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Leopold Koczan, Magda Sobczak-Kneć, Paweł Zaprawa

ON FUNCTIONS CONVEX IN THE DIRECTION OF THE IMAGINARY AXIS WITH REAL COEFFICIENTS

Abstract. Let $\mathcal{Y}$ be a subclass of the class of all analytic functions in the unit disk Δ having the normalization $f(0) = f'(0) - 1 = 0$. If there exists an analytic, univalent function m satisfying the following conditions: $m'(0) > 0$, $\bigwedge_{f\in\mathcal{Y}} m \prec f$ and for every analytic function k, $k(0) = 0$, there is $\left(\bigwedge_{f\in\mathcal{Y}} k \prec f\right) \Rightarrow k \prec m$, then this function is called the minorant of $\mathcal{Y}$. Similarly, if there exists an analytic, univalent function M such that $M'(0) > 0$, $\bigwedge_{f\in\mathcal{Y}} f \prec M$ and for every analytic function k, $k(0) = 0$, there is $\left(\bigwedge_{f\in\mathcal{Y}} f \prec k\right) \Rightarrow M \prec k$, then this function is called the majorant of $\mathcal{Y}$. It is possible to give a number of examples of classes of analytic functions for which the majorant or minorant does not exist. However, if these functions exist then $m(\Delta)$ and $M(\Delta)$ coincide with the Koebe domain and the covering domain for $\mathcal{Y}$, respectively.

In this paper we determine the Koebe domain and the covering domain as well as the minorant and the majorant for the class consisting of functions convex in the direction of the imaginary axis with real coefficients.

Introduction

In our research we use the concept of subordination. We say that an analytic function f is subordinated to an analytic and univalent function F in $\Delta \equiv \{\zeta \in C : |\zeta| < 1\}$ if and only if there exists an analytic function ω such that $\omega(0) = 0$, $\omega(\Delta) \subset \Delta$ and $f(z) = F(\omega(z))$ for $z \in \Delta$. Then we write $f \prec F$.

Let $\mathcal{A}$ denote the set of all functions f analytic in Δ and normalized by $f(0) = f'(0) - 1 = 0$, and let $\mathcal{Y}$ denote an arbitrary subclass of $\mathcal{A}$.

For a given $\mathcal{Y}$, if there exists an analytic and univalent function m satisfying the following conditions: $m'(0) > 0$,

$$\bigwedge_{f\in\mathcal{Y}} m \prec f \tag{1}$$

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and for every analytic function k, $k(0) = 0$, there is

$$\left(\bigwedge_{f\in\mathcal{Y}} k \prec f\right) \Rightarrow k \prec m, \tag{2}$$

then this function is called the minorant of $\mathcal{Y}$. The set $\bigcap_{f\in\mathcal{Y}} f(\Delta)$ is said to be the Koebe domain for $\mathcal{Y}$ and is denoted by $K_{\mathcal{Y}}$. Clearly, if the Koebe domain is a simply connected set, then the minorant exists and $K_{\mathcal{Y}} = m(\Delta)$.

If there exists an analytic and univalent function M such that $M'(0) > 0$,

$$\bigwedge_{f\in\mathcal{Y}} f \prec M \tag{3}$$

and for every analytic function k, $k(0) = 0$, there is

$$\left(\bigwedge_{f\in\mathcal{Y}} f \prec k\right) \Rightarrow M \prec k, \tag{4}$$

then this function is called the majorant of $\mathcal{Y}$. The set $\bigcup_{f\in\mathcal{Y}} f(\Delta)$ is said to be the covering domain for $\mathcal{Y}$ and is denoted by $L_{\mathcal{Y}}$. Notice that if the covering domain is a simply connected set then the majorant exists. In this case $L_{\mathcal{Y}} = M(\Delta)$.

Let Y denote the known class consisting of all functions which are univalent and convex in the direction of the imaginary axis and having real coefficients.

EXAMPLES.

1. $\mathcal{Y} = S$, where $S \subset \mathcal{A}$ is the class of all univalent functions in Δ. Then $m(z) = \frac{1}{4}z$, $z \in \Delta$. Hence $K_S = \Delta_{1/4}$ and the majorant does not exist ($L_S = C$).

2. $\mathcal{Y} = Y$. Then $m(z) = \frac{1}{2}z$, $z \in \Delta$, (McGregor, [4]) and the majorant does not exist ($L_Y = C$).

3. $\mathcal{Y} = CVR^{(2)}$, where $CVR^{(2)}$ is the class of univalent, convex and odd functions in Δ with real coefficients. The set $K_{CVR^{(2)}}$ was determined by Krzyż and Reade (see [1]). Then m maps Δ onto the set $K_{CVR^{(2)}}$ and $m'(0) > 0$. The majorant is given by $M(z) = \int_0^1 \frac{z}{\sqrt{(1-t^2)(1-t^2z^4)}}dt$ (see [3]).

Let $n \geq 2$ be a fixed integer. The aim of this paper is to determine the Koebe domain and the covering domain as well as the minorant and the majorant for $Y^{(n)}$. The class $Y^{(n)}$ is the set of n-fold symmetric functions from Y, i.e.

$$Y^{(n)} \equiv \{f \in Y : f(\varepsilon z) = \varepsilon f(z), z \in \Delta\}, \text{ where } \varepsilon = e^{\frac{2\pi i}{n}}.$$

We say that a set D is n-fold symmetric if for ε defined above we have $\varepsilon D = D$. The symbol λD is understood as $\{\lambda z : z \in D\}$.

