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THE RANGE OF NON-ATOMIC MEASURES ON EFFECT ALGEBRAS

Abstract. The present paper deals with the study of superior variation m^+, inferior variation m^- and total variation $|m|$ of an extended real-valued function m defined on an effect algebra L; having obtained a Jordan type decomposition theorem for a locally bounded real-valued measure m defined on L, we have observed that the range of a non-atomic function m defined on a D-lattice L is an interval $(-m^-(1),\ m^+(1))$. Finally, after introducing the notion of a relatively non-atomic measure on an effect algebra L, we have proved an analogue of Lyapunov convexity theorem for this measure.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $S(\mathcal{H})$ be a partially ordered group of all bounded self-adjoint operators on $\mathcal{H}$. Put $E(\mathcal{H}) = \{A \in S(\mathcal{H}) : 0 \leq A \leq I\}$. If a quantum mechanical system $\mathcal{F}$ is represented in the usual way by a Hilbert space $\mathcal{H}$, then the elements of $E(\mathcal{H})$ correspond to effects for $\mathcal{F}$ [29, 30]. Effects are of significance in representing unsharp measurements or observations on the system $\mathcal{F}$ [10], and effect valued measures play an important role in stochastic quantum mechanics [1, 36]. As a consequence, there have been a number of recent efforts to establish appropriate axioms for logics, algebras, or posets based on effects [13, 19]. In 1992, Kôpka defined D-posets of fuzzy sets in [18], which is closed under the formations of differences of fuzzy sets, while studying the axiomatical systems of fuzzy sets. A generalization of such structures to an abstract partially ordered set, where the basic operation is the difference, yields a very general and, at the same time, a very simple structure called a D-poset. A common generalization of orthomodular lattices and MV-algebras is termed as effect algebras introduced by Bennett and

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Foulis [5] in 1994, while working on quantum mechanical systems. Such structures are being frequently used because of their wide range of applications in quantum physics [6], mathematical economics [16], fuzzy theory [31] and functional analysis [37]. The equivalence of D-posets and effect algebras is proved by Foulis and Bennett [5] and independently by Pulmanová [32].

The decomposability of a vector measure was first studied by Rickart in 1943 [33], where he established a Lebesgue decomposition theorem for the class of "strongly bounded" additive measures. Several Jordan type decomposition theorems are exhibited by Diestel and Faires in [12]. Afterwards, Faires and Morrison [15] exposed conditions on a vector valued measures that ensure vector valued Jordan type decomposition theorem to hold. A Jordan type decomposition theorem for vector measures, defined on an algebra of sets, with values in an order complete Banach lattice is proved by Schmidt [34]. Up to slight modification, the result of [35] extends to the case where domain of the vector measure is a ring of sets. It is also possible to give a common approach to vector measures on a Boolean ring and linear operators on a vector lattice. A first step in this direction was done in [34], where real-valued case was studied. The method presented there is based on a common abstraction of Boolean rings and lattice ordered groups. This approach can be refined and fitted to the vector valued case, and it then yields results of [12, 15, 35] on Jordan decomposition without appeal to regularity of representing linear operators. Recently, a Jordan type decomposition for a weakly tight real-valued function defined on a sublattice of I^X has been studied in [22].

Suzuki in [38], for the first time introduced and investigated the concepts of atoms of fuzzy measures. Pap in [31], further introduced and studied atoms of null-additive set functions and proved a Saks type decomposition theorem. Also an illustrative study using the concepts of the atoms is done in [20, 21, 23-26]. An important problem in measure theory is to describe the properties of the range of the measures [9, 17]. One of the most famous theorem of measure theory is Lyapunov's convexity theorem, which states that the range of a non-atomic σ-additive measure on a σ-algebra with values in a finite dimensional vector space is convex. Blackwell [7] proved extensions of results of Lyapunov [27]. A simplified proof of Lyapunov's result has been given by Halmos [17]; it was shown that if each component of a completely additive set function is non-atomic, the range of this function is a convex set. Several authors [8, 11, 14, 28], devoted their study to range of certain measures. A generalization of Lyapunov's convexity thoerem for measures on effect algebras and for measures defined on a weaker algebraic structure than effect algebras has been proved by Barbieri [4].

