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ON HOMOTOPY CLASSIFICATION OF PHRASES AND ITS APPLICATION

Abstract. This paper is a survey on the recent study of homotopy theory of generalized phrases in Turaev's theory of words and phrases. In this paper, we introduce the homotopy classification of generalized phrases with some conditions on numbers of letters. The theory of topology of words and phrases are closely related to the theory of surface-curves. We also introduce applications of topology of words to the topology of surface-curves.

1. Introduction

The theory of topology of words and phrases was introduced by V. Turaev in the papers [15] and [16]. In this paper, we introduce results on homotopy classification of nanophrases and étale phrases in the theory of topology of words and phrases which was given in [15], [2], [3] and [4].

The homotopy theory of words and phrases is related to the theory of virtual links and multi-strings (see [7] and [16]), free knot and links (see [9] and [13]). Moreover, Turaev's theory of words and phrases is also used in the studies of Arnold basic invariants and finite type invariants of curves and fronts (see [1], [10] and [11]), and categorification of the Jones polynomial (see [5]).

An alphabet is a finite set and a letter is an element of an alphabet. A word on an alphabet $\mathcal{A}$ is finite sequence of letters in $\mathcal{A}$ and a phrase on $\mathcal{A}$ is a finite sequence of words on $\mathcal{A}$.

Turaev introduced generalized words and phrases (which is called étale words and étale phrases) as follows. Let α be an alphabet endowed with an involution τ. Then an α-alphabet is a pair of an alphabet $\mathcal{A}$ and a map $|\cdot|$

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from $\mathcal{A}$ to α (we call this map $|\cdot|$ projection). Then an étale words is a pair of an α-alphabet $\mathcal{A}$ and a word on $\mathcal{A}$. Similarly, an étale phrases is a pair of an α-alphabet $\mathcal{A}$ and a phrase on $\mathcal{A}$. Especially, if a word (respectively phrase) is a Gauss word (respectively a Gauss phrases), in other words, each letter appear exactly twice, then we call an étale word (respectively an étale phrase) a nanoword (respectively a nanophrase).

Moreover Turaev defined an equivalence relation which is called homotopy on nanophrases and étale phrases (see Section 2 for more details). Further, Turaev showed homotopy of nanowords and nanophrases are closed related to the topology of curves on surfaces. Indeed, Turaev showed that for an alphabet α_0 given by $\{a, b\}$ with an involution τ_0 permuting a and b, the homotopy theory of words and the theory of stable equivalence of ordered pointed multi-curves on surfaces are equivalent. Thus as an application of the homotopy classification of nanophrases, we obtain the classification of stable equivalence classes of multi-curves on surfaces. Homotopy of nanowords and nanophrases is an equivalence relation. Thus, to classify the nanowords, nanophrases, étale words and étale phrases up to homotopy is natural problem. In the paper [15], Turaev gave the homotopy classification of nanowords with less than or equal to six letters. Furthermore, in the same paper, Turaev gave the homotopy classification of étale words with less than or equal to five letters.

The purpose of this paper is to introduce the results on the homotopy classification of nanophrase and étale phrases which was given by the author in [2], [3] and [4]. To classify nanophrases and étale phrases up to homotopy, the author constructed some homotopy invariants for nanophrases (see Section 4 for more details). Moreover we introduce applications of the homotopy classification of nanophrases and étale phrases to the theory of the stable equivalence of curves on surfaces. We classify the pointed ordered multi-component curves on surfaces up to stably equivalent using the result of homotopy classification of nanophrases.

This paper is organised as follows. In section 1, we review the theory of topology of words. In Section 2, we give the homotopy classification theorem of nanowords, nanophrases, étale words and étale phrases with some conditions which were given in [15], [2], [3] and [4]. In Section 3, we introduce homotopy invariants of nanophrases which was introduced by the author and A. Gibson in [2], [3], [4] and [8]. In Section 4, using the results in Section 2, we classify the ordered pointed multi-curves on surfaces up to stably equivalent.