The very important property of the class $Y^{(n)}$ is given in

LEMMA 1. *If $f \in Y^{(n)}$ then the straight line $k : \zeta = e^{\frac{\pi i}{n}}t,\ t \in R$ is the symmetry axis of the set $f(\Delta)$.*

Proof. The symmetry with respect to the line $\zeta = e^{\frac{\pi i}{n}}t,\ t \in R$ means that for arbitrary $z, \zeta \in \Delta$ if

$$\overline{ze^{-\frac{\pi i}{n}}} = \zeta e^{-\frac{\pi i}{n}} \tag{5}$$

then

$$\overline{f(z)e^{-\frac{\pi i}{n}}} = f(\zeta)e^{-\frac{\pi i}{n}}. \tag{6}$$

Assume that the condition (5) is satisfied. We can write it equivalently in the form

$$\zeta = \overline{z}e^{\frac{2\pi i}{n}} = \overline{z}\varepsilon. \tag{7}$$

From properties of $f \in Y^{(n)}$ it follows that

$$\overline{f(z)}\varepsilon = f(\overline{z})\varepsilon = f(\overline{z}\varepsilon).$$

Applying (7) we obtain $\overline{f(z)}\varepsilon = f(\zeta)$. This condition is equivalent to (6). ∎

REMARK 1. The real axis is another symmetry axis of $f(\Delta)$ because of real coefficients of $f \in Y^{(n)}$. Moreover, from n-fold symmetry each straight line $\zeta = e^{\frac{\pi i}{n}k}t,\ t \in R,\ k = 0, 1, \ldots, 2n-1$ is also the symmetric axis of $f(\Delta)$.

The next lemma follows from Lemma 1 and from properties of the class $Y^{(n)}$.

LEMMA 2. *The Koebe domain and the covering domain for $Y^{(n)}$ are n-fold symmetric and symmetric with respect to the lines $\zeta = e^{\frac{\pi i}{n}k}t,\ t \in R,\ k = 0, 1, \ldots, 2n-1$.*

For a fixed n we use the notation: $\Lambda_j = \{\zeta \in C : 2(j-1)\pi/n \leq \operatorname{Arg}\zeta \leq 2j\pi/n\},\ j = 1, 2, ..., n$ and $\Lambda = \{\zeta \in C : 0 \leq \operatorname{Arg}\zeta \leq \pi/n\}$. Furthermore, we will write ∂D to denote the boundary of a set D.

From Lemma 2 it follows that we need to determine the boundaries of the Koebe domain and the covering domain only in the set Λ.

Koebe domain for odd n

LEMMA 3. *If $f \in Y^{(n)}$, $n \geq 3$ is odd and $w \in \partial f(\Delta)$, where $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$, then the rays $l_1 : \zeta = w + it,\ t \geq 0,\ l_2 : \zeta = w + e^{i(\frac{\pi}{n}-\frac{\pi}{2})}t,\ t \geq 0$ are disjoint from $f(\Delta)$.*

Proof. Let $f \in Y^{(n)}$ and $w \in \partial f(\Delta)$, $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$. From convexity of f in the direction of the imaginary axis we have $l_1 \cap f(\Delta) = \emptyset$. From n-fold symmetry we get that the point $w\varepsilon^k = we^{i(\pi-\frac{\pi}{n})}$, where $n = 2k+1$, does not belong to $f(\Delta)$. Hence the ray $\zeta = w\varepsilon^k + it,\ t \geq 0$ is disjoint from $f(\Delta)$.

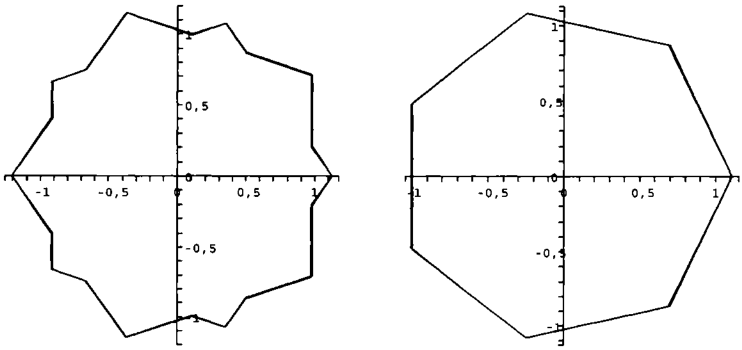

Fig. 1. Polygons: **a)** $n = 5$, $\operatorname{Arg} v = \frac{\pi}{15}$ **b)** $n = 7$, $\operatorname{Arg} v = \frac{\pi}{7}$.

Therefore, the ray $l_2 = \varepsilon^{-k}\,\{w\varepsilon^k + it,\ t \geq 0\} = \{w + it\varepsilon^{-k},\ t \geq 0\}$ is disjoint from $f(\Delta)$, too. ∎

From Lemma 3 it follows that the sector, which has vertex in the point w and radii l_1 and l_2, is disjoint from $f(\Delta)$. Hence we have

Corollary 1. *If $f \in Y^{(n)}$, $n \geq 3$ is odd then f is a starlike function.*

Let $n \geq 3$ be a fixed odd integer.
We consider a family of open and n-fold symmetric polygons which are symmetric with respect to the real axis and are such that their successive vertices u, v, w belong to Λ and $\operatorname{Arg} u = 0$, $\operatorname{Arg} v \in (0, \frac{\pi}{n})$, $\operatorname{Arg} w = \frac{\pi}{n}$. The polygons' interior angles are of the measure $\pi(1 - \frac{2}{n})$, $\pi(1 + \frac{1}{n})$ alternately. We start from the vertex on the real positive axis. From above it follows that these polygons have $4n$ sides.

