DEMONSTRATIO MATHEMATICA
Vol. XLIII No 2 2010

León Kushner and Marcelo José Saia

GEOMETRY OF PRE QUASI HOMOGENEOUS POLYNOMIALS OF TYPE (1/2, 1/4)

Abstract. In this article we study the geometry of the orbits of the space V which consists of pre quasi homogeneous polynomials of type $g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_4xy + a_5y^3 + a_6x + a_7y + a_8y^2$ with $a_i \in \mathbb{R}$, for all $i = 1, \dots, 8$ under the action of the group $G := \{h(x,y) = (\alpha x + \beta y^2, \delta y),$ with $\alpha, \delta > 0\}$. To study these orbits we observe first that there are three subspaces of dimension 5, $V_1 := \{g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_4xy + a_5y^3\}$, $V_2 := \{g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_6x + a_8y^2\}$ and $V_3 := \{g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_7y + a_8y^2\}$ of V which are also invariant under the action of this group. Then we describe the orbits which appear in these spaces and give the topological characterization of them by showing their stabilizers. We give a geometrical description of them inside $\mathbb{R}^5$. Moreover, we construct an appropriate map $h : \mathbb{R}^6 \to \mathbb{R}^5$ and prove that the fibers given by the inverse image of the orbits by h are two dimensional surfaces diffeomorphic to $\mathbb{R}^2 - (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$. We show that the points of these fibers which minimize the distance to the origin are indeed in the 3-torus $\Gamma^3 = S^1_{1/2} \times S^1_{1/2} \times S^1_{1/\sqrt{2}}$.

1. Introduction

The stratification of the real cubics plays an essential role in the theory of singularities of functions, as well as in many of its applications. The name "umbilic bracelet" given by E. C. Zeeman in its fundamental paper [7] derives from the fact that it gives the description of the relation between the umbilic catastrophes in a geometric way, (see [2], [6] and [7]). These are the strata consisting of degenerate cubics in the stratification of the space of real cubic forms in two variables under the classification by linear equivalence, or in other words, Zeeman considered the action of the general linear group of invertible matrices $GL(2)$ to obtain these orbits. Zeeman showed that the

Partial support: PAPIIT-IN118908-3.
Partial support: CNPq: 308153/2006-5.
2000 *Mathematics Subject Classification*: Primary 58K40, Secondary 13C05.
Key words and phrases: pre-quasi homogeneous map, stabilizer, orbit, finite relative determination.

representatives of the degenerate orbits under this action are x^2y, x^3 and moreover, Zeeman gave the geometric description of these orbits showing that they form a rotate hipocycloid, the famous "umbilic bracelet".

More recently, R. Bulajich, L. Kushner and S. López de Medrano in [3], gave a topological characterization of the strata consisting of degenerate cubics in both cases, the real and complex. In these works it was studied the orbits under the action of the group $GL(2)$ on the sets of real or complex cubics because these sets are invariant under this action, i.e., for any cubic f and any h in f this group, the polynomial $h.f$ is also a cubic.

The study of the orbits given by the action of different subgroups of $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$ opens a wide variety of problems but, to give a similar description for any other class of polynomials, one needs to find other convenient subgroup of invertible triangular matrices which leaves this fixed class invariant. This is done in [4] for the space of quasi homogeneous polynomials of type $(1/3,1/6)$ and degree 1, namely polynomials of type $ax^3+bx^2y^2+cxy^4+dy^6$. In this work the group considered is the subgroup of germs of diffeomorphisms $G := \{h(x,y) = (\alpha x+\beta y^2, \delta y)$, with $\alpha, \delta \neq 0\}$, which leaves this class of polynomials invariant. They give a topological characterization of all strata and it is also described a geometric characterization of the orbits inside a torus. It is interesting to remark that in the above cases the manifold which is considered to work is a product of spheres.

Another interesting class of polynomials is formed by the pre quasi homogeneous polynomials. Defined in [5], they can be written as $p(x) = q(x) + h(x)$, where $q(x)$ is a degree 1 quasi homogeneous polynomial of type (r,s) and each monomial of $h(x)$ has weighted degree less than 1 and greater than zero. For example, the simplest class consists of polynomials of type $p(x,y) = ax^2 + by^3 + cxy$, here $p = q + h$, where $q(x,y) = ax^2 + by^3$ is a quasi homogenous polynomial of type $(1/2,1/3)$ and $h(x,y) = cxy$ has weighted degree $5/6$. The main problem to study this class of polynomials is to find a convenient subgroup of invertible triangular matrices which leaves this class invariant.

