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A SURVEY ON POLYGONAL PORTRAITS OF MANIFOLDS

Abstract. Planar portraits are geometric representations of smooth manifolds defined by their generic maps into the plane. A simple subclass called the polygonal portraits is introduced, their realisations, and relations of their shapes to the topology of source manifolds are discussed. Generalisations and analogies of the results to other planar portraits are also mentioned. A list of manifolds which possibly admit polygonal portraits is given, up to diffeomorphism and up to homotopy spheres. This article is intended to give a summary on our research on the topic, and hence precise proofs will be given in other papers.

1. Introduction

For a smooth manifold M of dimension two or more, its planar portrait through $f : M \to \mathbf{R}^2$ is the pair $\mathcal{P} = (f(M), f(S_f))$, up to diffeomorphism of $\mathbf{R}^2$, where f is a generic map and S_f is the set of singular points. The second component, referred to as the critical loci of $\mathcal{P}$ or f, is a plane curve possibly disconnected and possibly with a finite number of (ordinary) cusps and normal crossings. A planar portrait can be regarded a natural, geometric representation of the manifold, but its relation to the topology of M is not straight and few is known about it. One can hence pose two basic problems as below. (A) *What topological properties of M are carried to a planar portrait (and how)?* (B) *Which compact set, bounded and separated into regions by a plane curve, can be a planar portrait of a manifold?* In this article, we introduce a special class of planar portraits named the *polygonal portraits*, and approach to these problems, especially to (B).

The critical loci of generic maps into the plane have been attractive for a long time, especially for maps from surfaces, as seen in *e.g.*, [Hae], [Ka], [OA] and [Ya]. For these maps from surfaces, important tools have been developed by S. Blank [Bl] (as exposed in [L2]) and Francis-Troyer [FT], etc. Recently, Hacon-Mendes-Fuster [HMF1, HMF2, HMF3] introduced an

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approach using a graph to represent a stable map of a surface, developed surgery techniques to calculate invariants, and obtained some classification results of fold maps. In contrast, our approach here are intended to treat manifolds which are not necessarily surfaces, and much more stress is placed on critical loci of uncertain manifolds. Our results are independent to these earlier works in appearance, but it is an interesting problem to consider the relevancy to them, especially to the last one.

We have four topics in this paper. First is on the realisation of polygonal *pre*-portraits by surfaces. For realisation by higher dimensional manifolds, we point out a difficulty and give only partial results. The second is on the shape of a polygonal portrait. In short, a polygonal portrait is formed from apparent contours of a set of homotopy subspheres in a manifold. The third is on the construction of generic maps of certain manifolds which produce polygonal portraits. The final one is the list of the diffeomorphism types of manifolds which possibly admit polygonal portraits, where homotopy spheres appear in the list as building blocks without specifying their diffeomorphism types. Our list is, in this sense, up to homotopy spheres. Into these topics, a section for toric 4-manifolds is inserted, since they, together with closed surfaces, are typical examples of manifolds admitting polygonal portraits. Our research on polygonal portrait is intended to be a step to approach to planar portrait in general. We pose some problems, from this viewpoint.

As seen, most part of the paper is devoted to the problem (B). On the problem (A), there is no explicit result, but one can guess what kind of properties are likely or not likely to be carried on the planar portraits from the results in the last three sections. We note that many results in this article are based on our previous observations in [K1]. Throughout, we assume that manifolds and surfaces are smooth, closed, and connected, unless otherwise mentioned.