The paper is organized as follows: Section 2 contains prerequisites and basic results on an effect algebra L. The notions of superior variation m^+, inferior variation m^- and total variation $|m|$ of an extended real-valued function m defined on L are studied elaborately in Section 3. In this section, we have given a Jordan type decomposition theorem for a locally bounded real-valued measure m defined on L, followed by various properties in the context of functions m^+, m^- and $|m|$ (cf. [3]). In Section 4, using the notion of an atom of a real-valued measure m [24, 25], we have showed that the range of a locally bounded real-valued σ-additive, non-atomic function m on a D-lattice L is an interval $(-m^-(1),\ m^+(1))$; characterizations of non-atomicity of m are established and used in obtaining this result (cf. [4]). In Section 5, we have proved an analogue of the Lyapunov convexity theorem for a relatively non-atomic measure defined on a σ-complete effect algebra L.

2. Preliminaries and basic results

First of all, we shall give some preliminaries and basic results from effect algebras, which can be found in [13] and the references therein.

An *effect algebra* $(L; \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for $a, b, c \in L$:

(1) $a \oplus b = b \oplus a$, if $a \oplus b$ is defined;
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side is defined;
(3) for every $a \in L$, there exists a unique $b \in L$ such that $a \oplus b = 1$ (we put $a^\perp = b$);
(4) if $a \oplus 1$ is defined, then $a = 0$.

For brevity, we denote an effect algebra $(L; \oplus, 0, 1)$ by L. In an effect algebra L, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a. We say that two elements $a, b \in L$ are *orthogonal*, and we write $a \perp b$, if $a \oplus b$ exists. If $a \oplus b = 1$, then b is called *orthocomplement* of a and write $b = a^\perp$. It is obvious that $1^\perp = 0$, $(a^\perp)^\perp = a$, $a \perp 0$ and $a \oplus 0 = a$, for all $a \in L$. Also, for $a, b \in L$, we define $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. It may be proved that $\leq$ is a partial ordering on L and $0 \leq a \leq 1$; $a \leq b \Leftrightarrow b^\perp \leq a^\perp$; and $a \leq b^\perp \Leftrightarrow a \perp b$ for $a, b \in L$. If $a \leq b$, then the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b^\perp)^\perp$ (we put $c = b \ominus a$).

In a natural way, the sum of more than two elements is obtained: If $a_1, a_2, \ldots, a_n \in L$, we inductively define $a_1 \oplus a_2 \oplus \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, a_2, \ldots, a_n\}$

of L is *orthogonal* if $a_1 \oplus a_2 \oplus \ldots \oplus a_n$ exists. For a sequence $\{a_n\}$, we say that it is *orthogonal* if, for every n, $\bigoplus_{i \leqslant n} a_i$ exists. If moreover, $\sup_n \bigoplus_{i \leqslant n} a_i$ exists, then the *sum* $\bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in L is defined as $\sup \bigoplus_{i \leqslant n} a_i$; we denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}^n$ the n-dimensional Euclidian space. An effect algebra L is called a *σ-complete effect algebra*, if every orthogonal sequence in L has its sum. If $(L, \leq)$ is a lattice, then we say that effect algebra is a *lattice ordered effect algebra* (or a *D-lattice*). The notion of σ-continuity of a D-lattice is, as usual, expressed in terms of monotone sequences: we write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in L and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is a decreasing sequence and $a = \inf_n a_n$). The lattice $(L, \leq)$ is said to be *σ-continuous* if $a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a$ implies $a_n \vee b \downarrow a \vee b$) for all $b \in L$. A function m defined on a D-lattice L with values in $\mathbb{R}^n$ is called *modular*, if $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$ for $a, b \in L$. A function m defined on an effect algebra L with values in $\mathbb{R}$, is called *locally bounded* if, for any $a \in L$, $\sup\{m(b) : b \leqslant a, b \in L\}$ exists. A function m defined on an effect algebra L with values in $\mathbb{R}^n$ is called a *measure* on L, if $a, b \in L$, $a \perp b$ implies $m(a \oplus b) = m(a) + m(b)$. It is clear that m is a measure if and only if $b \leq a$ implies $m(a) = m(b) + m(a \ominus b)$. We say that m is *σ-additive*, if for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $m(\bigoplus_n a_n) = \sum_{n=1}^{\infty} m(a_n)$. The function m is called *continuous from below* (respectively, *continuous from above*), if $a_n \in L$, $a_n \leqslant a_{n+1}$, $n \in \mathbb{N} \Rightarrow m\left(\bigvee_{n=1}^{\infty} a_n\right) = \lim_{n \to \infty} m(a_n)$, provided $\bigvee_{n=1}^{\infty} a_n$ exists (respectively, if $a_n \in L$, $a_n \geqslant a_{n+1}$, $n \in \mathbb{N}$ and $m(a_1) < \infty \Rightarrow m\left(\bigwedge_{n=1}^{\infty} a_n\right) = \lim_{n \to \infty} m(a_n)$, provided $\bigwedge_{n=1}^{\infty} a_n$ exists) (cf. [25]).

