i, $i \in L$.*

Proof. Define a relation γ on $(D, \prec, \succ)$ by

$a\gamma\, b$ if and only if there exists an
idempotent ε of the semigroup $(D, \prec)$ such
that $a^l = b^k = \varepsilon$ for some $l,\ k \in N$.

The fact that the relation γ is a semilattice congruence on the semigroup $(D, \prec)$ has been proved by Schwarz [21]. Let us show that γ is compatible concerning the operation $\succ$.

Let $a\gamma b$, $a, b, c \in D$. Then $a \prec c\gamma b \prec c$. It means that there exists an idempotent e of the semigroup $(D, \prec)$ such that

$$(a \prec c)^n = (b \prec c)^m = e$$

for some $n, m \in N$. Hence

$$(a \prec c)^n \prec (a \prec c)^n = (a \prec c)^{2n} = e, \tag{3}$$

$$(b \prec c)^m \prec (b \prec c)^m = (b \prec c)^{2m} = e. \tag{4}$$

By Lemma 1 from (3) and (4) it follows that $(a \succ c)^{2n} = (b \succ c)^{2m} = e$ and so, $a \succ c \, \gamma \, b \succ c$.

Dually, the left compatibility of the relation γ concerning the operation $\succ$ can be proved. So, γ is a congruence on $(D, \prec, \succ)$.

As $(D, \prec)/\gamma$ is a semilattice, then by Lemma 3(i) the operations of $(D, \prec, \succ)/\gamma$ coincide and so, it is a semilattice.

From [21] it follows that every class A of the congruence γ is a unipotent subsemigroup of the semigroup $(D, \prec)$. Let $e \in A$ and $e \prec e = e$. For an arbitrary element $a \in A$ there exists $p \in N$, $p > 1$ such that $a^p = e$. Hence

$$\begin{aligned} e \succ e &= a^p \succ a^p = a^p \succ (a^{p-1} \prec a) = (a^p \succ a^{p-1}) \prec a \\ &= a \prec (a^p \succ a^{p-1}) = (a \prec a^p) \prec a^{p-1} = (a \prec a^{p-1}) \prec a^p \\ &= a^p \prec a^p = e \prec e = e \end{aligned}$$

according to the commutativity of the operation $\prec$ and the axioms $(D1)$, $(D2)$, $(D3)$ of a dimonoid. So, e is an idempotent of the subsemigroup A of $(D, \succ)$. Thus, A is a unipotent subdimonoid of $(D, \prec, \succ)$. ∎

This theorem extends Schwarz's theorem [21] about the decomposition of commutative periodic semigroups into semilattices of unipotent semigroups.

Let $(D, \prec, \succ)$ be a dimonoid and $a \in D$. A dimonoid $(D, \prec, \succ)$ will be called a-connected, if semigroups $(D, \prec)$ and $(D, \succ)$ are a-connected (see section 2).

Theorem 17. *Let $(D, \prec, \succ)$ be a dimonoid with a commutative operation $\prec$ and let $a \in D$ be an arbitrary fixed element. Then $(D, \prec, \succ)$ is a semilattice R of a-connected subdimonoids D_i, $i \in R$.*

Proof. Define a relation ζ on $(D, \prec, \succ)$ by

$$x \zeta y \Leftrightarrow (\exists n \in N) \, (x \prec a)^n \in y \prec a \prec D, \\ (y \prec a)^n \in x \prec a \prec D.$$

By Protić and Stevanović [23] ζ is a semilattice congruence on the semigroup $(D, \prec)$. Let us show that ζ is a congruence on the semigroup $(D, \succ)$.

Let $x \zeta y$, $x, y, c \in D$. Then $x \prec c \zeta y \prec c$. It means that

$$(x \prec c \prec a)^m = y \prec c \prec a \prec t_1, \\ (y \prec c \prec a)^m = x \prec c \prec a \prec t_2$$

for some $m \in N, t_1, t_2 \in D$. Hence

$$\begin{aligned} (x \prec c \prec a)^m &= ((a \prec x) \prec c)^m = (a \prec (x \succ c))^m = ((x \succ c) \prec a)^m \\ &= y \prec c \prec a \prec t_1 = ((a \prec t_1) \prec y) \prec c = (a \prec t_1) \prec (y \succ c) \\ &= (y \succ c) \prec a \prec t_1 \end{aligned}$$

according to the commutativity of the operation $\prec$ and the axioms $(D1)$, $(D2)$ of a dimonoid. Analogously,

$$((y \succ c) \prec a)^m = (x \succ c) \prec a \prec t_2.$$

Consequently, $x \succ c \zeta y \succ c$.

Dually, the left compatibility of the relation ζ concerning the operation $\succ$ can be proved. So, ζ is a congruence on $(D, \prec, \succ)$.