2. Realisation problem of polygonal portraits

Let Δ_k, k a positive integer, be a polygon, or a 2-disc with k marked points on the boundary, placed in the plane. We duplicate each edge without changing the vertices (replace the edge with two arcs connecting the vertices, which may have mutual crossings) so that any of the 2^k combinations of k-edges chosen from each pair of duplicated edges bounds a polygon again (Figure 1, a and b). Then by performing a modification at each vertex as in c of the figure, we obtain a compact set K and a curve C, possibly disconnected and possibly with cusps and normal crossings, which bounds and separates K into regions. The pair (K, C), up to diffeomorphism of $\mathbf{R}^2$, is called a *polygonal pre-portrait* ([K4]). It can be constructed uniquely from Δ_k, if the number of crossings between the duplicated edges is given for each of
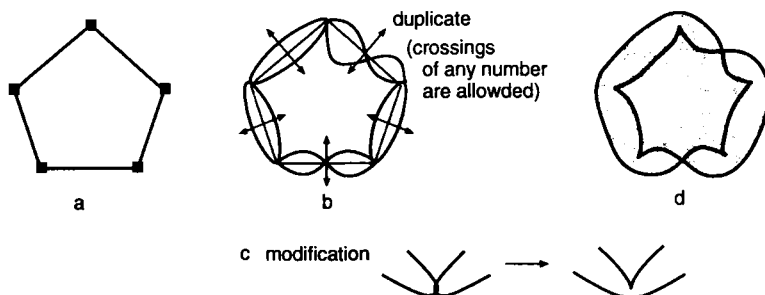
Fig. 1. A polygonal pre-portrait (d) obtained by duplications and modifications to a polygon (a)
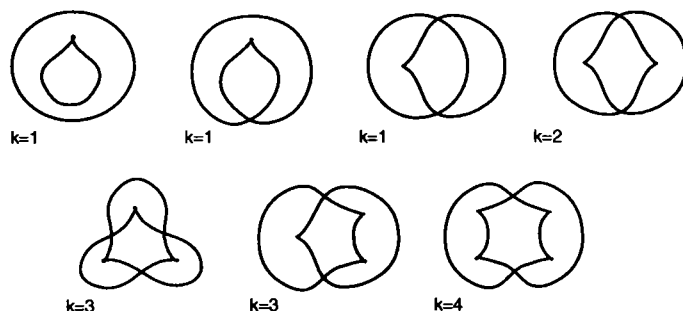

Fig. 2. Examples of polygonal pre-portraits

the k-edges. Some examples are shown in Figure 2, where every polygonal pre-portrait for a fixed k is not listed.

On their realisation as planar portraits, we can show the following:

THEOREM 2.1. ([K4]) *A polygonal pre-portrait is realised as a planar portrait of a manifold if and only if the number of cusps k is two or more.*

Note. In actual, such a polygonal pre-portrait in the theorem is realised at least by the closed surface whose total Betti number $b_1 + 2$ equals to k, as seen below.

A polygonal pre-portrait that is realised by a manifold is simply referred to as a polygonal portrait.

Proof. *In-realisation for $k = 1$.* Assume that a polygonal pre-portrait $\mathcal{P}$ of $k = 1$ is realised by a map f of an n-manifold of $n \geq 2$. If it has two or more crossings, it is easy to remove the crossings in pair by a homotopy of f, by using that fibres near a crossing is disconnected (refer to the *overhanging operation* for surfaces, mentioned later). Hence the problem is reduced to the case where $\mathcal{P}$ has at most one crossing, or it is either of the first two in Figure 2. But in either case, it contains "prohibited subpiece" listed in [K2],

which means that such a piece is impossible to be contained in any planar portraits. It is a contradiction. ∎

Proof. *Realisation for* $k \geq 2$. We show that $\mathcal{P}$ is realised by a closed surface.

LEMMA 2.2. *The polygonal pre-portraits* S, S' *and* P *in* Figure 3 *are realised by* S^2 *(for* S *and* S'*) and by* $\mathbf{R}P^2$ *(for* P*), respectively.*
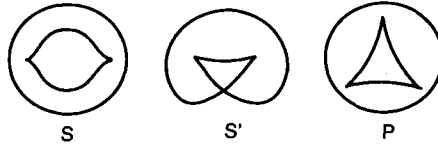

Fig. 3. The three elementary polygonal portraits

It is not difficult to see this (we mention this again later in this section, in more general settings; see the references there).

By using S, S' and P, we define three types of modifications (*addition of a cusp, making a crossing,* and *overhanging*) each of which changes a polygonal portrait X into another, as in the pictures below (Figure 4, 5, 6). The first two are both achieved just by a fibrewise connected sum and elimination of cusps. To eliminate the cusps, we must prepare the maps realising S' and P carefully so that the cusps have cancelable signs ([L1]). The third one is achieved by a homotopy of a map. The last one is a convertible operation, while the first two are not. It is obvious that, starting from S, one can achieve any $\mathcal{P}$ in the theorem by a sequence of these modifications. ∎