However, for some types of pre quasi homogeneous germs $g = f + h$, depending on the type of quasi homogeneity of the germ f, it is possible to find a subgroup such that this class of polynomials is invariant. For example, if we consider the space of pre quasi homogeneous germs of type $g = p + h$ where $p(x,y) = ax^3+bx^2y^2+cxy^4+dy^6$ is in the class of quasi homogenous germs studied in [4], we can consider the same subgroup G which is used in [4] for this class since it is also invariant under such action. The main problem in this example is that we have a total space of dimension 15 and as this dimension increases, the difficulty to find the orbits and to study the topology of the orbits also increases a lot.

In this article we study the space V which consists of pre quasi homogeneous polynomials of type $g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_4xy + a_5y^3 + a_6x + a_7y + a_8y^2$ with $a_i \in \mathbb{R}$, for all $i = 1, \ldots, 8$. This space is associate to the 3-dimensional space of quasi homogeneous polynomials of type $(1/2, 1/4)$ and degree 1, formed by the polynomials $f(x,y) = a_1x^2 + a_2xy^2 + a_3y^4$. Here again we fix the subgroup $G := \{h(x,y) = (\alpha x + \beta y^2, \delta y)$, with $\alpha, \delta \neq 0\}$ mentioned above as the subgroup acting on V, since this space is invariant under this action.

We remark here that, in fact we can treat the action of this group on the set $\tilde{V}$ of pre quasi homogeneous germs with two variables x, y of type $(\frac{1}{2}, \frac{1}{4})$ including the constant term. The space $\tilde{V}$ can be understood as the space with Newton polygon

$$\Delta = \left\{ (n,m) \in \mathbb{Z}^2 \mid n \geq 0,\, m \geq 0\ \frac{1}{2}n + \frac{1}{4}m \leq 1 \right\}.$$

The dimension of the space $\tilde{V}$ is 9 and the constant function is G invariant. Therefore the study of G-orbit decomposition of $\tilde{V}$ is reduced to that of V, which is 8-dimensional.

To study these orbits we observe first that there are three subspaces of dimension 5, V_1, V_2 and V_3 of V which are also invariant under the action of this group. V_1 consists of the polynomials of type $g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_4xy + a_5y^3$, V_2 is formed by the polynomials of type $g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_6x + a_8y^2$ and V_3 is the set of polynomials $g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_7y + a_8y^2$, then we describe the orbits which appear in these spaces and give the topological characterization of them by showing their stabilizers.

We shall give a geometrical description of them inside $\mathbb{R}^5$. Moreover, we construct an appropriate map $h : \mathbb{R}^6 \to \mathbb{R}^5$, show that the fibers given by the inverse image of the orbits by h are two dimensional surfaces diffeomorphic to $\mathbb{R}^2 - (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$ and we also show that the points of these fibers which minimize the distance to the origin and also are in the 5-sphere are indeed in the 3-torus $\Gamma^3 = S^1_{1/2} \times S^1_{1/2} \times S^1_{1/\sqrt{2}}$.

2. The pre quasi-homogeneous space, the group action and the orbits

We fix the set of weights $(\frac{1}{2}, \frac{1}{4})$ and the vector space of quasi homogeneous polynomials in two variables of degree 1, namely polynomials $f(x,y) = a_1x^2 + a_2xy^2 + a_3y^4$.

We shall study the space V of pre quasi homogeneous polynomials of type $g(x,y) = a_1x^2 + a_2xy^2 + a_3y^4 + a_4xy + a_5y^3 + a_6y^2 + a_7x + a_8y$, which we identify with $\mathbb{R}^8$ by taking the coefficients as variables. On this space we consider the action of the group $G = \{h(x,y) = (\alpha x + \beta y^2, \delta y)$, with $\alpha, \delta \neq 0\}$, acting on V on the right. In fact we see that the group G is invariant on the following subspaces of V: $W_1 = \langle x^2, xy^2, y^4 \rangle$, $W_2 = \langle xy, y^3 \rangle$, $W_3 = \langle x, y^2 \rangle$ and $W_4 = \langle y, y^2 \rangle$. To describe the orbits of V we will consider the following 5-dimensional subspaces: $V_1 = \langle x^2, xy^2, y^4, xy, y^3 \rangle$, $V_2 = \langle x^2, xy^2, y^4, x, y^2 \rangle$ and $V_3 = \langle x^2, xy^2, y^4, y, y^2 \rangle$.