As $(D, \prec)/\zeta$ is a semilattice, then by Lemma 3(i) the operations of $(D, \prec, \succ)/\zeta$ coincide and so, it is a semilattice.

Let A be an arbitrary class of the congruence ζ. By the definition of ζ the class A is a a-connected semigroup concerning the operation $\prec$. From Lemma 6(i) it follows that A is a a-connected semigroup concerning the operation $\succ$. Thus, A is a a-connected subdimonoid of $(D, \prec, \succ)$. ∎

We say that a dimonoid is rectangular, if its both semigroups are rectangular bands. Define a relation $\Im$ on the dimonoid $(D, \prec, \succ)$ with an idempotent operation $\prec$ by

$$a \Im b \text{ if and only if } a = a \prec b \prec a, \quad b = b \prec a \prec b.$$

Theorem 18. *The relation $\Im$ on the dimonoid $(D, \prec, \succ)$ with an idempotent operation $\prec$ is the least semilattice congruence. Every idempotent dimonoid $(D, \prec, \succ)$ is a semilattice Ω of rectangular subdimonoids D_i, $i \in \Omega$.*

Proof. The fact that the relation $\Im$ is a semilattice congruence on the semigroup $(D, \prec)$ has been proved by McLean [22]. Let us show that $\Im$ is compatible concerning the operation $\succ$.

Let $a \Im b$, $a, b, c \in D$. Then $a \prec c \Im b \prec c$. It means that

$$(a \prec c) \prec (b \prec c) \prec (a \prec c) = a \prec c, \tag{5}$$

$$(b \prec c) \prec (a \prec c) \prec (b \prec c) = b \prec c. \tag{6}$$

Multiply both parts of the equality (5) by $a \succ c$ and of the equality (6) by $b \succ c$:

$$\begin{aligned}
&(a \succ c) \prec ((a \prec c) \prec (b \prec c) \prec (a \prec c)) \\
&= ((a \succ c) \prec (a \prec c)) \prec (b \prec c) \prec (a \prec c) \\
&= ((a \succ c) \prec (a \succ c)) \prec (b \prec c) \prec (a \prec c) \\
&= (a \succ c) \prec (b \prec c) \prec (a \prec c) \\
&= ((a \succ c) \prec (b \succ c)) \prec (a \prec c) \\
&= (a \succ c) \prec (b \succ c) \prec (a \succ c) \\
&= (a \succ c) \prec (a \prec c) = (a \succ c) \prec (a \succ c) = a \succ c,
\end{aligned}$$

$$\begin{aligned}
&(b \succ c) \prec ((b \prec c) \prec (a \prec c) \prec (b \prec c)) \\
&= ((b \succ c) \prec (b \prec c)) \prec (a \prec c) \prec (b \prec c) \\
&= ((b \succ c) \prec (b \succ c)) \prec (a \prec c) \prec (b \prec c) \\
&= (b \succ c) \prec (a \prec c) \prec (b \prec c) \\
&= ((b \succ c) \prec (a \succ c)) \prec (b \prec c) \\
&= (b \succ c) \prec (a \succ c) \prec (b \succ c) \\
&= (b \succ c) \prec (b \prec c) = (b \succ c) \prec (b \succ c) = b \succ c
\end{aligned}$$

according to the axioms $(D1)$, $(D2)$ of a dimonoid and the idempotent property of the operation $\prec$. Consequently, $a \succ c \Im b \succ c$.

Dually, the left compatibility of the relation $\Im$ concerning the operation $\succ$ can be proved. So, $\Im$ is a congruence on $(D, \prec, \succ)$.

As $(D, \prec)/\Im$ is a semilattice, then according to Lemma 3(i) the operations of $(D, \prec, \succ)/\Im$ coincide and so, it is a semilattice.

The proof of the first statement of the theorem will be completed, if we show that $\Im$ is contained in every semilattice congruence ρ on $(D, \prec, \succ)$. Let $a \Im b$, $a, b \in D$. Then $a \prec b \prec a = a, \quad b \prec a \prec b = b$. As ρ is a semilattice congruence, then $a = a \prec b \prec a \rho\, b \prec a \prec b = b$. So, $a \rho b$ and $\Im \subseteq \rho$.

Now we shall prove the second statement of the theorem.

Since $\Im$ is a congruence on $(D, \prec, \succ)$ and $(D, \prec, \succ)/\Im$ is a semilattice, then

$$(D, \prec, \succ) \to (D, \prec, \succ)/\Im : x \mapsto [x]$$

is a homomorphism ($[x]$ is a class of the congruence $\Im$, which contains x). From McLean's theorem [22] it follows that every class A of the congruence $\Im$ is a rectangular band concerning the operation $\prec$. According to Lemma 7 A is a rectangular band concerning the operation $\succ$. Thus, A is a rectangular subdimonoid of $(D, \prec, \succ)$. ■

This theorem extends McLean's description [22] of the least semilattice congruence on bands and McLean's decomposition [22] of bands into semilattices of rectangular bands.