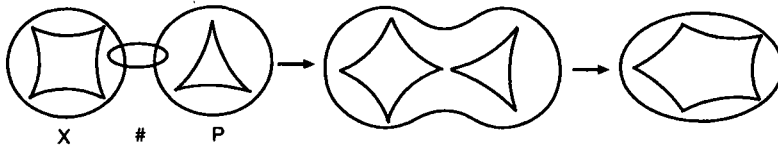*Addition of a cusp* (Figure 4):

Fig. 4. Addition of a cusp by a sum with P

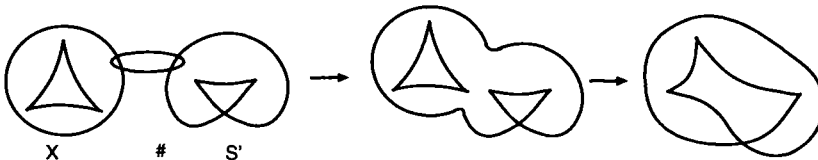*Making a crossing* (Figure 5):

Fig. 5. Making a crossing by a sum with S'
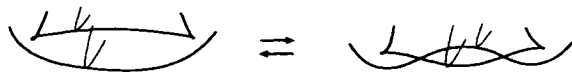
Overhanging (Figure 6):

Fig. 6. Making or deleting a pair of crossings by a homotopy

We next consider the realisation of polygonal pre-portraits by manifolds of a given, higher dimension. A polygonal (pre-) portrait is *strict* if crossed duplications do not adjoin for two edges.

LEMMA 2.3. *Any polygonal portrait of an n-manifold of $n \geq 3$ is strict.*

Proof. Assume that a polygonal portrait is realised by an n-manifold. The indices of folds on the two branches abutting a cusp are τ for folds on a branch, and $n - 2 - \tau$ for those on the other, as seen by the normal form of a cusp, where indices are counted according to the normal direction pointed inward of the portrait. If crossed duplications occur for adjoining two edges, then the cusp corresponding to the common vertex of the edges has two branches which are both made of folds of definite types $(u, |x|^2)$. Hence both $\tau = 0$ and $n - 2 - \tau = 0$ hold, which implies $n = 2$. ∎

LEMMA 2.4. *A polygonal pre-portrait with odd number of cusps is impossible to be realised by any odd-dimensional manifolds.*

Proof. It is a direct application of the mod 2 congruence of Thom [Th, theorem 9] between the Euler characteristic of manifolds $\chi(M)$ and the number of cusps. ∎

These two lemmas give basic restrictions on dimensions of realising manifolds of a polygonal pre-portrait. The following observation suggests a more complicated aspect of the problem.

FACT 2.5. ([K4]) *There is no 6- and 12-manifolds admitting the polygonal portrait* P *in Figure 3. More generally,* P *can not be realised by any n-manifolds if either* (a) *n is odd, or* (b) $n \neq 2^r$, $r = 1, 2, 3, \ldots$.

Proof. If P is realised by an n-manifold of $n \geq 2$, then n is even, by Lemma 2.4. Hence we assume that $n = 2m$ below. Divide P into two half discs so that one contains a cusp and the other does two cusps. According to the division, M is also divided into two pieces M_1 and M_2 with common boundary $\partial M_1 = \partial M_2$. The piece M_1 that contains exactly one cusp is diffeomorphic to D^n by [K1, Theorem 2.2], while the other is the two copies of $D^m \times D^m$ pasted along $D^m \times \partial D^m$ in a way that map-fibres are preserved. Lemma 3.4 stated later shows it to be a tubular neighbourhood of a homotopy subsphere Σ^m. Hence the boundary ∂M_2 is a sphere S^{2m-1} and, at the

same time, a sphere bundle with fibre S^{m-1} over Σ^m. A homotopy result by J. Adem ([Ad, Corollary 2.3]) implies that $m = 2^r, r = 0, 1, 2, \cdots$, which contradicts to our assumption on n. ∎

Returning to realisation of strict polygonal pre-portraits by higher dimensional manifolds, we can see that the elementary three pieces in Figure 3 are realised also by manifolds of dimensions $4, 8$ and 16; in strict, S and S' by spheres of any dimensions ([K1, Example 7.1, K5]), and P by the projective planes over $\mathbf{C}, \mathbf{H}$ ([K1, Example 7.2]) and the Keyley numbers. Hence by following essentially the same argument as before, we obtain:

THEOREM 2.6. ([K4]) *A strict polygonal pre-portrait of two or more cusps can be realised as a planar portrait by a manifold of any of the following dimensions:* $2, 4, 8, 16$.