We order our bases as $\{x^2, xy^2, y^4, xy, y^3, x, y, y^2\}$ to get $V_1 = \mathbb{R}^5 \times \{\bar{0}\}$, $V_2 = \mathbb{R}^3 \times \{\bar{0}\} \times \mathbb{R} \times \{\bar{0}\} \times \mathbb{R}$ and $V_3 = \mathbb{R}^3 \times \{\bar{0}\} \times \mathbb{R}^2$.

THEOREM 1. *The orbits in each V_i have measure zero and in fact we have a finite number of models, parametrized each of them by at most two parameters.*

Proof. Since $\dim G = 3$ and $\dim V_i = 5$ the orbits have measure zero, then the result will finish from the fact that after classifying the quasi homogeneous maps of type $f(x,y) = a_1x^2 + a_2xy^2 + a_3y^4$, to get the orbits in each V_i we only have to add a sum of two terms which are invariant under G. ∎

PROPOSITION 1. *Consider the general quasi homogeneous map of type $f(x,y) = a_1x^2 + a_2xy^2 + a_3y^4$ and let $\Delta = a_2^2 - 4a_1a_3$ its discriminant, then*

1. *If $\Delta \geq 0$, f is G-equivalent to one of the following models: $\pm y^4$, xy^2, $\pm x(x+y^2)$, $\pm x^2$.*
2. *If $\Delta < 0$, f is G-equivalent to: $\pm(x^2 + y^4)$.*

Proof. Consider the equalities $f(x,y) = (ax+by^2)(cx+dy^2)$ for $\Delta \geq 0$ and for $\Delta < 0$, let

$$f(x,y) = a_1\left(x + \frac{a_2}{2a_1}y^2\right)^2 + \left(\frac{4a_1a_3 - a_2^2}{4a_1^2}\right)y^4.$$

Then, to show the item 1, we analyze the constants a, b, c, d.

1. If $a = c = 0$, then $f(x,y) = \pm y^4$.
2. If $a = 0$ and $c \neq 0$, $f(x,y) = xy^2$.
3. If $a \neq 0$, $c \neq 0$ and $ad - bc \neq 0$, $f(x,y) = \pm x(x+y^2)$.
4. If $a \neq 0$, $c \neq 0$ and $ad - bc = 0$, $f(x,y) = \pm x^2$.

We only have to prove the cases 3 and 4, then let $x' = ax + by^2$ and $y' = y$, hence

$$f(x',y') = x'\left(\frac{c}{a}x' + \left(\frac{ad-bc}{a}\right)y'^2\right)$$

and the result follows easily. ∎

We describe now the orbits in the subspaces V_1, V_2 and V_3.

We show the representative, the stabilizer and the dimension of the orbit. First we show the orbits of V_1.

Representative	Stabilizer	Orbit dimension
$\pm y^4$	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_4$	1
$\pm(y^4 + ey^3)$, $e \neq 0$	$\mathbb{R}^* \times \mathbb{R}$	1
$\pm(y^4 + xy + ey^3)$	$\mathbb{Z}_4$	3
xy^2	$\mathbb{R}^*$	2
$xy^2 + y^3$	$\mathbb{Z}_3$	3
$xy^2 + xy + dy^3$	$\{Id\}$	3
$x^2 + xy^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
$x^2 + xy^2 + dxy$	$\{Id\}$	3
$x^2 + xy^2 + dxy + ey^3$ $(d \neq 2e)$	$\{Id\}$	3
$\pm x^2$	$\mathbb{Z}_2 \times \mathbb{R}^*$	2
$\pm(x^2 + y^3)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	3
$\pm(x^2 + xy + ey^3)$	$\{Id\}$	3
xy	$\mathbb{R}^*$	2
y^3	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_3$	1
$\pm(x^2 + y^4)$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
$\pm(x^2 + y^4 + ey^3)$	$\mathbb{Z}_2$	3
$\pm(x^2 + y^4 + dxy)$	$\mathbb{Z}_2$	3
$\pm(x^2 + y^4 + dxy + ey^3)$	$\{Id\}$	3

Next we give the table II, with the orbits of the space V_2.