This set of polygons is extended on limiting cases. If $v = u$ ($\operatorname{Arg} v = 0$) or $v = w$ ($\operatorname{Arg} v = \frac{\pi}{n}$), then we obtain regular polygons having n sides and all angles measuring $\pi(1 - \frac{2}{n})$. In case $v = u$, one of vertices belongs to the real positive semi-axis, and if $v = w$, then one of vertices belongs to the real negative semi-axis.

We denote this family of polygons by $\mathcal{V}$.

The Schwarz–Christoffel formulae confirm the existence of exactly one analytic function which maps Δ univalently onto a fixed polygon of the family $\mathcal{V}$ and has positive derivative in 0. This function is of the form

$$\text{(8)}\qquad \Delta \ni z \mapsto A\int_0^z \sqrt[n]{\frac{(\zeta^n - e^{in\varphi})(\zeta^n - e^{-in\varphi})}{(\zeta^n - 1)^2(\zeta^n + 1)^2}}\,d\zeta,$$

for suitable $\varphi \in \left[0, \frac{\pi}{n}\right]$ and $A > 0$.

From now on we choose the principal branch of the n-th root.

Taking $A = 1$ in (8) we get the function with classical normalization. We denote this function by F_φ and the polygon $F_\varphi(\Delta)$ by A_φ. With this notation all polygons of the family $\mathcal{V}$ can be written as λA_φ, $\lambda > 0$.

Moreover, let

$$(9) \qquad v_1(\varphi) \equiv F_\varphi(e^{i\varphi}).$$

For a fixed φ, the point $v_1(\varphi)$ coincides with the vertex of the polygon A_φ such that its argument is from the range $[0, \frac{\pi}{n}]$. Hence v_1 is given by formula

$$(10) \qquad v_1 \ : \ \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1-t^n)(1-t^n e^{2in\varphi})}{(1-t^n e^{in\varphi})^2(1+t^n e^{in\varphi})^2}} dt,$$

and it is an injective function on $[0, \frac{\pi}{n}]$.

Theorem 1. *Let $n \geq 3$ be odd. The Koebe domain for $Y^{(n)}$ is bounded, n-fold symmetric and symmetric with respect to the real axis. Its boundary in the set Λ is $v_1([0, \frac{\pi}{n}])$, where v_1 is given by* (10).

Proof. Let $f \in Y^{(n)}$, $n \geq 3$ be odd.

Assume that $\lambda v_1(\varphi) \in \partial f(\Delta)$, $\lambda > 0$, $\varphi \in [0, \frac{\pi}{n}]$.

Lemma 3, Lemma 1 and n-fold symmetry of f give us

$$f(\Delta) \subset \lambda A_\varphi = \lambda F_\varphi(\Delta).$$

This and univalence of F_φ lead to

$$f \prec \lambda F_\varphi.$$

Hence $1 = f'(0) \leq \lambda F'_\varphi(0) = \lambda$, so $\lambda \geq 1$. Therefore, if $0 < \lambda < 1$ then $[0, \lambda v_1(\varphi)] \subset f(\Delta)$. Moreover, $v_1(\varphi) \in \partial F_\varphi(\Delta)$, so $v_1(\varphi) \in \partial K_{Y^{(n)}}$.

From the facts given above we conclude that $v_1([0, \frac{\pi}{n}])$ is the Koebe domain's boundary of $Y^{(n)}$ in Λ. Using Lemma 2 completes the proof. ∎

The condition

$$\bigcap_{f \in Y^{(n)}} f(\Delta) = \bigcap_{\varphi \in [0, \frac{\pi}{n}]} F_\varphi(\Delta)$$

results from Theorem 1. Since all functions belonging to the class $Y^{(n)}$ are convex in the direction of the imaginary axis and starlike (by Corollary 1), we have

Corollary 2. *The Koebe domain for $Y^{(n)}$ and odd $n \geq 3$ is convex in the direction of the imaginary axis and starlike.*

Theorem 2. *The function*

$$F(z) = z\int_0^1 \sqrt[n]{\frac{(1-t^n)(1-t^nz^{2n})}{(1-t^nz^n)^2(1+t^nz^n)^2}}dt$$

is the minorant of the class $Y^{(n)}$ *for odd* $n \geq 3$.

Proof. For a fixed $\varphi \in [0, \frac{\pi}{n}]$ we have

$$F(e^{i\varphi}) = e^{i\varphi}\int_0^1 \sqrt[n]{\frac{(1-t^n)(1-t^ne^{2in\varphi})}{(1-t^ne^{in\varphi})^2(1+t^ne^{in\varphi})^2}}dt.$$

Values $F(e^{i\varphi})$ coincide with the values of the function $v_1(\varphi)$. Moreover, F is an n-fold symmetric and injective function on $\partial\Delta$, so it is univalent in Δ. Hence F is the minorant of $Y^{(n)}$. ∎

Corollary 3. *The minorant* F *of the class* $Y^{(n)}$ *for odd* $n \geq 3$ *is convex in the direction of the imaginary axis* ($F/F'(0) \in Y^{(n)}$) *and starlike.*

Koebe domain for even n

Lemma 4. *If* $f \in Y^{(n)}$, n *is even and* $w \in \partial f(\Delta)$, *where* $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$, *then the rays* $l_1 : \zeta = w + it$, $t \geq 0$, $l_2 : \zeta = w + e^{i(\frac{2\pi}{n} - \frac{\pi}{2})}t$, $t \geq 0$ *are disjoint from* $f(\Delta)$.