The choice of dimensions above is somehow *ad hoc*, but it is impossible to drop the restrictions entirely, as seen in the fact above. The key to know the realisable dimensions is the following problem:

PROBLEM 2.7. *Determine the dimensions of manifolds which admit the piece* P *in Figure 3 as a planar portrait.*

In later sections, we give some partial answers to our realisation problem of strict polygonal pre-portrait, approached from different viewpoints.

3. Cores of source manifolds

An *outline* of a polygonal (pre-) portrait is the chain of loops, up to isotopy, obtained from the duplication of edges of underlying polygon by replacing each pair of duplicated edges with a slightly bigger loop so that adjoining loops have small overlaps as in Figure 7.
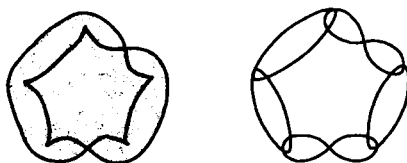

Fig. 7. Polygonal portrait (left) and its outline (right)

THEOREM 3.1. ([K4]) *Let* $\mathcal{P}$ *be a polygonal portrait of a manifold* M *through* f. *There exists a collection of* k *embedded homotopy spheres* $\{S_i\}$ *of* $\dim S_i \geq 1$ *in* M, *called a* core, *where* k *is the number of cusps, such that:*

1. *the restriction* $f|S_i$ *is either a generic map onto an embedded 2-disc only with definite fold singularities (if* $\dim S_i \geq 2$), *or an immersion of a loop (otherwise), and*

2. *the collection of apparent contours of the restrictions $f|S_i$ is an outline of $\mathcal{P}$.*

An *apparent contour* of $f|S_i$ above stands for the critical loci, or the boundary curve of $f(S_i)$ in case $\dim S_i \geq 2$, or just the image $f(S_i)$ in case $\dim S_i = 1$.

Before giving an outline of the proof, we state a basic property of the core.

PROPOSITION 3.2. ([K4]) *The core $\{S_i\}$ above is accompanied by a Morse function $H : M \to \mathbf{R}$ so that:*

1. *H has the critical points exactly at the intersections of components,*
2. *the critical points of $H|S_i$ are critical points of H also, and*
3. *the Morse index of each critical point of H agrees with the sum of $\dim S_i$ through S_i that contains this point as the maximum of $H|S_i$.*

We define in general a set of homotopy subspheres $\{S_i\}$ of a manifold M to be a core if there exists a Morse function $H : M \to \mathbf{R}$ of the three properties above. Twins, or a pair of transverse equators S^p and S^q with $p+q = n$ in the sphere S^n, the union of spheres $S^p \times \{N, P\}$ and $\{N', P'\} \times S^q$ in $S^p \times S^q$, where $N', P' \in S^p$ and $N, P \in S^q$ are both pair of distinct points, and the canonical divisor $3P^1$ of the projective plane (over $\mathbf{R}, \mathbf{C}, \mathbf{H}$) are typical examples of cores. The core is an essential part for the topology of M; in these examples, M can be decomposed into a tubular neighbourhood of the core and a "standard" piece diffeomorphic to $S^{p-1} \times S^{q-1} \times D^2$, where p, q in the last example are both half of the (real) dimension of M.
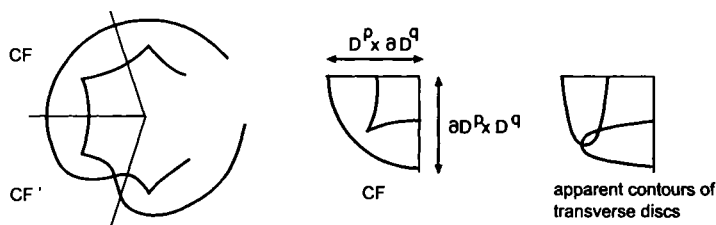

Fig. 8. Decomposition of a polygonal portrait and the cusped fan projection

Outline of the proof of Theorem 3.1. We can divide $\mathcal{P}$ into k pieces of two kinds; a cusped fan (CF) and a crossed cusped fan (CF') of any number of crossings, as in Figure 8. According to the subdivision, M can be divided into k pieces M_i of manifolds with corner. Let τ and $n - 2 - \tau$ be the indices of folds between two cusps counted as before (refer to the proof of Lemma 2.3). It follows from [K1, Theorem 2.2] that:

LEMMA 3.3.