Representative	Stabilizer	Orbit dimension
$\pm y^4$	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_4$	1
$\pm y^4 + x$	$\mathbb{Z}_4$	3
$\pm(y^4 + ey^2)$, $e \neq 0$	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_2$	1
xy^2	$\mathbb{R}^*$	2
$xy^2 \pm y^2$	$\mathbb{Z}_2$	3
$xy^2 + x$	$\mathbb{Z}_2$	3
$xy^2 + x + ey^2$	$\mathbb{Z}_2$	3
$xy^2 + x^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
$xy^2 + x^2 + ey^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
$xy^2 + x^2 + dx$	$\mathbb{Z}_2$	3
$xy^2 + x^2 + dx + ey^2$	$\mathbb{Z}_2$ or $\mathbb{Z}_4$ if $d = 2e$	3
$\pm y^2$	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_2$	1
x	$\mathbb{R}$	2
$\pm(x + y^2)$	$\mathbb{R}$	2
$x^2 + y^4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
$x^2 + y^4 + dx$	$\mathbb{Z}_4$	3
$x^2 + y^4 + ey^2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	3
$x^2 + y^4 + dx + ey^2$	$\mathbb{Z}_2$	3

The table III gives the orbits of the subspace V_3.

Representative	Stabilizer	Orbit dimension
$\pm y^4$	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_4$	1
$\pm(y^4+dy)$, $e \neq 0$	$\mathbb{R}^* \times \mathbb{R}$	1
$\pm(y^4+ey^2)$	$\mathbb{R}^* \times \mathbb{R} \times \mathbb{Z}_2$	1
$\pm(y^4+dy+ey^2)$	$\mathbb{R}^* \times \mathbb{R}$	1
xy^2	$\mathbb{R}^*$	2
xy^2+y	$\{Id\}$	3
xy^2+y^2	$\mathbb{Z}_2$	3
xy^2+y+ey^2	$\{Id\}$	3
x^2+xy^2	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
x^2+xy^2+dy	$\mathbb{Z}_2$	3
$x^2+xy^2+ey^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
$x^2+xy^2+dy+ey^2$	$\{Id\}$	3
x^2+y^4	$\mathbb{Z}_2 \times \mathbb{Z}_4$	3
x^2+y^4+dy	$\mathbb{Z}_2$	3
$x^2+y^4+ey^2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	3
$x^2+y^4+dxy+y^3$	$\mathbb{Z}_2$	3

3. The geometry of the orbits

As in [3] we have to choose a quartic that will parametrize our vector space in pieces which are the three vector spaces V_1, V_2 and V_3. Our quartic should be of six coefficients and easy to handle. For this, first we consider the quartic in two variables given by

$$f(x,y) = (ax+by)(cx+dy)(ex+fy)^2$$

which in coordinates $(x^4, x^3y, x^2y^2, xy^3, y^4)$ induces a map $h : \mathbb{R}^6 \to \mathbb{R}^5$, given by

$$h(a,b,c,d,e,f) =$$
$$(ace^2, 2acef+(ad+bc)e^2, acf^2+2(ad+bc)ef+bde^2, (ad+bc)f^2+2bdef, bdf^2).$$

Now we understand the images in $\mathbb{R}^5$ whose coordinates are in the ordered basis of the spaces V_1, V_2 and V_3. Our purpose is

1. Calculate the fibers of h and the nearest points of them to the origin.
2. Prove that the intersection of these points with the 5-sphere is contained in the 3-torus $\Gamma^3 = S^1_{1/2} \times S^1_{1/2} \times S^1_{1/\sqrt{2}}$.
3. Give the canonical identification in M (taking out some set) such that the induced map $\hat{h} : M \to \mathbb{R}^5$ is injective and calculate its singularities.

First Step:

We have the following system of equations in the variables (a,b,c,d,e,f):

(1) $$ace^2 = \alpha\gamma\varphi^2$$

(2) $$2(acef) + (ad+bc)e^2 = 2\alpha\gamma\varphi\eta + (\alpha\delta+\beta\gamma)\varphi^2$$

(3) $$acf^2 + 2(ad+bc)ef + bde^2 = \alpha\gamma\eta^2 + 2(\alpha\delta+\beta\gamma)\varphi\eta + \beta\delta\varphi^2$$

(4) $$2(bdef) + (ad+bc)f^2 = 2(\beta\delta\varphi\eta) + (\alpha\delta+\beta\gamma)\eta^2)$$

(5) $$bdf^2 = \beta\delta\eta^2$$

(I) The case $cdef \neq 0$. From (1) and (5) we obtain: $a = \dfrac{\alpha\gamma\varphi^2}{ce^2}$, $b = \dfrac{\beta\delta\eta^2}{df^2}$.