Proof. Let $f \in Y^{(n)}$ and $w \in \partial f(\Delta)$, $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$. By Lemma 1 points $\overline{w}$ and $\overline{w}\varepsilon = \overline{w}e^{\frac{2\pi i}{n}}$ do not belong to $f(\Delta)$. The function f is convex in the direction of the imaginary axis so $l_1 \cap f(\Delta) = \emptyset$ and $\{\overline{w}\varepsilon + it,\ t \geq 0\} \cap f(\Delta) = \emptyset$. Since $f(\Delta)$ is symmetric with respect to the line $\zeta = e^{\frac{\pi i}{n}}t$, $t \in R$, we have $l_2 \cap f(\Delta) = \emptyset$. ∎

From Lemma 4 we conclude that for $n \geq 4$ the sector, which has vertex in the point w and radii l_1 and l_2, is disjoint from $f(\Delta)$. Observe that if $n = 2$ then the line l_1 coincides with l_2. We have

Corollary 4. *If* $f \in Y^{(n)}$, $n \geq 4$ *is even, then* f *is starlike.*

Let n be a fixed positive even integer.
For $n \geq 4$ we consider a family of open and n-fold symmetric polygons, which are symmetric with respect to the real axis and such that their successive vertices u, v, w belong to Λ and $\operatorname{Arg} u = 0$, $\operatorname{Arg} v \in (0, \frac{\pi}{n})$, $\operatorname{Arg} w = \frac{\pi}{n}$. The polygons' interior angles at these vertices are of the measure $\pi(1 - \frac{4}{n})$, $\pi(1 + \frac{2}{n})$, $\pi(1 - \frac{2}{n})$, respectively. From above it follows that these polygons have $4n$ sides.

We extend this set of polygons on limiting cases. If $v = u$ ($\operatorname{Arg} v = 0$) then we obtain a regular polygon having n sides and all angles measuring

$\pi(1-\frac{2}{n})$. The arguments of the polygon's vertices are equal to $\frac{\pi}{n}(2j+1)$, $j=0,1\dots,n-1$. In case $v=w$ ($\operatorname{Arg} v=\frac{\pi}{n}$), the polygon has $2n$ sides and its angles are $\pi(1-\frac{4}{n})$, $\pi(1+\frac{2}{n})$, alternately. Moreover, the argument of the polygon's vertices are equal to $\frac{\pi}{n}j$, $j=0,1,\dots,2n-1$.

We denote this family of polygons by $\mathcal{W}$.

For $n=4$ the sets of the family $\mathcal{W}$ are unbounded. Every fourth vertex of such a polygon is extended to infinity. For this reason both sides adjacent to every such vertex are parallel. In this way we obtain a star-shaped set with four unbounded strips. The thickness of strips is growing as $\operatorname{Arg} v$ tends to $\frac{\pi}{4}$. In cases $\operatorname{Arg} v=0$ and $\operatorname{Arg} v=\frac{\pi}{4}$ these sets become four-pointed unbounded stars.

By the Schwarz–Christoffel formulae there exists exactly one analytic function which maps Δ univalently onto a fixed polygon of the family $\mathcal{W}$ and which has positive derivative in 0. This function is of the form

$$\Delta \ni z \mapsto B\int_0^z \sqrt[n]{\frac{(\zeta^n-e^{in\varphi})^2(\zeta^n-e^{-in\varphi})^2}{(\zeta^n-1)^4(\zeta^n+1)^2}}\,d\zeta, \tag{11}$$

for suitable $\varphi\in\left[0,\frac{\pi}{n}\right]$ and $B>0$.

Taking $B=1$ in (11) we get the function with classical normalization. This function is denoted by G_φ and the polygon $G_\varphi(\Delta)$ by B_φ. With this notation all polygons of the family $\mathcal{W}$ can be written as λB_φ, $\lambda>0$.

Moreover, let us denote

$$v_2(\varphi)\equiv G_\varphi(e^{i\varphi}). \tag{12}$$

This means that $v_2(\varphi)$, when φ varies in $\left[0,\frac{\pi}{n}\right]$, coincide with the vertices of polygons B_φ for which arguments are from the range $\left[0,\frac{\pi}{n}\right]$. Hence v_2 is

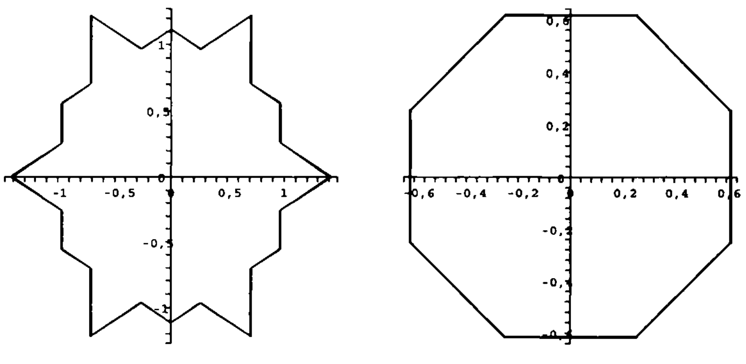

Fig. 2. Polygons: **a)** $n=6$, $\operatorname{Arg} v=\frac{\pi}{12}$ **b)** $n=8$, $\operatorname{Arg} v=0$.

given by the formula

$$(13)\qquad v_2\ :\ \left[0,\frac{\pi}{n}\right]\ni\varphi\mapsto e^{i\varphi}\int_0^1\sqrt[n]{\frac{(1-t^n)^2(1-t^ne^{2in\varphi})^2}{(1-t^ne^{in\varphi})^4(1+t^ne^{in\varphi})^2}}dt$$

and it is an injective function on $[0,\frac{\pi}{n}]$.