2. Topology of words and phrases

In this section, we review the theory of topology of words and phrases which was introduced by V. Turaev in [15] and [16].

2.1. Nanowords, nanophrases and their homotopy

In this paper, an *alphabet* means a finite set and *letters* are its elements. For a positive integer n, a *word* of length n on an alphabet $\mathcal{A}$ is a mapping $w : \{1, 2, \ldots, n\} \to \mathcal{A}$. We denote a word of length n as follows: $w(1)w(2)\cdots w(n)$. If all letters in an alphabet $\mathcal{A}$ appear exactly twice in a word w, then we call the word w is a *Gauss word.* So if $\mathcal{A}$ has n elements, then length of a Gauss word on $\mathcal{A}$ is $2n$ (C. F. Gauss studied plane curves by using Gauss words. See [6]).

Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. An *α-alphabet* is an alphabet $\mathcal{A}$ endowed with a map $|\cdot| : \mathcal{A} \to \alpha$. We call this map $|\cdot|$ a *projection.* An isomorphism of α-alphabets $\mathcal{A}_1$, $\mathcal{A}_2$ is a bijection $f : \mathcal{A}_1 \to \mathcal{A}_2$ such that $|A|$ is equal to $|f(A)|$.

Now we define étale phrases and nanophrases. An *étale word* over α is a pair (an α-alphabet $\mathcal{A}$, a word on $\mathcal{A}$). A *nanoword* over α is a pair (an α-alphabet $\mathcal{A}$, a Gauss word on $\mathcal{A}$). We call nanoword in an empty α-alphabet the *empty nanoword* and we denote this empty word $\emptyset$. For a positive integer k, an *étale phrase* of length k over α is a pair consisting of an α-alphabet $\mathcal{A}$ and a sequence of words $w_1, w_2, \ldots, w_k$ on $\mathcal{A}$. We denote this étale phrase by ($\mathcal{A}$, $(w_1|w_2|\cdots|w_k)$), or we denote it simply by $(w_1|w_2|\cdots|w_k)$. A *nanophrase* of length k over α is an étale phrase ($\mathcal{A}$, $(w_1|\cdots|w_k)$) such that $w_1\cdots w_k$ is a Gauss word. Note that we can consider a nanoword w to be a nanophrase (w) of length 1. A map $f : \mathcal{A}_1 \to \mathcal{A}_2$ is an *isomorphism* of two nanophrases ($\mathcal{A}_1$, $(w_1|\cdots|w_k)$) and ($\mathcal{A}_2$, $(v_1|\cdots|v_k)$) if f is an isomorphism of α-alphabets such that v_j is equal to $f \circ w_j$ for all $j \in \{1, \ldots, k\}$.

Next we define homotopy moves of nanophrases.

Definition 2.1. We define *homotopy moves* (1)–(3) of nanophrases as follows:

(1) $(\mathcal{A}, (xAAy)) \longrightarrow (\mathcal{A} \setminus \{A\}, (xy))$
for all $A \in \mathcal{A}$ and x, y are sequences of letters in $\mathcal{A} \setminus \{A\}$, possibly including the | character.

(2) $(\mathcal{A}, (xAByBAz)) \longrightarrow (\mathcal{A} \setminus \{A, B\}, (xyz))$
if $A, B \in \mathcal{A}$ satisfy $|B| = \tau(|A|)$. x, y, z are sequences of letters in $\mathcal{A} \setminus \{A, B\}$, possibly including | character.

(3) $(\mathcal{A}, (xAByACzBCt)) \longrightarrow (\mathcal{A}, (xBAyCAzCBt))$
if $A, B, C \in \mathcal{A}$ satisfy $|A| = |B| = |C|$. x, y, z, t are sequences of letters in $\mathcal{A}$, possibly including | character.

Now we define homotopy of nanophrases.

Definition 2.2. Two nanophrases $(\mathcal{A}_1, P_1)$ and $(\mathcal{A}_2, P_2)$ over α are *homotopic* (denoted $(\mathcal{A}_1, P_1) \simeq (\mathcal{A}_2, P_2)$) if $(\mathcal{A}_2, P_2)$ can be obtained from $(\mathcal{A}_1, P_1)$ by a finite sequence of isomorphism, homotopy moves (1)–(3) and the inverse of moves (1)–(3).