1. *A subpiece M_i over CF is diffeomorphic to $D^p \times D^q$, where $p = \tau + 1$ and $p + q = n$.*
2. *f restricted to each M_i is uniquely determined by τ, up to right-left equivalence.*
3. *(Under the identification of M_i with $D^p \times D^q$) f restricted to each of the transverse discs $D_i = D^p \times \{0\}$ and $D'_i = \{0\} \times D^q$ is either a generic map having only definite folds as singular points (if $\dim D_i$ or $\dim D'_i$ is two or more), or an immersion (otherwise).*
4. *The apparent contour of the above restriction is contained in the outline of $\mathcal{P}$.*

We refer to the generic map $f|M_i$ over CF as a *cusped fan projection*. For M_i over CF', we can see the similar things by using that fibres near crossings are disconnected and hence that one can remove the crossings in CF' by a homotopy of $f|M_i$ as mentioned in the first part of the proof of Theorem 2.1. One can thus apply the lemma to M_i over CF' after removed the crossings.

The boundaries $\partial D^p \times D^q$ and $D^p \times \partial D^q$ of M_i are mapped to the cut loci of $\mathcal{P}$. With the help of the following lemma, we can paste the transverse pair $D_i \cup D'_i$ in each M_i all together to obtain the required collection of homotopy spheres.

LEMMA 3.4. ([K4], a special version in [K3]) *By a self-diffeomorphism of $\partial D^p \times D^q$ (resp. $D^p \times \partial D^q$) which preserves f, the boundary ∂D_i (resp. $\partial D'_i$) is preserved up to f-preserving isotopy.*

In this way we can obtain the theorem. It is not a direct answer to the problem (A), but one can know what the shape of our planar portrait reflects, which will be helpful in answering to the problem.

Outline of the proof of Proposition 3.2. We mention briefly the construction of the accompanying Morse function H. Take a submersion $r : \mathbf{R}^2 \to \mathbf{R}$ so that it is transverse both to the critical loci of f and to the chain of apparent contours of $f|S_i$, both off a set of small neighbourhoods of overlaps of the chain (Figure 9). Outside the inverse image of the neighbourhoods, we take $H = r \circ f$. On the other hand, the restriction of $\mathcal{P}$ to each of these neighbourhoods is a CF with its corner smoothed, and hence we can identify the inverse image with a smoothing of $D^p \times D^q$. We take $H = |x|^2 + |y|^2, x \in D^p, y \in D^q$ in the inverse image, if $r \circ f$ restricted to the two components S_i, S_j in intersection take both maximum or both minimum in this neighbourhood, or $H = |x|^2 - |y|^2$, otherwise. The two manners of definition of H above is consistent, as seen by the normal form of the cusped fan projection in [K1, Theorem 2.2].
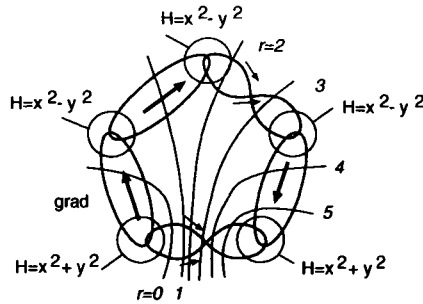
Fig. 9. An accompanying Morse function to a core

We give a picture of another example (Figure 10) that a planar portrait is produced by apparent contours of a core (refer to [K3] for details).