We consider the function G_φ in case $n=2$. We have

$$G_\varphi(z)=\int_0^z\frac{(1-\zeta^2e^{-2i\varphi})(1-\zeta^2e^{2i\varphi})}{(1-\zeta^2)^2(1+\zeta^2)}d\zeta=\alpha\frac{z}{1-z^2}+(1-\alpha)\frac{1}{2i}\log\frac{1+iz}{1-iz},$$

where $\alpha=\sin^2\varphi\in[0,1]$. This function belongs to $Y^{(2)}$. Moreover, for a fixed $\varphi\in[0,\frac{\pi}{2})$

$$(14)\quad G_\varphi(\Delta)=C\setminus\left\{\pm\frac{\pi}{4}\cos^2\varphi\pm ib,b\geq\frac{1}{4}\left(2\sin\varphi-\cos^2\varphi\ln\frac{1-\sin\varphi}{1+\sin\varphi}\right)\right\}.$$

Furthermore,

$$(15)\qquad G_\varphi(e^{i\varphi})=\frac{\pi}{4}\cos^2\varphi+\frac{i}{4}\left(2\sin\varphi-\cos^2\varphi\ln\frac{1-\sin\varphi}{1+\sin\varphi}\right).$$

In case $\varphi=\frac{\pi}{2}$ we have

$$G_{\frac{\pi}{2}}(\Delta)=C\setminus\left\{\pm ib,b\geq\frac{1}{2}\right\}\quad\text{and}\quad G_{\frac{\pi}{2}}(i)=\frac{1}{2}i.$$

THEOREM 3. *Let n be even. The Koebe domain for $Y^{(n)}$ is bounded, n-fold symmetric and symmetric with respect to the real axis. Its boundary in Λ is $v_2([0,\frac{\pi}{n}])$, where v_2 is given by* (13).

Proof. Let $f\in Y^{(n)}$, n be even.

I. $n=2$. Assume that $f\in Y^{(2)}$ and $w\in\partial f(\Delta)$, where $\operatorname{Re}w\geq 0$, $\operatorname{Im}w\geq 0$. Hence the points $-w$, $\overline{w}$, $-\overline{w}$ do not belong to $f(\Delta)$. From properties of $Y^{(2)}$ it follows that four rays $k_1=\{w+it,t\geq 0\}$, $k_2=\{-\overline{w}+it,t\geq 0\}$, $k_3=-k_1$, $k_4=-k_2$ are disjoint from $f(\Delta)$. Consequently, f is subordinated to G_φ for a suitable $\varphi\in[0,\frac{\pi}{2}]$. Therefore, $1=f'(0)\leq G'_\varphi(0)=1$ and $f=G_\varphi$.

From the above we conclude that

$$\bigcap_{f\in Y^{(2)}}f(\Delta)=\bigcap_{\varphi\in[0,\frac{\pi}{2}]}G_\varphi(\Delta).$$

The assertion follows immediately from (14) and (15).

II. For $n\geq 4$ the proof is similar to the proof of Theorem 1. ∎

COROLLARY 5. *The boundary of the Koebe domain for $Y^{(2)}$ in Λ consists of the curve given by the parametric equation*

$$\begin{cases} x(\alpha) = (1-\alpha)\frac{\pi}{4} \\ y(\alpha) = \frac{1}{2}\left[\sqrt{\alpha} - \frac{1}{2}(1-\alpha)\ln\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right], \end{cases} \qquad \alpha \in [0,1)$$

and the point $(0, \frac{1}{2})$.

The condition

$$\bigcap_{f\in Y^{(n)}} f(\Delta) = \bigcap_{\varphi\in[0,\frac{\pi}{n}]} F_\varphi(\Delta)$$

for even n results from Theorem 3. Since all functions of the class $Y^{(n)}$ are convex in the direction of the imaginary axis and starlike (by Corollary 4) we have

COROLLARY 6. *The Koebe domain for $Y^{(n)}$ and even n is convex in the direction of the imaginary axis and starlike.*

Proof. We shall prove this fact only for $n = 2$. Otherwise, this corollary is the simple consequence of the definition of $Y^{(n)}$ and Corollary 4.

Assume that the Koebe domain for $Y^{(2)}$ is not starlike. It means that there are two points w_1, w_2 belonging to the boundary of this set such that $\operatorname{Arg} w_1 = \operatorname{Arg} w_2 \in (0,\pi)$ and $|w_1| < |w_2|$. Hence, there exist two functions $G_{\varphi_1}, G_{\varphi_2} \in Y^{(2)}$ such that

$$G_{\varphi_1}(\partial\Delta)\cap\Lambda_1 = \{w_1 + it\,,\, t \geq 0\}$$

and

$$G_{\varphi_2}(\partial\Delta)\cap\Lambda_1 = \{w_2 + it\,,\, t \geq 0\}.$$

Therefore,

$$G_{\varphi_2} = \frac{|w_2|}{|w_1|}G_{\varphi_1}$$

and

$$G'_{\varphi_2}(0) = \frac{|w_2|}{|w_1|}G'_{\varphi_1}(0).$$

Combining it with the normalization of G_{φ_1}, G_{φ_2} we obtain $|w_1| = |w_2|$, a contradiction. ∎

THEOREM 4. *The function*

$$G(z) = z\int_0^1 \sqrt[n]{\frac{(1-t^n)^2(1-t^nz^{2n})^2}{(1-t^nz^n)^4(1+t^nz^n)^2}}\,dt$$

is the minorant of the class $Y^{(n)}$, for even n.