We also have the solutions $d = (\delta/\gamma)c$ and $f = (\eta/\varphi)e$, therefore our fiber is the surface given by:

$$(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\alpha\delta\phi^2}{ce^2}, c, \frac{\beta}{\alpha}c, e, \frac{\eta}{\varphi}e\right).$$

If we substitute $u = e/f$ and $v = c/d$, in (3), (4) and (5) we have the solutions; $d = (\delta/\gamma)c$ and $f = (\eta/\varphi)e$, our fiber is the surface given by:

$$(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\beta\gamma\phi^2}{ce^2}, c, \frac{\delta}{\gamma}c, e, \frac{\eta}{\varphi}e\right)$$

with domain $\mathbb{R}^2 - (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$, $(\gamma.\varphi \neq 0)$.

i) In the case $\gamma = 0$, the parametrization of the surface is given as

$$(c,e) \to \left(0, \frac{\alpha\delta\varphi^2}{ce^2}, c, \frac{\beta}{\alpha}c, e, \frac{\eta}{\varphi}e\right).$$

ii) In the case $\varphi = 0$, we have

$$(d,e) \to \left(0, \frac{\alpha\gamma\eta^2}{de^2}, 0, d, e, \left(\frac{\alpha\delta+\beta\gamma}{2\alpha\gamma}\right)e\right), \text{ with } \alpha\delta = \beta\gamma.$$

Next we give a complete list for all cases, including the above three, all of them with domain $\mathbb{R}^2 - (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$.

Denoting by: I) $cdef \neq 0$, II) $c = 0$, III) $f = 0$, IV) $d = 0$ and V) $e = o$.

Parametrization of the fibers

I) a: $(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\beta\delta\varphi^2}{ce^2}, c, \frac{\delta}{\gamma}c, e, \frac{\eta}{\varphi}e\right)$,
b: $(c,e) \to \left(0, \frac{\beta\delta\varphi}{ce^2}, c, \frac{\beta}{\alpha}c, e, \frac{\eta}{\varphi}e\right)$ and
c: $(d,e) \to \left(0, \frac{\alpha\gamma\eta^2}{de^2}, 0, d, e, \frac{\alpha\delta+\beta\gamma}{2\alpha\gamma}e\right)$, with $\alpha\delta = \beta\gamma$.

II) a: $(d,e) \to \left(\frac{\alpha\delta\varphi^2}{de^2}, \frac{\beta\delta\varphi^2}{de^2}, 0, d, e, \frac{\eta}{\varphi}e\right)$,
b: $(d,e) \to \left(\frac{\alpha\delta\varphi^2}{de^2}, \frac{\beta\delta\varphi^2}{de^2}, 0, d, e, \frac{\eta}{\varphi}e,\right)$,
c: $(d,f) \to \left(0, \frac{\alpha\gamma\eta^2}{df^2}, 0, d, \frac{\alpha\delta+\beta\gamma}{2\beta\delta}f, f\right)$,

d: $(d,f) \to \left(\frac{\beta\gamma\eta^2}{df^2}, \frac{\beta\delta\eta^2}{df^2}, 0, d, 0, f\right)$ and

e: $(d,f) \to \left(\frac{\alpha\delta\eta^2}{df^2}, \frac{\beta\delta\eta^2}{df^2}, 0, d, 0, f\right)$.

III) a: $(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\alpha\gamma\varphi\eta}{ce^2}, c, \frac{\eta}{\varphi}c, e, \right)$,

b: $(d,e) \to \left(\frac{\alpha\delta\varphi^2}{ce^2}, 0, \frac{\gamma}{\delta}d, d, e, 0\right)$,

c: $(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\alpha\delta\varphi^2}{ce^2}, c, 0, e, 0, \right)$,

d: $(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\beta\gamma\varphi^2}{ce^2}, c, \frac{\delta}{\gamma}c, e, 0, \right)$ and

e: $(c,e) \to \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\alpha\delta\varphi^2}{ce^2}, c, \frac{\beta}{\alpha}c, e, 0, \right)$.

To obtain IV) {or V } we exchange in II) {or in III)} respectively: α by β, γ by δ, a by b, c by d and e by f.

Now we show in each orbit the nearest points of the origin.