PROBLEM 3.5. *Is any planar portrait of a manifold "produced" by the loops which are apparent contours of core components ?*

For our case of the polygonal portraits, the word "produced" above is, in rough, modification of chains at each overlaps, from the right picture in Figure 8 to the middle one. But in other examples, it happens that more than three components of a core meet at a critical point of the accompanying Morse function H, and that some components may have no contribution to the critical loci, as seen in Figure 10. Hence the word "produced" should be defined to cover these situations.
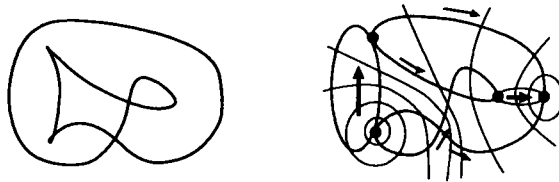

Fig. 10. A planar portrait and apparent contours of a core

4. Construction of polygonal portraits

A core $\{S_i\}$ of a manifold M is a *cyclic core* if each component has intersections with two other components. In the previous section, we have seen that: *If a manifold admits a polygonal portrait, then it has a cyclic core.* One can see the converse:

THEOREM 4.1. ([K5]) *If a manifold of dimension two or more admits a cyclic core, then it has a polygonal portrait whose number of cusps is the number of components of the core.*

Outline of the proof. Denote by M the manifold, and by N the tubular neighbourhood of the core. We first use the fact that M is decomposed into N and $S^{p-1} \times S^{q-1} \times D^2$ as mentioned before, where p and q are the two dimensions of the components of the cyclic core. Note that N is decomposed into k copies of $D^p \times D^q$ so that the transverse pair $D^p \times \{0\} \cup \{0\} \times D^q$ is identified with a subpiece of core components, where k is the number of intersection points of the core, which equals to the number of components. We construct the projection of N into $\mathbf{R}^2$ by using the cusped fan projections $D^p \times D^q \to \mathbf{R}^2$ as building blocks so that its image is an annulus and produces the same critical loci as a polygonal portrait. On the second piece $S^{p-1} \times S^{q-1} \times D^2$, we take the canonical projection onto D^2. They are well-pasted, as can be seen from the restrictions of the accompanying Morse function H to the core, to the boundary ∂N.

As mentioned, the theorem, together with Theorem 3.1 gives an answer to the realisation problem of polygonal portraits:

COROLLARY 4.2. *A manifold of dimension two or more admits a polygonal portrait if and only if it admits a cyclic core.*

It is but still unclear that what are the manifolds which admit cyclic cores, to which we show a little, as seen in Theorem 6.1 stated later. The theorem claims also that: an outline of a planar portrait is provided by apparent contours of core components. We note that these loops are in a special arrangement that is derived from a polygon. One can pose a problem on a generalisation of this result;

PROBLEM 4.3. *Find a condition on the arrangement of apparent contours of core components so that they give "outlines" of a planar portrait.*

An answer in a special case, including what "outlines" stands for, can be seen in [K3].

5. Toric 4-manifolds

A toric 4-manifold (a quasitoric 4-manifolds, in another terminology; refer to [BP]) is a 4-manifold with a locally standard T^2-action. It has a momentum map, or the quotient map of the action, onto a polygon Δ_k of some k. Divide Δ_k into k-rectangles by a barycentric subdivision, and according to it, divide M into k-pieces $M_i, i = 1, 2, \cdots, k$. Each piece is diffeomorphic to $D^2 \times D^2$ and the momentum map on a piece is given by $(|z|^2, |w|^2)$, where z, w are complex numbers in D^2 identified with the unit disc in $\mathbf{C}$. We replace the momentum map on M_i with the cusped fan projection of $\tau = 1$. Since the last map is a perturbation of $(|z|^2, |w|^2)$, one can thus obtain a global perturbation of the momentum map ([K1, example 7.3]),

which clearly produces a polygonal portrait. From the previously mentioned results and argument, we obtain the following:

COROLLARY 5.1. ([K4]) *A toric 4-manifold admits a polygonal portrait such that:*

1. *the number of cusps k is the total Betti number,*
2. *the core producing the portrait is the union of 1- and 0-dimensional orbits,*
3. *the map producing the portrait is a stable perturbation of the momentum map.*

Proof. One can obtain a core of M by pasting the transverse discs $D^2 \times \{0\} \cup \{0\} \times D^2$ in each M_i identified with $D^2 \times D^2$, as shown in section 3. It is a k collection of 2-spheres. Proposition 3.2, 3 shows that the Morse function $H : M \to \mathbf{R}$ which accompanies to the core has one critical point of index 0, another one of index 4, and $k-2$ points of index 2, which implies 1. Assertion 2 is clear, since the transverse two discs above is the union of 1 and 0-dimensional orbits of the standard T^2 action on $D^2 \times D^2$. Assertion 3 is just a repetition of what mentioned before. One can also apply Theorem 4.1 and its proof to show this, since the core above is cyclic. ∎