Let us recall the following results, which we shall use in subsequent sections:

2.1. Assume that a, b, c are elements of an effect algebra L.

(i) If $a \leqslant b$, then $b = a \oplus (b \ominus a)$.
(ii) If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.
(iii) If $a \leqslant b \leqslant c$, then $(b \ominus a) \leq (c \ominus a)$.
(iv) If $a \leqslant b$ and $c \leq b \ominus a$, then $a \perp c$ and $a \oplus c \leq b$.

2.2. [2] Let L be a σ-complete effect algebra. If $\{a_n\}$ is an increasing (respectively, decreasing) sequence, then $\sup_n a_n$ (respectively, $\inf_n a_n$) exists.

2.3. [2] Let $a_0, a_1, \ldots, a_n$ be in L with $a_0 \leq a_1 \leq \ldots \leq a_n$ and let $b_i = a_i \ominus a_{i-1}$, for every $i \in \{1, 2, \ldots, n\}$. Then $\{b_1, b_2, \ldots, b_n\}$ is orthogonal and $b_1 \oplus b_2 \oplus \ldots \oplus b_n = a_n \ominus a_0$.

2.4. [2] Let $m : L \to \mathbb{R}^n$ be a measure. Then the following assertions are equivalent:

(i) m is σ-additive.
(ii) m is continuous from below.
(iii) m is continuous from above.
(iv) $a_n \downarrow 0$ implies $\lim_{n\to\infty} m(a_n) = 0$.

3. A Jordan type decomposition theorem

DEFINITION 3.1. (cf. [24]) Let m be an extended real-valued function defined on an effect algebra L, that is, $m : L \to [-\infty, \infty]$, with $m(0) = 0$. Then for $a \in L$,

(i) *superior variation of* m is defined by

$$m^+(a) = \sup\{m(b) : b \leqslant a, b \in L\};$$

(ii) *inferior variation of* m is defined by

$$\begin{aligned} m^-(a) &= -\inf\{m(b) : b \leqslant a, b \in L\} \\ &= \sup\{-m(b) : b \leqslant a, b \in L\}; \end{aligned}$$

(iii) *total variation of* m is defined by

$$|m| = m^+ + m^-.$$

REMARK 3.1. (i) $0 \leq m^+(a) \leq \infty$, $0 \leq m^-(a) \leq \infty$, $0 \leq |m|(a) \leq \infty$, $a \in L$;

(ii) $m^+(0) = 0 = m^-(0)$, $|m|(0) = 0$;
(iii) $m^- = (-m)^+$, $m^+ = (-m)^-$;
(iv) $-m^-(a) \leqslant m(a) \leqslant m^+(a)$, $|m(a)| \leq |m|(a)$, $a \in L$.
(v) $m^+(a) \leq m^+(b)$, $m^-(a) \leq m^-(b)$, $|m|(a) \leq |m|(b)$, whenever $a, b \in L$, $a \leq b$.

THEOREM 3.1 (Jordan type decomposition theorem). *If m is a locally bounded real-valued measure defined on an effect algebra L, then m can be written as*

$$m = m^+ - m^-.$$

If m is a real-valued modular measure defined on a lattice ordered effect algebra L, then the decomposed parts m^+ and m^- are measures on L (and hence $|m|$ is also a measure on L). Furthermore, if m is a locally bounded real-valued σ-additive function defined on a σ-continuous D-lattice L, then the decomposed parts m^+, m^-, and $|m|$ are also σ-additive.

Proof. Let $\varepsilon > 0$, and let $a \in L$. Then there exists $b \in L$ such that $b \leqslant a$ and $m^+(a) - \varepsilon < m(b)$. Since $a \ominus b \leq a$, we have $-m(a \ominus b) \leq m^-(a)$. Thus, we get

$$m^+(a) - \varepsilon - m^-(a) < m(b) + m(a \ominus b),$$

which yields that

$$m^+(a) - m^-(a) - \varepsilon < m(a).$$

Letting $\varepsilon \to 0$, we obtain

$$m^+(a) - m^-(a) \leq m(a). \tag{1}$$

Further, since (1) is true for any m, with the aid of Remark 3.1(iii), we have

$$m^+(a) - m^-(a) \geq m(a).$$

Thus, $m(a) = m^+(a) - m^-(a)$, that is, $m = m^+ - m^-$. Next, we will prove that m^+ is a measure. We have proved in [24], that for $a, b \in L$ with $a \perp b$,

$$m^+(a \oplus b) \leq m^+(a) + m^+(b). \tag{2}$$

(For completeness, we give the proof of (2): let $a, b \in L$ such that $a \perp b$, and let $c \in L$ such that $c \leq a \oplus b$. Set $d = c \wedge a$ and $e = (c \vee a) \ominus a$. The relation $e \leq a^\perp \leq a^\perp \vee c^\perp = (a \wedge c)^\perp$, yields that $d \perp e$. Using modularity of m, we have $m(d \oplus e) = m(d) + m(e) = m(c \wedge a) + m((c \vee a) \ominus a) = m(c)$. Since $d \leq a$ and $e \leq ((a \oplus b) \vee a) \ominus a = (a \oplus b) \ominus a = b$, then from $m(c) = m(d \oplus e)$, the assertion follows.)