In section 3, we gave examples of commutative dimonoids (see also [10], [15]). We finish this section with the construction of different examples of dimonoids with one and two idempotent operations.

a) Let $(X, \prec)$ be a left zero semigroup, $(X, \succ)$ be a zero semigroup. Then $(X, \prec, \succ)$ is a dimonoid with the idempotent operation $\prec$. It is easy to see that the least semilattice congruence $\Im = X \times X$ on $(X, \prec, \succ)$.

b) Let X^* be the set of finite nonempty words in the alphabet X. Recall that we denote the first (respectively, the last) letter of a word $w \in X^*$ by $w^{(0)}$ (respectively, by $w^{(1)}$).

Assuming the operations $\prec$ and $\succ$ on the set X^* by

$$w \prec u = w, \quad w \succ u = w^{(0)}u^{(1)}$$

for all $w, u \in X^*$, we obtain a dimonoid with the idempotent operation $\prec$. At that $\Im = X^* \times X^*$.

c) Let $(X, \prec)$ be a left zero semigroup, $(X, \succ)$ be a rectangular band. Then $(X, \prec, \succ)$ is an idempotent dimonoid. It is easy to see that the least semilattice congruence $\Im = X \times X$ on $(X, \prec, \succ)$.

d) Let $(X, \prec)$ be a rectangular band, $(X, \succ)$ be a right zero semigroup. Then $(X, \prec, \succ)$ is an idempotent dimonoid. It is easy to see that the least semilattice congruence $\Im = X \times X$ on $(X, \prec, \succ)$.

e) We prove the following statement.

Proposition 19. *Let $(D, \prec, \succ)$ be a dimonoid, $H(D) = \{e \in D \mid e \succ z = z \prec e$ for all $z \in D\}$. If $H(D) \neq \varnothing$, then $H(D)$ is a subdimonoid of $(D, \prec, \succ)$.*

Proof. If $e, \varepsilon \in H(D)$, then $e \succ z = z \prec e,\ \varepsilon \succ z = z \prec \varepsilon$ for all $z \in D$. For all $z \in D$ we have

$$\begin{aligned} z \prec (e \prec \varepsilon) &= (z \prec e) \prec \varepsilon = (e \succ z) \prec \varepsilon = e \succ (z \prec \varepsilon) \\ &= e \succ (\varepsilon \succ z) = (e \prec \varepsilon) \succ z, \\ z \prec (e \succ \varepsilon) &= (z \prec e) \prec \varepsilon = (e \succ z) \prec \varepsilon = e \succ (z \prec \varepsilon) \\ &= e \succ (\varepsilon \succ z) = (e \succ \varepsilon) \succ z \end{aligned}$$

according to the preceding equalities and the axioms $(D1)$–$(D5)$ of a dimonoid. It means that $e \prec \varepsilon, e \succ \varepsilon \in H(D)$. So, $H(D)$ is a subdimonoid of $(D, \prec, \succ)$. ∎

Let $R(D) = \{e \in D \mid e \succ z = z = z \prec e$ for all $z \in D\}$. From Proposition 19 it follows that $R(D)$ is a subdimonoid of $H(D)$ (if $R(D) \neq \varnothing$). Moreover, it is easy to see that $R(D)$ is an idempotent dimonoid. Obviously, its least semilattice congruence $\Im$ coincides with the universal relation on $R(D)$.

f) Let S be an arbitrary idempotent semigroup, R be a rectangular band. Define the operations $\prec$ and $\succ$ on the set $S \times R$ by

$$(s_1, p_1) \prec (s_2, p_2) = (s_1 s_2, p_1 p_2), \qquad (s_1, p_1) \succ (s_2, p_2) = (s_1 s_2, p_2)$$

for all $(s_1, p_1), (s_2, p_2) \in S \times R$. It is not difficult to check that $(S \times R, \prec, \succ)$ is an idempotent dimonoid. We denote this dimonoid by S^R.

Define a relation $\overline{\Im}$ on S by $a \overline{\Im} b$ if and only if $aba = a,\ bab = b$. By McLean's theorem [22] $\overline{\Im}$ is a congruence on S. From Theorem 18 it follows that

$$(s_1, p_1) \Im (s_2, p_2) \Leftrightarrow s_1 \overline{\Im} s_2$$

for all $(s_1, p_1),\ (s_2, p_2) \in S^R$.

We denote the semigroup $S/\overline{\Im}$ by P. According to [22] S is a semilattice P of rectangular bands S_i, $i \in P$.

Using Theorem 18 it is easy to prove the following statement.

Proposition 20. *The dimonoid S^R is a semilattice P of rectangular subdimonoids S_i^R, $i \in P$.*

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