The proof of this theorem is similar to the proof of Theorem 2.

In case $n = 2$ the function G can be written in the form

$$G(z) = \frac{(1+z^2)^2}{4z^2}\arctan z - \frac{1-z^2}{4z} = \frac{2}{3}z + \frac{2}{15}z^3 + \dots$$

Corollary 7. *The minorant G of the class $Y^{(n)}$ for even n is convex in the direction of the imaginary axis ($G/G'(0) \in Y^{(n)}$) and starlike.*

Covering domain for odd n

Lemma 5. *If $f \in Y^{(n)}$, $n \geq 3$ is odd and $w \in f(\Delta)$, where* $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$, *then the segments*

$$\{\zeta = w - it,\ t \geq 0\} \cap \Lambda$$

$$\left\{\zeta = w + e^{i(\frac{\pi}{2}+\frac{\pi}{n})}t,\ t \geq 0\right\} \cap \Lambda$$

are contained in $f(\Delta)$.

Proof. Let $f \in Y^{(n)}$ and $w \in f(\Delta)$. From n-fold symmetry of f the point $w\varepsilon^k = we^{i(\pi - \frac{\pi}{n})}$, where $n = 2k+1$, belongs to $f(\Delta)$. Since the function f is convex in the direction of the imaginary axis and has real coefficients, the segments $[\overline{w}, w]$ and $[\overline{w\varepsilon}^k, w\varepsilon^k]$ are contained in $f(\Delta)$. Using n-fold symmetry of f once again the segment $[\overline{w\varepsilon}^{2k}, w] = [\overline{w}\varepsilon, w]$ is included in $f(\Delta)$, too. ∎

Let $n \geq 3$ be a fixed odd integer.

We consider a family of open and n-fold symmetric polygons, which are symmetric with respect to the real axis. For each polygon both sides diverging from the only vertex u in Λ are orthogonal to the radii of Λ in such a way that we obtain the polygon with $2n$ sides. The polygon's all interior angles are of the measure $\pi(1 - \frac{1}{n})$.

We extend the polygons' family as follows. If $\operatorname{Arg} u = 0$ or $\operatorname{Arg} u = \frac{\pi}{n}$ then we obtain a regular polygon having n sides and all angles of the measure $\pi(1 - \frac{2}{n})$. The arguments of the polygon's vertices are equal to $\frac{2j\pi}{n}$, $j = 0, 1 \dots, n-1$ or $\frac{(2j+1)\pi}{n}$, $j = 0, 1 \dots, n-1$, respectively. Hence in all cases the polygons are convex.

It means that the covering domain for $Y^{(n)}$ is the same as the covering domain for $CVR^{(n)}$, which was determined in [3].

Theorem 5. *Let $n \geq 3$ be odd. The covering domain for $Y^{(n)}$ is bounded, n-fold symmetric and symmetric with respect to the real axis. Its boundary in Λ is $u_1([0, \frac{\pi}{n}])$, where*

$$u_1 : \left[0, \frac{\pi}{n}\right] \in \varphi \to e^{i\varphi} \int_0^1 \frac{1}{\sqrt[n]{(1-t^n)(1-t^n e^{2in\varphi})}}\,dt. \tag{16}$$

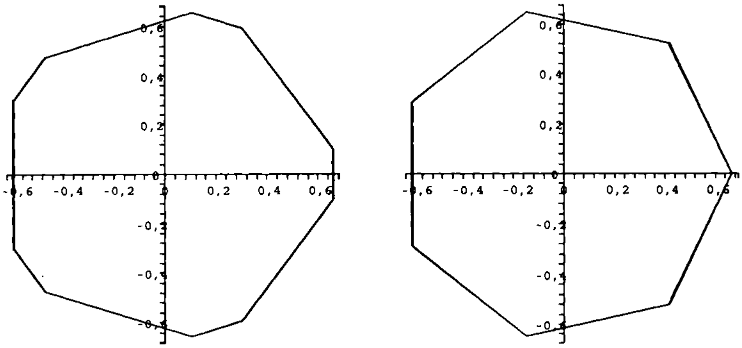

Fig. 3. Polygons: **a)** $n = 5$, $\operatorname{Arg} u = \frac{\pi}{20}$ **b)** $n = 7$, $\operatorname{Arg} u = 0$.

The covering domain possesses a similar property to the property of the Koebe domain (compare to Corollary 2).

COROLLARY 8. *The covering domain for $Y^{(n)}$ and odd $n \geq 3$ is convex in the direction of the imaginary axis and starlike.*

The next corollary results from Theorem 5

COROLLARY 9. *The function*

$$J(z) = z \int_0^1 \frac{1}{\sqrt[n]{(1-t^n)(1-t^n z^{2n})}} dt$$

is the majorant of the class $Y^{(n)}$, for odd $n \geq 3$.