By Definition 3.1(i), (ii), there are sequences $\{a_n\}$ and $\{b_n\}$ of elements from L such that $a_n \leq a$, $b_n \leq b$ with $\lim_{n\to\infty} m(a_n) = m^+(a)$, $\lim_{n\to\infty} m(b_n) = m^+(b)$. Obviously, $a_n \perp b_n$ for each n. Therefore, from $m(a_n \oplus b_n) = m(a_n) + m(b_n)$, we have $\lim_{n\to\infty} m(a_n \oplus b_n) = m^+(a) + m^+(b)$. Further, $a_n \oplus b_n \leq a \oplus b$ yields that

$$m^+(a \oplus b) \geq m^+(a) + m^+(b). \tag{3}$$

From (2) and (3), we get $m^+(a \oplus b) = m^+(a) + m^+(b)$, that is, m^+ is a measure. By similar argument, we can show that m^- is a measure, and so also $|m|$.

Now, we will prove σ-additivity of m^+. Let $a_n \uparrow a$, $a, a_n \in L$. Then $m^+(a_n) \leq m^+(a)$, for every n. Thus the increasing sequence $\{m^+(a_n)\}$ converges to a limit l, say, where $l \leq m^+(a)$. For any element $b \in L$, $b \leq a$, we have $m(b \wedge a_n) \leq m^+(b \wedge a_n) \leq m^+(a_n)$. Further, since L is σ-continuous, we get $b \wedge a_n \uparrow b$, and therefore σ-additivity of m yields that $\lim_{n\to\infty} m(b \wedge a_n) = m(b)$. Hence, $m(b) \leq l$. As $b \in L$ is arbitrary, we get $m^+(a) \leq l$. It follows that $m^+(a) = l$, that is, $\lim_{n\to\infty} m^+(a_n) = m^+(a)$. Further, since m^+ is a measure, in view of 2.4, m^+ is σ-additive. The σ-additivity of m^- and $|m|$ are obvious. ∎

From now onwards, we shall study various properties in the context of functions m^+, m^- and $|m|$, which are consequences of their respective definitions. For this, let m be a real-valued function defined on an effect algebra L.

DEFINITION 3.2. m is called *monotone increasing* (*monotone decreasing*) in L, if for every $a, b \in L$ with $b \leq a$, we have $m(b) \leq m(a)$ ($m(b) \geq m(a)$).

REMARK 3.2. (i) If m is a measure on L, then m is monotone increasing (monotone decreasing) in L if and only if $m(a) \geq 0$ ($m(a) \leq 0$) for every $a \in L$.

(ii) m^+, m^- and $|m|$ are monotone increasing.

(iii) The representation $m = m^+ - m^-$ is distinguished by the following property: if m is a locally bounded measure on L and $m = m_1 - m_2$, where m_1 and m_2 are measures and monotone increasing, then for every $a \in L$, we have:

$$m^+(a) \leq m_1(a), \quad m^-(a) \leq m_2(a).$$

PROPOSITION 3.1. (i) *If m is a measure and monotone increasing (monotone decreasing) in L, then $m^+ = |m| = m$, $m^- = 0$ ($m^- = |m| = -m$, $m^+ = 0$).*

(ii) *Let m be a measure. If $m^- = 0$ ($m^+ = 0$), then m is monotone increasing (monotone decreasing).*

(iii) *If $m = m_1 + m_2$, then $m^+ \leq m_1^+ + m_2^+$, $m^- \leq m_1^- + m_2^-$ and $|m| \leq |m_1| + |m_2|$.*

(iv) *If $m = m_1 - m_2$, then $m^+ \leq m_1^+ + m_2^-$, $m^- \leq m_1^- + m_2^+$ and $|m| \leq |m_1| + |m_2|$.*

(v) *If $m_1 \leq m_2$, then $m_1^+ \leq m_2^+$ and $m_1^- \geq m_2^-$.*

DEFINITION 3.3. Let m be a real-valued function defined on an effect algebra L with $m(0) = 0$. An element $a \in L$ is called a *null element* for m (symbolically written as: $a =_m 0$), if for all $b \leq a$, $b \in L$, we have $m(b) = 0$.