We shall describe here the case I) a. We get a map

$$\Theta : \mathbb{R}^2 - (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}) \to \mathbb{R}^6$$

given by

$$\Theta(c,e) = \left(\frac{\alpha\gamma\varphi^2}{ce^2}, \frac{\beta\gamma\varphi^2}{ce^2}, c, \frac{\delta}{\gamma}c, e, \frac{\eta}{\varphi}e\right).$$

When we minimize the distance to the origin we obtain

$$c = \frac{\pm\gamma(\varphi^2+\eta^2)^{1/4}(\alpha^2+\beta^2)^{1/8}}{2^{1/4}(\gamma^2+\delta^2)^{3/8}} \quad \text{and} \quad e = \frac{\pm 2^{1/4}\varphi(\alpha^2+\beta^2)^{1/8}(\gamma^2+\delta^2)^{1/8}}{(\varphi^2+\eta^2)^{1/4}}.$$

Intersecting with S^5 we obtain $(\alpha^2+\beta^2)^{1/4}(\gamma^2+\delta^2)^{1/4}(\varphi^2+\eta^2)^{1/2} = \frac{1}{2^{3/2}}$, therefore the points in the image of Θ can be seen in $\Gamma^3 = S^1_{1/2} \times S^1_{1/2} \times S^1_{1/\sqrt{2}}$.

We summarize our results in the following

THEOREM 2. *Let $h : \mathbb{R}^6 \to \mathbb{R}^5$ given as before, then:*

1) *The fibers are two dimensional surfaces diffeomorphic to $\mathbb{R}^2 - (\mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R})$.*
2) *The points of these fibers which minimize the distance to the origin and also are in the 5-sphere are indeed in the 3-torus $\Gamma^3 = S^1_{1/2} \times S^1_{1/2} \times S^1_{1/\sqrt{2}}$.*
3) *The map h induces a map $\hat{h}$ by restriction $\hat{h} : \Gamma^3 \to \mathbb{R}^5$.*

LEMMA 3. *For an $\hat{h}$ as above, its fibers $\hat{h}(\hat{h}^{-1}(\theta_0, \varphi_0, \psi_0))$ consist of at most eight points:*

1. $(\theta_0, \varphi_0, \psi_0)$, $(\theta_0+\pi, \varphi_0+\pi, \psi_0)$, $(\theta_0, \varphi_0, \psi_0+\pi)$, $(\theta_0+\pi, \varphi_0+\pi, \psi_0+\pi)$,
2. $(\varphi_0, \theta_0, \psi_0)$, $(\varphi_0+\pi, \theta_0+\pi, \psi_0)$, $(\varphi_0, \theta_0, \psi_0+\pi)$, $(\varphi_0+\pi, \theta_0+\pi, \psi_0+\pi)$.

Proof. From the equations given by $\hat{h}$ in coordinates (θ, φ, ψ) we get the coordinates in $\mathbb{R}^5$.

1. $\cos\theta \cos\varphi \cos^2\psi$,

2. $2\cos\theta\cos\varphi\sin\psi\cos\psi + \sin\theta\cos\varphi\cos^2\psi + \cos\theta\sin\varphi\cos^2\psi$,
3. $\cos\theta\cos\varphi\sin^2\psi + 2\sin\theta\cos\varphi\cos\psi\sin\psi + 2\cos\theta\sin\varphi\cos\psi\sin\psi + \sin\theta\sin\varphi\cos^2\psi$,
4. $2\sin\theta\sin\varphi\sin\psi\cos\psi + \sin\theta\cos\varphi\sin^2\psi + \cos\theta\sin\varphi\sin^2\psi$,
5. $\sin\theta\sin\varphi\sin^2\psi$.

It is clear that the eight points are in a fiber (it might be repetitions), moreover if we set

$$(*) \qquad \cos\theta = \frac{\cos\theta_0\cos\varphi_0\cos^2\psi_0}{\cos\varphi\cos^2\psi} \quad \text{and} \quad \sin\theta = \frac{\sin\theta_0\sin\varphi_0\sin^2\psi_0}{\sin\varphi\sin^2\psi}$$

when we exchange in the equations 2, 3 and 4 and set $U = \frac{\sin\varphi}{\cos\varphi}$, $V = \frac{\sin\psi}{\cos\psi}$, we get the solutions $U = \tan\varphi_0$, $V = \tan\psi_0$ and $U = \tan\theta_0$, $V = \tan\psi_0$. Now we finish using the equalities given in the equations $(*)$.