Remark 2.3. In [15] and [16], Turaev gave more general definition of homotopy moves of nanowords and nanophrases (called S-homotopy moves). See [15] and [16] for more detail.

A nanoword which is homotopic to the empty nanoword is called *contractible* nanoword. We denote the set of homotopy classes of nanophrases over α of length k by $\mathcal{P}_k(\alpha)$.

Recall following lemmas from [16].

Lemma 2.4. (V. Turaev [16]) *Let $\mathcal{A}$ be an α-alphabet. Then*

(i) $(\mathcal{A}, (xAByACzBCt)) \simeq (\mathcal{A}, (xBAyCAzCBt))$ *if* $|A| = \tau(|B|) = |C|$,
(ii) $(\mathcal{A}, (xAByCAzBCt)) \simeq (\mathcal{A}, (xBAyACzCBt))$ *if* $\tau(|A|) = \tau(|B|) = |C|$,
(iii) $(\mathcal{A}, (xAByACzCBt)) \simeq (\mathcal{A}, (xBAyCAzBCt))$ *if* $\tau(|A|) = |B| = |C|$,

where x, y, z, t are sequences of letters in $\mathcal{A} \setminus \{A, B, C\}$, possibly including | character.

Lemma 2.5. (V. Turaev [16]) *Let $\mathcal{A}$ be an α-alphabet. Then*

$$(\mathcal{A}, (xAByABz)) \simeq (\mathcal{A} \setminus \{A, B\}, (xyz))$$

if $|A| = \tau(|B|)$; where x, y, z are words in $\mathcal{A} \setminus \{A, B\}$ possibly including the | character.

We define homotopy of étale phrases via desingularization of étale phrases. For an étale phrase $(\mathcal{A}, P = (w_1|\cdots|w_k))$ over α, we define *desingularization* of $(\mathcal{A}, P)$ as follows: Let $\mathcal{A}^d$ be an α-alphabet $\{A_{i,j} := (A, i, j) \mid A \in \mathcal{A},\ 1 \leq i < j \leq m_P(A)\}$ with the projection $|A_{i,j}| := |A|$ for all $A_{i,j}$ where $m_P(A)$ is defined by $Card((w_1 \cdots w_k)^{-1}(A))$. The phrase P^d is obtained from P by first deleting all $A \in \mathcal{A}$ with $m_P(A)$ is less than or equal to one. Then for each $A \in \mathcal{A}$ with $m_P(A)$ is greater than or equal to two and each $i = 1, 2, \ldots m_P(A)$, we replace the i-th entry of A in P by

$$A_{1,i}A_{2,i} \ldots A_{i-1,i}A_{i,i+1}A_{i,i+2} \ldots A_{i,m_P(A)}.$$

The resulting $(\mathcal{A}^d, P^d)$ is a nanophrase with $\sum_{A \in \mathcal{A}} m_P(A)(m_P(A)-1)$ letters and called a *desingularization of* $(\mathcal{A}, P)$.

Then we define two étale phrases are *homotopic as étale phrases* if their desingularizations are homotopic as nanophrases.

3. Homotopy classification of nanophrases and étale phrases

In this section, we introduce the classification theorems of nanophrases and étale phrases. Geometric applications of the classification of nanophrases is stated in Section 5.

For nanowords, Turaev gave the classification of nanowords with less than or equal to six letters in [15]. In this paper we introduce classification theorem of nanowords with less than or equal to four letters.

THEOREM 3.1. (V. Turaev [15]) *Let w be a nanoword of length four over α. Then w is either homotopic to the empty nanoword or isomorphic to the nanoword $w_{a,b} := (\mathcal{A} = \{A, B\}, ABAB)$ where $|A| = a, |B| = b \in \alpha$ with $a \neq \tau(b)$. Moreover for $a \neq \tau(b)$, the nanoword $w_{a,b}$ is non-contractible and two nanowords $w_{a,b}$ and $w_{a',b'}$ are homotopic if and only if $a = a'$ and $b = b'$.*

Moreover Turaev gave the classification of étale words with less than or equal to five letters in [15].