Proof. Let $n \geq 3$ be a fixed odd integer. For an arbitrary fixed $\varphi \in [0, \frac{\pi}{n}]$ we have

$$J(e^{i\varphi}) = e^{i\varphi} \int_0^1 \frac{1}{\sqrt[n]{(1-t^n)(1-t^n e^{2in\varphi})}} dt.$$

Hence the values of $J(e^{i\varphi})$ coincide with $u_1(\varphi)$ for all $\varphi \in [0, \frac{\pi}{n}]$. From univalence and n-fold symmetry it follows that J is the majorant of the class $Y^{(n)}$. ∎

COROLLARY 10. *The majorant J of the class $Y^{(n)}$ for odd $n \geq 3$ is convex in the direction of the imaginary axis ($J/J'(0) \in Y^{(n)}$) and starlike.*

Applying the hypergeometric function one can write J in the form

$$J(z) = {}_2F_1(\tfrac{1}{n}, \tfrac{1}{n}, 1; z^{2n}) \cdot \frac{\pi/n}{\sin(\pi/n)} z.$$

Since for all $z \in \Delta$ there is $|J(z)| < |J(1)| = {}_2F_1(\frac{1}{n}, \frac{1}{n}, 1; 1) \cdot \frac{\pi/n}{\sin(\pi/n)} = \frac{1}{n} B(\frac{1}{n}, 1 - \frac{2}{n})$, where B means the beta Euler function, so we have

Corollary 11. *For odd $n \ge 3$ we have* $\sup\{|f(z)| \,:\, f \in Y^{(n)}, z \in \Delta\} = u_1(0) = \frac{1}{n}B(\frac{1}{n}, 1 - \frac{2}{n})$.

Covering domain for even n

Lemma 6. *If $f \in Y^{(n)}$, n is even and $w \in f(\Delta)$, where* $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$, *then the segments*

$$\{\zeta = w - it,\ t \ge 0\} \cap \Lambda$$

$$\left\{\zeta = w + e^{i(\frac{\pi}{2} + \frac{2\pi}{n})}t,\ t \ge 0\right\} \cap \Lambda$$

are contained in $f(\Delta)$.

Proof. Let $f \in Y^{(n)}$ and $w \in f(\Delta)$, where $\operatorname{Arg} w \in [0, \frac{\pi}{n}]$. By Lemma 1 the point $\overline{w}\varepsilon = \overline{w}e^{\frac{2\pi i}{n}}$ belongs to $f(\Delta)$. From convexity of $f(\Delta)$ in the direction of the imaginary axis and symmetry with respect to the real axis we obtain the segments $[\overline{w}, w]$ and $[w\overline{\varepsilon}, \overline{w}\varepsilon]$ are contained in $f(\Delta)$. Moreover, using n-fold symmetry of f once again we get $[w, \overline{w}\varepsilon^2] \subset f(\Delta)$. ∎

Let n be a fixed positive even integer.

For $n \ge 4$ we consider a family of open and n-fold symmetric polygons, which are symmetric with respect to the real axis and such that two successive vertices u, v belong to Λ and $\operatorname{Arg} u \in (0, \frac{\pi}{n})$, $\operatorname{Arg} v = \frac{\pi}{n}$. Moreover, for each polygon one of its sides diverging from the vertex u is orthogonal to the real axis. The polygon's interior angles adjacent to these vertices are of the measure $\pi(1 - \frac{2}{n})$, $\pi(1 + \frac{2}{n})$. Hence these polygons have $3n$ sides.

For $n \ge 6$ we extend the family of polygons on limiting cases. If $\operatorname{Arg} u = 0$ then we obtain a regular polygon having $2n$ sides and angles $\pi(1-\frac{4}{n})$, $\pi(1+\frac{2}{n})$ alternately, counting from the vertex on the real positive semi-axis. In case $u = v$ the polygon is regular and has n sides and angles of the measure $\pi(1 - \frac{2}{n})$. The arguments of the vertices of such a polygon are equal to $\frac{\pi}{n}(2j + 1)$, $j = 0, 1 \ldots, n - 1$. This family of polygons will be denoted by $\mathcal{U}$.

From the Schwarz–Christoffel formulae there exists exactly one analytic function, such that its derivative in 0 is positive and it maps Δ univalently onto a fixed polygon of the family $\mathcal{U}$. This function is of the form

$$\Delta \ni z \mapsto D\int_0^z \sqrt[n]{\frac{(\zeta^n + 1)^2}{(\zeta^n - e^{in\varphi})^2(\zeta^n - e^{-in\varphi})^2}}\,d\zeta, \tag{17}$$

for suitable $\varphi \in \left[0, \dfrac{\pi}{n}\right]$ and $D > 0$.

Taking $D = 1$ in (17) we get the function with classical normalization. We denote this function by H_φ and the polygon $H_\varphi(\Delta)$ by D_φ. With this notation all polygons of the family $\mathcal{U}$ can be written as λD_φ, $\lambda > 0$.

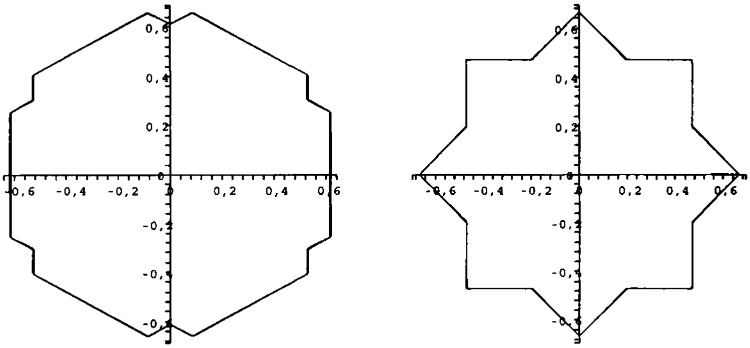

Fig. 4. Polygons: **a)** $n = 6$, $\operatorname{Arg} u = \frac{\pi}{8}$ **b)** $n = 8$, $\operatorname{Arg} u = 0$.