PROPOSITION 2. *The rank of $D\hat{h}$ is:*

a) *if $\theta = \varphi = \psi$,*
b) *if $\theta = \varphi$ and $\theta \neq \psi$; or $\theta \neq \varphi$ and $\theta = \psi$; or $\psi = \varphi$ and $\theta \neq \varphi$,*
c) *in all other cases.*

We remark that we understand the equalities module the points in the fibers. We shall denote the curves of the first case by $\{c_i\}$ and the tori in the second case by $\{S_j\}$.

In the next proposition we show that the kernel of the map $D\hat{h}$ is transversal to the tangent spaces of these curves and tori in the torus Γ^3.

PROPOSITION 3. *For the map $D\hat{h}$ we get the following equalities:*

$$\ker D\hat{h} \oplus TC_i = T(\Gamma^3) \quad \textit{and} \quad \ker D\hat{h} \oplus TS_j = T(\Gamma^3).$$

The proof of these results are done by some simple calculation.

Next we show how the torus Γ^3 remains when we use the identification given by the Lemma 3.

PROPOSITION 4. *Consider the torus Γ^3 and the identification given in the Lemma 3, i.e., the identification $\sim$ which collapses each fiber in one point, then the quotient space $\frac{T^3}{\sim}$ remains as the following figure times S^1.*

Proof. The map $\sigma : S^1 \times S^1 \to S^1 \times S^1$ given by $\sigma(z, w) = (-z, -w)$ is an involution with no fixed points and in fact preserves orientation and the quotient $\frac{S^1 \times S^1}{\sigma}$ is $\mathbb{R}P^1 \times \mathbb{R}P^1$, hence we obtain that the identification of type (i) in the Lemma 3 is diffeomorphic to $S^1 \times S^1 \times S^1$.

For the case (ii), we can see for $(\theta, \varphi) \sim (\varphi, \theta)$, as the identification of the square $[-\pi, \pi] \times [-\pi, \pi]$ to the torus restricted to the triangle $T = \{y \geq x\}$ with diagonal $\theta = \varphi$. Therefore the identification of the vertices gives our figure. ∎

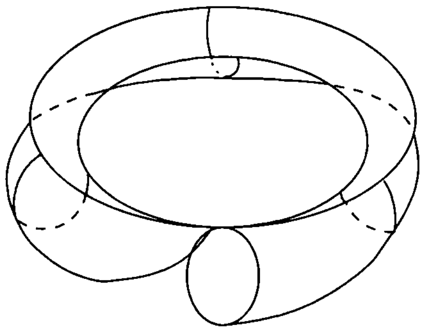

Fig. 1.

We remark that our "bad" points are $\overline{(\pi,\pi,\psi)}$, with $\psi \in S^1$ and their image under $\hat{h}$ is, in our coordinates, given by the equation

$$(\cos^2\psi)x^2 + (2\sin\psi\cos\psi)xy^2 + (\sin^2\psi)y^4 = ((\cos\psi)x + (\sin\psi)y^2)^2,$$

or $(\alpha x + \beta y^2)^2$ with $\alpha^2 + \beta^2 = 1$.

Now we show the images of $\hat{h}$.

i) $\hat{h}(\pi,\pi,\psi) = (\cos^2\psi, 2\cos\psi\sin\psi, \sin^2\psi, 0, 0)$ in the case $\psi = \pi$, the image is x^2 and in the case $\psi = \pm\pi/2$, the image is y^4.

The image is in the set $x_1^2 + x_2^2/2 + x_3^2 = 1$.

ii) $\hat{h}(\theta,\theta,\theta) = (\cos^4\theta, 4\cos^3\theta\sin\theta, 6\cos^2\theta\sin^2\theta, 4\cos\theta\sin^3\theta, \sin^4\theta)$ which satisfy the equation $x_1^2 + x_2^2/4 + x_3^2/6 + x_4^2/4 + x_5^2 = 1$.

iii) $\hat{h}(\theta,\theta,\psi) = h(\cos\theta, \sin\theta, \cos\theta, \sin\theta, \cos\psi, \sin\psi) =$
$(\cos^2\theta\cos^2\psi, 2(\cos^2\theta\sin\psi\cos\psi + \cos\theta\sin\theta\cos\psi), \cos^2\theta\sin^2\psi$
$+ 4\cos\theta\sin\theta\cos\psi\sin\psi + \sin^2\theta\cos^2\psi, 2(\cos\theta\sin\theta\sin^2\psi$
$+ \sin^2\theta\cos\psi\sin\psi, \sin^2\theta\sin^2\psi)$.