THEOREM 3.2. (V. Turaev [15]) *A multiplicity-one-free word of length less than or equal to four in the alphabet α has one of the following forms: aa, aaa, aaaa, aabb, abba, abab with distinct $a, b \in \alpha$. The words aa, aabb, abba are contractible. The words aaa and aaaa are contractible if and only if $\tau(a) = a$. The word abab is contractible if and only if $\tau(a) = b$. Non-contractible words of type aaa, aaaa and abab are homotopic if and only if they are equal.*

For nanophrases, using the Theorem 3.1 and some homotopy invariants of nanophrases which are introduced in the next section the author gave the classification of nanophrases with less than or equal to four letters.

First we describe the classification theorem of nanophrases with less than or equal to two letters. Set $P_a^{1,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{A}}|\emptyset|\cdots|\emptyset)$ with $|A|$ is equal to a for $1 \leq l_1 < l_2 \leq k$.

THEOREM 3.3. ([3]) *Let P be a nanophrase of length k with two letters. Then P is either homotopic to $(\emptyset|\cdots|\emptyset)$ or isomorphic to $P_a^{1,1;l_1,l_2}$ for some $l_1, l_2 \in \{1, \ldots, k\}$, $a \in \alpha$. Moreover $P_a^{1,1;l_1,l_2}$ and $P_{a'}^{1,1;l'_1,l'_2}$ are homotopic if and only if $l_1 = l'_1$, $l_2 = l'_2$ and $a = a'$.*

To describe the classification theorem of nanophrases with less than or equal to four letters, we prepare following notations.

$$P_{a,b}^{4;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{A}BAB}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{3,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{ABA}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{2,2I;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}B}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{A}B}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{2,2II;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}B}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}A}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,3;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{B\check{A}B}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{2,1,1I;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}B}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{2,1,1II;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{B}A}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,2,1I;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{A}B}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,2,1II;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}A}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,1,2I;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{A}B}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,1,2II;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}A}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,1,1,1I;l_1,l_2,l_3,l_4} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{l_4}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,1,1,1II;l_1,l_2,l_3,l_4} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_4}{\check{B}}|\emptyset|\cdots|\emptyset),$$

$$P_{a,b}^{1,1,1,1III;l_1,l_2,l_3,l_4} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{A}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{B}}|\emptyset|\cdots|\emptyset|\overset{l_4}{\check{A}}|\emptyset|\cdots|\emptyset),$$

with $|A|$ is equal to a and $|B|$ is equal to b. If a is equal to $\tau(b)$, then nanophrases $P_{a,b}^{4;l}$, $P_{a,b}^{2,2I;l_1,l_2}$ and $P_{a,b}^{2,2II;l_1,l_2}$ are homotopic to the nanophrase $(\emptyset|\cdots|\emptyset)$. So when we write $P_{a,b}^{4;l}$, $P_{a,b}^{2,2I;l_1,l_2}$, $P_{a,b}^{2,2II;l_1,l_2}$ we always assume that a is not equal to $\tau(b)$.

Under the above preparation, we can describe the classification theorem as follows.

THEOREM 3.4. ([3]) *Let P be a nanophrase of length k with four letters. Then P is either homotopic to nanophrase with less than or equal to two letters or isomorphic to $P_{a,b}^{X;Y}$ for some $X \in \{4,(3,1),\ldots,(1,1,1,1III)\}$, $Y \in \{1,\ldots,k,(1,2),\ldots,(k-3,k-2,k-1,k)\}$. Moreover $P_{a,b}^{X;Y}$ and $P_{a',b'}^{X';Y'}$ are homotopic if and only if $X = X'$, $Y = Y'$, $a = a'$ and $b = b'$.*