6. List of manifolds admitting polygonal portraits

Any polygonal pre-portrait can be written in either form $D(a_I)$ or $D_0(a_0)$ defined below, where $I = (1, 2, \cdots, s)$ is a multi-index of length $s \geq 1$, $a_I = (a_1, a_2, \cdots, a_s)$ a sequence of non-negative integers with a positive total $(\Sigma a_i > 0)$, and a_0 a positive integer. We denote by $D(a_I)$ the polygonal pre-portrait with s crossings that has $a_1, a_2, \cdots, a_s$ cusps between the adjoining two crossings, and by $D_0(a_0)$ that with two disjoint components of critical loci, having a_0 cusps on the inside loci (see Figure 11). We put here a list of manifolds which possibly admit either type of the polygonal portrait above.
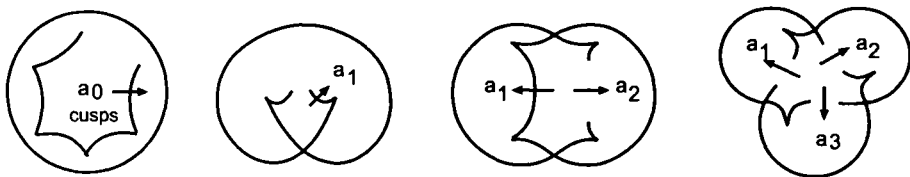

Fig. 11. $D_0(a_0)$, and $D(a_I)$ for $I = (a_1), (a_1, a_2)$ and (a_1, a_2, a_3)

THEOREM 6.1. ([K5])

1. *If an orientable n-manifold M $(n \geq 2)$ admits the polygonal portrait $D(a_I)$ for some I, then $k = \Sigma a_i \geq 2$ and M is either of the below, up to diffeomorphism:*

$$\begin{cases} \text{(a)} \quad S^n \natural_l (S^1 \times S^{n-1}) & (n \le 6) \quad or \\ \text{(b)} \quad S^n \natural_{i=1}^l (S^1 \times \Sigma_i^{n-1}) \natural \Sigma^n & (n \ge 7), \end{cases}$$

where Σ_i^{n-1} and Σ^n are homotopy spheres.

2. If an orientable n-manifold M of dimension $2, 3$ or 4 admits the polygonal portrait $D_0(a_0)$ for some a_0, then $a_0 \ge 2$ and it is either (a) in 1 above or (c) below, up to diffeomorphism;

$$\text{(c)} \quad \natural_l S^2 \times S^2 \natural_m \mathbf{C}P^2 \natural_n \overline{\mathbf{C}P^2}.$$

Note. In case dim $M = 4$ and the polygonal portrait is $D_0(a_0)$, the manifold M is (a), if folds between some cusps are definite, and is (c), otherwise.

The detection of M for $D_0(a_0)$ is much more difficult than for $D(a_I)$, since in the second case, half or more of the arcs between cusps are made by definite folds, which works effectively to simplify the argument. To cover the disadvantage, we put a technical assumption that dim $M = 2, 3, 4$ in the second assertion.

Outline of the proof. To show the first assertion, we reduce M together with the polygonal portrait one after another, to a manifold with $D(2)$ portrait. Each step of the reduction is to make a loop by elimination of cusps ([L1]), followed by removing of the loop by a surgery mentioned in [K2]. The source manifolds of $D(2)$ are homotopy spheres as seen easily, and then we trace the way backward, to the original manifold. The second assertion is clear for $n = 2$, and the proof for $n = 3$ is the same as in the previous assertion. For $n = 4$, we see that M is obtained by plumbing over a linear graph. Then the conclusion follows from Neumann-Weintraub [NW].

It is not confirmed yet but is likely that each manifold listed in the theorem admits the polygonal portraits of the prescribed type, by using Theorem 4.1. It gives affirmative answer to our realisation problem of $D(a_I)$ type strict portrait by n-manifolds, if it is the case. But for $D(a_0)$ type, the problem is still widely open, as mentioned for $a_0 = 3$.

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