These points satisfy

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 2x_1x_3 - 2x_3x_5 - 2x_2x_4 + 2x_1x_5 = 1.$$

iv) $\hat{h}(\theta,\varphi,\theta) = h(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi, \cos\theta, \sin\theta) =$
$(\cos^3\theta\cos\varphi, 2\cos^2\theta\sin\theta\cos\varphi + \cos^2\theta(\cos\theta\sin\varphi + \sin\theta\cos\varphi),$
$\cos\theta\cos\varphi\sin^2\theta + 2(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\sin\theta\cos\theta + \sin\theta\sin\varphi\cos^2\theta,$
$(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\sin^2\theta + 2\sin^2\theta\cos\theta\sin\varphi, \sin^3\theta\sin\varphi)$.

For this case the associate quadric in 5 variables is

$$x_1^2 + x_2^2 + x_3^2/3 + x_4^2 + x_5^2 - x_1x_3 - 3x_1x_5 - x_3x_5 = 1.$$

v) $\hat{h}(\theta,\varphi,\varphi) = h(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi, \cos\varphi, \sin\varphi) =$
$(\cos\theta\cos^3\varphi, 2\cos\theta\cos^2\varphi\sin\varphi + (\cos\theta\sin\varphi + \sin\theta\cos\varphi)\cos^2\varphi,$
$\cos\theta\cos\varphi sin^2\varphi + 2(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\sin\varphi\cos\varphi,$
$+ \sin\theta\sin\varphi\cos^2\varphi(\cos\theta\sin\varphi + \sin\theta\cos\varphi)\sin^2\varphi,$
$+ 2\sin\theta\sin^2\varphi\cos\varphi, \sin\theta\sin^3\varphi)$.

Here, the associate quadric in 5 variables is similar to the one of the case iv).

As an immediate consequence of the proposition given above we obtain the following

PROPOSITION 5. *The map* $h : \mathbb{R}^6 \to \mathbb{R}^5$ *induces an injective map* $\bar{h} : \mathrm{X} \to \mathbb{R}^5$ *and we have a circle* S^1 *such that* $X - S^1$ *is a* 3*-manifold diffeomorphic to* $S \times S^1$*, where* S *is diffeomorphic to a smooth triangle without a vertex embedded in a torus* T^2.

Proof. From the figure in the previous proposition, the result is clear. ∎

References

[1] E. Andrade, L. Kushner Schnur, *Finite determination*, Finite relative determination and the Artin-Rees Lemma. Notas ICMC-USP, n: 218. 2006.

[2] V. I. Arnol'd, A. N. Varchenko, S. M. Goussein-Zadé, *Singularités des Applications Différentiables*, Editions Mir, Moscou, 1986.

[3] R. Bulajich, L. Kushner Schnur, S. López de Medrano, *Cubics in* $\mathbb{R}$ *ans* $\mathbb{C}$, Real and Complex Singularities, D. Mond and M. J. Saia (eds.), Lecture Notes Series in Pure and Applied Maths, Marcel-Decker, 237–244, 2003.

[4] R. Bulajich, L. Kushner, S. López de Medrano, *The space of quasi homogeneous maps in two variables of type* $(1/3, 1/6)$ *and degree* 1, preprint, 2006.

[5] M. J. Saia, *Pre-weighted homogeneous map germs, finite determinacy and topological triviality*, Nagoya Math. J. 151 (1998), 209–220.

[6] R. Thom, *Structural Stability and Morphogenesis*, An Outline of a General Theory of Models, W. A. Benjamin, Inc., 1975.

[7] E. C. Zeeman, *The Umbilic Bracelet and the Double Cusp Catastrophe*, in: Catastrophe Theory, Selected Papers 1972-77, Addison-Wesley, 1977, pp. 563–601.

León Kushner
FACULTAD DE CIENCIAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, D. F.
E-mail: kushner@servidor.unam.mx

Marcelo José Saia
INSTITUTO DE CINCIAS MATEMÁTICAS E DE COMPUTAÇÃO
UNIVERSIDADE DE SÃO PAULO
AVENIDA TRABALHADOR SÃOCARLENSE, 400
13566-590, SÃO CARLOS, SP, BRASIL
E-mail: mjsaia@icmc.usp.br

Received November 28, 2009.