Next we introduce the classification of étale phrases with less than or equal to three letters which was given by the author in [4]. First we prepare some notations. Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. Then we set

$$P_a^{1,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset),$$

$$P_a^{3;l} := (\emptyset|\cdots|\emptyset|\overset{l}{\check{a}^3}|\emptyset|\cdots|\emptyset),$$
$$P_a^{2,1;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}^2}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset),$$
$$P_a^{1,2;l_1,l_2} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_2}{\check{a}^2}|\emptyset|\cdots|\emptyset),$$
$$P_a^{1,1,1;l_1,l_2,l_3} := (\emptyset|\cdots|\emptyset|\overset{l_1}{\check{a}}|\emptyset|\cdots|\overset{l_2}{\check{a}}|\emptyset|\cdots|\emptyset|\overset{l_3}{\check{a}}|\emptyset|\cdots|\emptyset),$$

where $a \in \alpha$ and $l, l_1, l_2, l_3 \in \{1, \dots, k\}$ with $l_1 < l_2 < l_3$. If a is equal to $\tau(a)$, then we can easily check that $P_a^{3;l}$ is homotopic to the nanophrase $(\emptyset|\cdots|\emptyset)$. So when we use the notation $P_a^{3;l}$, we always assume that a is not equal to $\tau(a)$.

Then the classification of étale phrases with less than or equal to three letters is described as follows.

THEOREM 3.5. ([4]) *Let P be a multiplicity-one-free étale phrase over α with less than or equal to three letters. Then P is either homotopic to $(\emptyset|\dots|\emptyset)$ or isomorphic to one of the following étale phrases: $P_a^{1,1;l_1,l_2}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$, $P_a^{1,1,1;l_1,l_2,l_3}$ for some l_1, l_2, $l_3 \in \{1, \dots, k\}$ and $a \in \alpha$. Moreover $P_a^{1,1;l_1,l_2}$, $P_a^{3;l}$, $P_a^{2,1;l_1,l_2}$, $P_a^{1,2;l_1,l_2}$, $P_a^{1,1,1;l_1,l_2,l_3}$ are homotopic if and only if they are equal.*

In the next section, we introduce invariants of nanophrases which was defined in papers [2], [3] and [8].

4. Homotopy invariants of nanophrases

In this section we discuss homotopy invariants of nanophrases defined by the author in [2] and [3] and A. Gibson in [8] independently at the same time.

4.1. Some simple invariants

Let $(w_1|w_2|\cdots|w_k)$ be a nanophrase over α. For $l \in \{1, \dots, k\}$, we define $w(l) \in \mathbb{Z}/2\mathbb{Z}$ by the length of w_l. We call the vector $(w(1), \dots, w(k)) \in (\mathbb{Z}/2\mathbb{Z})^k$ the *component length vector.*

PROPOSITION 4.1. (A. Gibson [8], see also [2]) *The component length vector is a homotopy invariant of nanophrases.*

REMARK 4.2. Gibson proved the component length vector is also invariant under the shift move. See [8] for more detail.

EXAMPLE 4.3. Consider a nanophrases $(A|A)$ over α. Then the component length vector of this nanophrase is $(1, 1)$. On the other hand, the component length vector of the nanophrase $(\emptyset|\emptyset)$ is $(0, 0)$. So $(A|A)$ is not homotopic to $(\emptyset|\emptyset)$ by Proposition 4.1.

Next we define another invariant of nanophrases. Let π be the group which is defined as follows:

$$\pi := (a \in \alpha | a\tau(a) = 1, ab = ba \ for \ all \ a, b \in \alpha \ ).$$

Let $(w_1|w_2|\cdots|w_k)$ be a nanophrase of length k over α. We define $(w_i, w_j)_P \in \pi$ for $i < j$ by

$$(w_i, w_j)_P := \prod_{A \in Im(w_i) \cap Im(w_j)} |A|.$$

We call a vector $((w_i, w_j)_P)_{i<j} \in \pi^{\frac{1}{2}k(k-1)}$ the *linking vector.*

PROPOSITION 4.4. ([3]) *The linking vector of nanophrases is a homotopy invariant of nanophrases.*