Note that $0 =_m 0$.

PROPOSITION 3.2. (i) *$a =_m 0$ if and only if $|m|(a) = 0$, $a \in L$.*

(ii) *$a =_m 0$ if and only if $m^+(a) = m^-(a) = 0$, $a \in L$.*

(iii) *$a =_m 0$ if and only if $a =_{|m|} 0$, $a \in L$.*

(iv) *$a =_m 0$ if and only if $a =_{m^+} 0$ and $a =_{m^-} 0$, $a \in L$.*

(v) *If m is monotone increasing and monotone decreasing both, then $a =_m 0$ if and only if $m(a) = 0$, $a \in L$.*

(vi) *If $m = m_1 + m_2$ (or $m = m_1 - m_2$), then $a =_{m_1} 0$ and $a =_{m_2} 0$ imply that $a =_m 0$, $a \in L$.*

(vii) *If $a =_m 0$ and $b \leq a$, $a, b \in L$, then $b =_m 0$.*

4. Non-atomic measures

Let m be a real-valued function defined on an effect algebra L. Firstly, we shall recall the notion of an atom of a measure m defined on an effect algebra L, which has been studied in [24, 25].

Definition 4.1. An element $a \in L$ with $m(a) \neq 0$ is called an *atom of* m (or an m-*atom*), if

(i) $m(b) = 0$ for all $b \leq a$, $b \in L$ (that is, $a =_m 0$) or
(ii) $m(a) = m(b)$ for all $b \leq a$, $b \in L$.

In case there are no atoms of m in L, m is called *non-atomic* on L.

Theorem 4.1. *Let m be a locally bounded real-valued measure defined on an effect algebra L. Then the following conditions are equivalent:*

(i) *m^+ and m^- are non-atomic.*
(ii) *$|m|$ is non-atomic.*
(iii) *m is non-atomic.*

Proof. (i) $\Rightarrow$ (ii): Let $a \in L$ be a $|m|$-atom. Let $b \leq a$, $b \in L$ with $m^+(b) \neq 0$. Obviously, $|m|(b) \neq 0$ and hence $|m|(a) = |m|(b)$, which yields that $a \in L$ is an m^+-atom.

(ii) $\Rightarrow$ (iii): See proof of Theorem 5.5 of [24].

(iii) $\Rightarrow$ (i): Let $a \in L$ be an m^+-atom, and let $b \leq a$, $b \in L$ with $m(b) \neq 0$. Obviously, $m^+(b) \neq 0$ and hence $m^+(a) = m^+(b)$, which yields that

$$m(a) \leq m(b). \tag{4}$$

From (4) and Theorem 3.1, we have

$$m^+(a) - m^-(a) \leq m(b). \tag{5}$$

Replacing m by $-m$ in (5), we get

$$m(a) \geq m(b). \tag{6}$$

From (4) and (6), $a \in L$ is an m-atom. ∎

Theorem 4.2. *Let m be a $[0, \infty)$-valued σ-additive function defined on a σ-complete effect algebra L. Then m is non-atomic on L if and only if for a given element $a \in L$ with $m(a) > 0$ and $\varepsilon > 0$, there exists $b \in L$, $b \leq a$, such that $0 < m(b) < \varepsilon$.*

Proof. The *if* part: Obvious.

The *only if* part: Suppose the contrary and choose an element $a \in L$ with $m(a) > 0$ and $t_0 > 0$, for which $m(b) \geqslant t_0$ holds if $b \leqslant a$, $b \in L$ and $m(b) > 0$. Define

$$t_1 = \inf\{m(b) : b \in L, b \leqslant a, m(b) > 0\}.$$

Then obviously $0 < t_0 \leqslant t_1$. Take $a_1 \leqslant a$, $a_1 \in L$ with $t_1 \leqslant m(a_1) < t_1 + 1$ and setting

$$t_2 = \inf\{m(b) : b \in L, b \leqslant a_1, m(b) > 0\}.$$

Choose $a_2 \leqslant a_1$ with $t_2 \leqslant m(a_2) < t_2 + \frac{1}{2}$. Continuing the process in the same manner, we obtain sequences $\{t_n\}$ and $\{a_n\}$ such that $t_0 \leqslant t_1 \leqslant t_2 \leqslant$