Moreover, let us denote

$$u_2(\varphi) \equiv H_\varphi(e^{i\varphi}). \tag{18}$$

With this notation $u_2(\varphi)$ is the vertex of polygon D_φ, whose argument is in $[0, \frac{\pi}{n}]$. Hence

$$u_2 \;:\; \left[0, \frac{\pi}{n}\right] \ni \varphi \mapsto e^{i\varphi} \int_0^1 \sqrt[n]{\frac{(1 + t^n e^{in\varphi})^2}{(1 - t^n)^2 (1 - t^n e^{2in\varphi})^2}}\, dt. \tag{19}$$

Obviously, u_2 is an injective function on $[0, \frac{\pi}{n}]$.

THEOREM 6. *Let $n \geq 6$ be even. The covering domain for the class $Y^{(n)}$ is bounded, n-fold symmetric and symmetric with respect to the real axis. Its boundary in Λ is $u_2([0, \frac{\pi}{n}])$, where u_2 is given by* (19).

Proof. Let $f \in Y^{(n)}$, $n \geq 6$ be even.
Assume that $\lambda u_2(\varphi) \in \partial f(\Delta)$, $\lambda > 0$, $\varphi \in [0, \frac{\pi}{n}]$. By Lemma 6

$$f(\Delta) \supset \lambda D_\varphi = \lambda H_\varphi(\Delta).$$

Combining it with univalence of H_φ it follows that

$$\lambda H_\varphi \prec f.$$

Hence $\lambda = \lambda H'_\varphi(0) \leq f'(0) = 1$. Therefore, if $\lambda > 1$ then $\lambda u_2(\varphi) \notin f(\Delta)$ and if $0 < \lambda < 1$ then $[0, \lambda u_2(\varphi)] \subset f(\Delta)$. Moreover, $u_2(\varphi) \in \partial H_\varphi(\Delta)$, which means that $u_2(\varphi) \in \partial L_{Y^{(n)}}$. Hence $u_2([0, \frac{\pi}{n}])$ is the boundary of the covering domain for $Y^{(n)}$ in Λ. From Lemma 2 we get the conclusion of this theorem. ■

THEOREM 7. *The covering domain for $Y^{(4)}$ is unbounded, 4-symmetric and symmetric with respect to the real axis. Its boundary in Λ is $u_2((0, \frac{\pi}{4}])$, where u_2 is given by* (19).

The proof of Theorem 7 is similar to the proof of Theorem 6.

Theorem 8. *The covering domain for $Y^{(2)}$ is the whole plane C.*

Proof. Let $n = 2$. For the function H_φ, which was defined above, we have

$$H_0(z) = \frac{z}{1-z^2} \quad \text{and} \quad H_{\frac{\pi}{2}}(z) = \tfrac{1}{2i}\log\frac{1+iz}{1-iz}.$$

Hence

$$C = H_0(\Delta) \cup H_{\frac{\pi}{2}}(\Delta) \subset \bigcup_{f\in Y^{(2)}} f(\Delta). \ ■$$

For even $n \geq 4$ by Theorem 6 and Theorem 7

$$\bigcup_{f\in Y^{(n)}} f(\Delta) = \bigcup_{\varphi\in[0,\frac{\pi}{n}]} H_\varphi(\Delta).$$

All functions belonging to the class $Y^{(n)}$ are convex in the direction of the imaginary axis and starlike (by Corollary 4) so we have

Corollary 12. *The covering domain for $Y^{(n)}$ and even $n \geq 4$ is convex in the direction of the imaginary axis and starlike.*

The proof of the following corollary is similar to the proof of Corollary 9.

Corollary 13. *The function*

$$H(z) = z\int_0^1 \sqrt[n]{\frac{(1+t^n z^n)^2}{(1-t^n)^2(1-t^n z^{2n})^2}}\,dt$$

is the majorant of the class $Y^{(n)}$ for even $n \geq 4$.

Corollary 14. *The majorant H of the class $Y^{(n)}$ for even $n \geq 4$ is convex in the direction of the imaginary axis ($H/H'(0) \in Y^{(n)}$) and starlike.*

Since $|H(z)| < |H(1)| = \int_0^1 \sqrt[n]{\frac{(1+t^n)^2}{(1-t^n)^4}}\,dt$, we have

Corollary 15. *For even n we have*

$$\sup\{|f(z)| \ : \ f \in Y^{(n)}, z \in \Delta\} = \begin{cases} u_2(0) = \frac{B\left(\frac{1}{n},\frac{1}{2}-\frac{2}{n}\right)}{n\sqrt[n]{4}} & \textit{for } n \geq 6 \\ \infty & \textit{for } n = 2 \textit{ or } n = 4. \end{cases}$$

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DEPARTMENT OF APPLIED MATHEMATICS
LUBLIN UNIVERSITY OF TECHNOLOGY
Nadbystrzycka 38D
20-618 LUBLIN, POLAND
E-mail: l.koczan@pollub.pl
m.sobczak-knec@pollub.pl
p.zaprawa@pollub.pl

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