EXAMPLE 4.5. Consider a nanophrase $(AB|AC|BC)$ over α with $|A|$ is equal to a, $|B|$ is equal to b and $|C|$ is equal to c. Then the linking vector of this nanophrase is (a, b, c). On the other hand, the linking vector of the nanophrase $(\emptyset|\emptyset|\emptyset)$ is $(1, 1, 1)$. In the group π, an element $a \in \pi$ is not equal to the unit element of π. So (a, b, c) is not equal to $(1, 1, 1)$. Therefore $(AB|AC|BC)$ and $(\emptyset|\emptyset|\emptyset)$ are not homotopic each other. Note that we can not distinguish these two nanophrases by the component length vector of nanophrases.

REMARK 4.6. In [8], Gibson defined an equivalent invariant for nanophrases over the one-element set and proved this invariant is also invariant under the shift move.

4.2. Invariant γ for nanophrases

In [15], Turaev defined a homotopy invariant of nanowords called γ. The author extended this invariant for nanophrases. Let Π be the group which is defined as follows:

$$\Pi := (\{z_a\}_{a \in \alpha} | z_a z_{\tau(a)} = 1 \ for \ all \ a \in \alpha).$$

DEFINITION 4.7. Let $P = (w_1|w_2|\cdots|w_k)$ be a nanophrase of length k over α and n_i the length of nanoword w_i. Set $n = \sum_{1 \leq i \leq k} n_i$. Then we define n elements $\gamma_1^i, \gamma_2^i, \cdots$, and $\gamma_{n_i}^i$ for $i \in \{1, \ldots, k\}$ of Π by $\gamma_i^j := z_{|w_j(i)|}$ if $w_j(i) \neq w_l(m)$ for all $l < j$ and for all $m < i$ when $l = j$. Otherwise $\gamma_i^j := z_{\tau(|w_j(i)|)}$. Then we define $\gamma(P) \in \Pi^k$ by

$$\gamma(P) := (\gamma_1^1 \gamma_2^1 \cdots \gamma_{n_1}^1, \gamma_1^2 \gamma_2^2 \cdots \gamma_{n_2}^2, \cdots, \gamma_1^k \gamma_2^k \cdots \gamma_{n_k}^k).$$

Then we obtain a following proposition.

PROPOSITION 4.8. ([2]) *γ is a homotopy invariant of nanophrases.*

Example 4.9. Consider a nanophrase $(AB|BA)$ with $|A| \neq \tau(|B|)$. Then $\gamma((AB|BA))$ is $(z_a z_b, z_{\tau(b)} z_{\tau(a)})$. This is not equal to $(1,1)$ in Π^2. So $(AB|BA)$ is not homotopic to $(\emptyset|\emptyset)$.

4.3. Invariant T for nanophrases

We define an invariant of nanophrases T. First we prepare some notations. Consider an orbit decomposition of the τ : $\alpha/\tau = \{\widehat{a_{i_1}}, \widehat{a_{i_2}}, \ldots, \widehat{a_{i_l}}, \widehat{a_{i_{l+1}}}, \ldots, \widehat{a_{i_{l+m}}}\}$, where $\widehat{a_{i_j}} := \{a_{i_j}, \tau(a_{i_j})\}$ such that $Card(\widehat{a_{i_j}}) = 2$ for all $j \in \{1, \ldots, l\}$ and $Card(\widehat{a_{i_j}}) = 1$ for all $j \in \{l+1, \ldots, l+m\}$ (we fix a complete representative system $\{a_{i_1}, a_{i_2}, \ldots, a_{i_l}, a_{i_{l+1}}, \ldots, a_{i_{l+m}}\}$ which satisfy the above condition). Let $\mathcal{A}$ be an α-alphabet. For $A \in \mathcal{A}$ we define $\varepsilon(A) \in \{\pm 1\}$ by

$$\varepsilon(A) := \begin{cases} 1 & (\text{if } |A| = a_{i_j} \text{ for some } j \in \{1, \ldots, l+m\}), \\ -1 & (\text{if } |A| = \tau(a_{i_j}) \text{ for some } j \in \{1, \ldots l\}). \end{cases}$$