$\ldots \leqslant m(a)$ and $a \geqslant a_1 \geqslant a_2 \geqslant \ldots$ with

$$t_n \leqslant m(a_n) < t_n + \frac{1}{2^n},$$

for all n. Using 2.2, put $a_0 = \bigwedge_{n=1}^{\infty} a_n$. Clearly, in view of 2.4, we have $m(a_0) = \lim_{n\to\infty} m(a_n) = \lim_{n\to\infty} t_n > 0$. Let $b \leqslant a_0$ with $m(b) > 0$. Then $\mu(a_0) \geqslant \mu(b) \geqslant t_n$, for any n and hence $\mu(b) = \mu(a_0)$. This gives that $a_0 \in L$ is an atom of m, a contradiction. ∎

THEOREM 4.3. *Let m be a locally bounded real-valued σ-additive function defined on a σ-continuous, σ-complete D-lattice L. If m is non-atomic on L, then m takes every value between $-m^-(1)$ and $m^+(1)$.*

Proof. Firstly, we will prove that if m is a $[0,\infty)$-valued σ-additive, non-atomic function defined on a σ-complete effect algebra L, then m takes every value between 0 and $m(1)$. For this, let $0 < t < m(1)$. According to Theorem 4.2, there are elements $c \in L$ such that $0 < m(c) < t$. Let

$$s_1 = \sup\{m(c) : c \in L, m(c) \leqslant t\}.$$

(Obviously $0 < s_1 \leqslant t$). Then there exists an element $c_1 \in L$ such that $\frac{s_1}{2} < m(c_1) \leqslant s_1$. Let

$$s_2 = \sup\{m(c) : c \in L, c_1 \leqslant c, m(c) \leqslant t\}.$$

Then there exists an element $c_2 \in L$ such that $c_2 \geqslant c_1$ and $s_2 - \frac{s_1}{2^2} < m(c_2) \leqslant s_2$. Continue this construction inductively to obtain

$$s_n = \sup\{m(c) : c \in L, c_{n-1} \leqslant c, m(c) \leqslant t\},$$

and then there exists $c_n \geqslant c_{n-1}$, $c_n \in L$ such that

$$s_n - \frac{s_1}{2^n} < m(c_n) \leqslant s_n.$$

It is clear that $\{s_n\}$ is a decreasing sequence and $\{c_n\}$ is an increasing sequence of elements in L such that $d = \bigvee_{n=1}^{\infty} c_n \in L$ (using 2.2) and therefore, in view of 2.4, we get $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \mu(c_n) = \mu(d)$. Therefore $\mu(d) = \lim_{n\to\infty} s_n = s$ (let). Clearly $s \leqslant t$. Now we claim that $s = t$. For, otherwise, let us suppose that $s < t$. Since $0 < t < \mu(1)$, we get $\mu(1 \ominus d) > 0$, $d \in L$ and therefore, by Theorem 4.2, we obtain an element b of L such that $b \leqslant (1 \ominus d)$ and $s < \mu(d \oplus b) < t$. But then $d \oplus b \geqslant c_{n-1}$, for all $n > 1$, which yields that $\mu(d \oplus b) \leqslant s_n$, for all n. This will further imply that $\mu(d \oplus b) \leqslant s$, a contradiction. Thus $\mu(d) = t$ as required. Finally, the conclusion follows from Theorem 3.1 and Theorem 4.1. ∎

5. Relatively non-atomic measures

Let $\mathcal{F}$ be the class of Borel subsets of the real line $\mathbb{R}$ and $m = (m_1, m_2)$, where m_1 is the Lebesgue measure and $m_2(A)$ counts the number of integers in the set $A \in \mathcal{F}$. Then the range $\mathcal{R}$ of m is not convex but one may observ that if a point (a, b) in $\mathcal{R}$ lies on the line segment connecting the zero vector and $m(A)$, then there exists $A' \subseteq A$, $A' \in \mathcal{F}$, such that $m(A') = (a, b)$. In this section, we have introduced the notion of relative non-atomicity in an effect algebra which covers the situation of this kind and proved analogue of Lyapunov convexity theorem. For this let us assume that m is a σ-additive function defined on an effect algebra L, whose range $\mathcal{R}$ is a subset on n-dimensional Euclidian space $\mathbb{R}^n$ and each component of m is non-negative.

Definition 5.1. The function m is said to be *non-atomic relative to* $\mathcal{R}$ if for every element $a \in L$ and every number α with $0 < \alpha < 1$ such that $\alpha m(a) \in \mathcal{R}$, there exists $a' \leq a$, $a' \in L$ and a number α' with $0 < \alpha' < 1$ such that $m(a') = \alpha' m(a)$.

Definition 5.2. Let $a \in L$. Let us define $\mathcal{R}(a) = \{r \in \mathcal{R} : r = \alpha m(a)$ with $0 < \alpha \leq 1\}$. Let $\alpha_0 = \inf\{\alpha : \alpha > 0, \alpha m(a) \in \mathcal{R}\}$. Then we define $r_0(a) = \alpha_0 m(a)$.

Lemma 5.1. *In* Definition 5.1, *we may choose $a' \in L$ and α' such that $\alpha' \leq \alpha$.*

Proof. Let us suppose that the conclusion of the lemma is false. Let $a' \leq a$, $a', a \in L$, such that $m(a') = \alpha' m(a)$ with $0 < \alpha' < 1$. Since m is a measure, $m(a \ominus a') = (1 - \alpha') m(a)$ and $\alpha' > \alpha$, $1 - \alpha' > \alpha$. Hence $\alpha < \frac{1}{2}$. Now, $r = \alpha m(a) = \left(\frac{\alpha}{\alpha'}\right) \alpha' m(a) = \left(\frac{\alpha}{\alpha'}\right) m(a')$. Similarly, $r = \left(\frac{\alpha}{1-\alpha'}\right) m(a \ominus a')$. Hence using the definition and the above procedure separately to the elements a' and $a \ominus a'$, we obtain that $\alpha < \frac{1}{4}$. The application of the same procedure indefinitely yields that $\alpha = 0$, which is a contradiction. ∎

Lemma 5.2. *Let m be non-atomic relative to $\mathcal{R}$ defined on a σ-complete effect algebra L. Then for every $a \in L$, there exists $b \leq a$, $b \in L$ such that $m(b) = r_0(a) = \alpha m(a)$ for some α, with $0 \leq \alpha \leq 1$.*

Proof. Let $a \in L$. If $r_0(a)$ is a null vector, then we can choose $b = 0 \in L$ and $\alpha = 0$, and if $r_0(a) = \alpha m(a)$, then we can choose $b = a$ and $\alpha = 1$. If $r_0(a) \in \mathcal{R}(a)$, then the conclusion is obvious from Lemma 5.1. Otherwise, we can obtain a strictly decreasing sequence of numbers $\{\alpha_n\}$ with $0 < \alpha_n < 1$ such that $r_0(a) = \lim_{n\to\infty} \alpha_n m(a)$, and such that $\alpha_n m(a) \in \mathcal{R}(a)$. But again the usage of Lemma 5.1, yields a sequence of numbers $\{\alpha'_n\}$ and a sequence of elements $\{b_n\}$ from L, which may be chosen as $\cdots b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1 \leq a$ and such that $\lim_{n\to\infty} m(b_n) = \lim_{n\to\infty} \alpha'_n m(a) = r_0(a)$. Using 2.2, put $b = \bigwedge_{n=1}^{\infty} b_n$. In view of 2.4, clearly $b \in L$ satisfies the conclusion of the lemma. ∎

LEMMA 5.3. *Let m be non-atomic relative to $\mathcal{R}$ defined on a σ-complete effect algebra L. Let $a \in L$ and suppose $b \leq a$, $b \in L$ satisfies the conclusion of* Lemma 5.2, *that is, $m(b) = r_0(a)$. Then either $m(b)$ is a null vector or $m(a)$ is an integral multiple of $m(b)$.*

Proof. Let $m(b)$ be not a null vector, that is, $m(b) = \alpha m(a)$ for some α with $0 < \alpha \leq 1$. If $\alpha = 1$, then there is nothing to do. Now, let us suppose that $0 < \alpha < 1$. To satisfy the conclusion of Lemma 5.2, there exists $b' \leq a \ominus b$, $b' \in L$ such that $m(b') = r_0(a \ominus b) = \alpha' m(a \ominus b) = \alpha'(1-\alpha)m(a)$ for some α' with $0 \leq \alpha' \leq 1$. From the definition of r_0, it follows that $\alpha'(1-\alpha) \geq \alpha$. If $\alpha < \alpha'(1-\alpha)$, then $m(b) = \left[\frac{\alpha}{\alpha'(1-\alpha)}\right] m(b')$. Now, since m is non-atomic relative to $\mathcal{R}$, there exists $b'' \leq b'$, $b'' \in L$ and α'' with $0 < \alpha'' < 1$ such that $m(b'') = \alpha'' m(b')$. But this is a contradiction to the fact that $b' \in L$ is minimal for $a \ominus b$, that is, $m(b') = r_0(a \ominus b)$. Hence $m(b') = m(b)$. Further, if $m(a \ominus b \ominus b')$ is a null vector, then there is nothing to do. Otherwise, we repeat this process. Clearly this process must stop in a finite number of steps, and the lemma is proved. ∎

LEMMA 5.4. *Let m be non-atomic relative to $\mathcal{R}$ defined on a σ-complete effect algebra L. Let $a \in L$ and suppose $r_0(a)$ is not a null vector. Then $a \in L$ can be written as the sum of finitely many orthogonal elements, each having measure $r_0(a)$.*

Proof. Follows by the usage of the same technique as in the proof of the Lemma 5.3. ∎

LEMMA 5.5. *Let m be non-atomic relative to $\mathcal{R}$ defined on a σ-complete effect algebra L. Let $a \in L$ and suppose $r_0(a)$ is not a null vector. Then every $r \in \mathcal{R}(a)$ is a positive integral multiple of $r_0(a)$.*

Proof. Let $r \in \mathcal{R}(a)$, that is, $r = \alpha m(a)$ with $0 < \alpha \leq 1$. From Lemma 5.3, there exists $b \leq a$, $b \in L$ such that $m(a) = nm(b) = nr_0(a)$ for some positive integer n. If $\alpha = 1$, then there is nothing to prove. Now let us suppose that $0 < \alpha < 1$ and that $\alpha = \left\{\frac{k+l}{n}\right\}$, where k is an integer with $1 \leq k < n$ and l is a number with $0 < l < 1$. The case $k = 0$ is impossible for in that case $\alpha < \frac{1}{n}$. Now $r \in \mathcal{R}$ implies that there exists $c \in L$ such that $r = m(c)$. Now consider $r_0(c) = \alpha' m(c) = \alpha' \alpha m(a)$ for some α with $0 \leq \alpha' \leq 1$. If $\alpha' = 0$, then $r_0(a)$ is a null vector, contradicting to the hypothesis. Consequently, from Lemma 5.3, $\alpha' > 0$ and $m(c) = im(c_1)$ for some $c_1 \leq c$, $c_1 \in L$ and some positive integer i. Now $m(b) = \left(\frac{1}{n}\right) m(a) = \left(\frac{1}{k+l}\right) m(c)$ and hence $\left(\frac{1}{i}\right) \leq \left(\frac{1}{k+l}\right)$. Since $k+l$ is not an integer, we have $k+l < i$. But then $m(c_1) = \left(\frac{k+l}{in}\right) m(a)$ and $\left(\frac{k+l}{in}\right) < \left(\frac{1}{n}\right)$, which is a contradiction to the fact that $\left(\frac{1}{n}\right)$ is minimal. Hence $k+l$ must be an integer and the lemma is proved. ∎

THEOREM 5.1 (Lyapounov's convexity type theorem). *Let m be a non-atomic function relative to $\mathcal{R}$ defined on a σ-complete effect algebra L. Let $a \in L$ and $r \in \mathcal{R}(a)$. Then there exists $b \leq a$, $b \in L$ with $r = m(b)$.*

Proof. Let $a \in L$. If $r_0(a)$ is not a null vector, then conclusion of the theorem follows from Lemma 5.4 and Lemma 5.5. Now let us suppose that $r_0(a)$ is a null vector. Since $r \in \mathcal{R}(a)$, we have $r = \alpha m(a)$ for some α with $0 \leq \alpha \leq 1$. If $\alpha = 0$ or $\alpha = 1$, then the conclusion is obvious. Let us assume now that $0 < \alpha < 1$. Consider the family $\mathcal{F} = \{b \in L : b \leq a$ and $m(b) = \beta m(a)$ with $0 < \beta \leq \alpha\}$. Let us define a partial order $\leq$ on $\mathcal{F}$ by $b_1 \leq b_2$, b_1, $b_2 \in \mathcal{F}$ and corresponding β_1 and β_2 satisfy $\beta_1 \leq \beta_2$. Let us define $a_i = b_i \ominus b_{i-1}$; $b_0 = 0$. In view of 2.3 and by Zorn's lemma, $\mathcal{F}$ has a maximal element, say b, and we may suppose $m(b) = \beta m(a)$. Now we shall show that $\beta = \alpha$. If possible, let us suppose that $\beta < \alpha$. Since $r_0(a)$ is a null vector, it follows easily that $r_0(a \ominus b)$ is also a null vector. Hence we can find an arbitrary small positive number γ and a corresponding element $b' \leq a \ominus b$, $b' \in L$ such that $m(b') = \gamma m(a \ominus b) = \gamma(1 - \beta)m(a)$. Let $b'' = b \oplus b'$. Then $m(b'') = [\beta + \gamma(1 - \beta)]\, m(a)$ and the choice of sufficiently small γ contradicts to maximality of b. Thus $\beta = \alpha$ and the theorem is proved. ∎

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