Let $P = (\mathcal{A}, (w_1|\cdots|w_k))$ be a nanophrase over α and A, $B \in \mathcal{A}$. Let $K_{(i,j)}$ be $\mathbb{Z}$ if $i \leq l$ and $j \leq l$, otherwise $\mathbb{Z}/2\mathbb{Z}$. We denote $K_{(1,1)} \times K_{(1,2)} \times \cdots K_{(1,l+m)} \times K_{(2,1)} \times \cdots \times K_{(l+m,l+m)}$ by $\prod K_{(i,j)}$. Then we define $\sigma_P(A,B) \in \prod K_{(i,j)}$ as follows: If A and B form $\cdots A \cdots B \cdots A \cdots B \cdots$ in P, $|A| \in \widehat{a_{i_p}}$and $|B| = a_{i_q}$ for some $m, n \in \{1, \ldots, l+m\}$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P, $|A| \in \widehat{a_{i_p}}$ and $|B| = \tau(a_{i_q})$ for some $p, q \in \{1, \ldots, l+m\}$, then $\sigma_P(A,B) := (0, \cdots, 0, \overset{(p,q)}{\check{1}}, 0, \cdots, 0)$. If $\cdots A \cdots B \cdots A \cdots B \cdots$ in P, $|A| \in \widehat{a_{i_p}}$ and $|B| = \tau(a_{i_q})$, or $\cdots B \cdots A \cdots B \cdots A \cdots$ in P, $|A| \in \widehat{a_{i_p}}$ and $|B| = a_{i_q}$, then $\sigma_P(A,B) := (0, \cdots, 0, \overset{(p,q)}{\check{-1}}, 0, \cdots, 0)$. Otherwise $\sigma_P(A,B) := (0, \ldots, 0)$.

Under the above preparation, we define the invariant T as follows.

Definition 4.10. Let $P = (\mathcal{A}, (w_1|w_2|\cdots|w_k))$ be a nanophrase of length k over α. For $A \in \mathcal{A}$ such that there exist $i \in \{1, 2, \cdots, k\}$ with $\text{Card}(w_i^{-1}(A)) = 2$, we define $T_P(A) \in \prod K_{i,j}$ by

$$T_P(A) := \varepsilon(A) \sum_{B \in \mathcal{A}} \sigma_P(A,B),$$

and $T_P(w_i) \in \prod K_{i,j}$ by

$$T_P(w_i) := \sum_{A \in \mathcal{A},\, \text{Card}(w_i^{-1}(A))=2} T_P(A).$$

Then we define $T(P) \in (\prod K_{i,j})^k$ by

$$T(P) := (T_P(w_1), T_P(w_2), \cdots, T_P(w_k)).$$

Proposition 4.11. ([3]) *T is a homotopy invariant of nanophrases over α.*

Example 4.12. Set $\alpha = \{a, b\}$ with involution τ permuting a and b. We choose $a \in \alpha$ as the representative element of the orbit. Consider nanophrases $(ABA|B)$ and $(B|ABA)$ over α with $|A|$ is equal to a and $|B|$ is equal to b. Then $T((ABA|B))$ is $(-1, 0) \in \mathbb{Z}^2$. On the other hand $T((A|BAB))$ is $(0, -1) \in \mathbb{Z}^2$. This implies $(ABA|B)$ is not homotopic to $(B|ABA)$.

Remark 4.13. In [8], Gibson constructed a new homotopy invariant of nanophrases over the one-element set called S_o invariant, and proved Gibson's S_o invariant is stronger than the invariant T for nanophrases over the one-element set. See [8] for more details.

Using these invariants and some consideration concerning nanophrase, we can prove Theorem 3.4 (see [2] and [3] for more details).

5. Applications to topology of surface-curves

In this section, we introduce applications of Theorems 3.3 and 3.4 to the topology of surface-curves. First, we review the stable equivalence of multi-component surface-curves.

5.1. Stable equivalence of multi-component surface-curves

In this paper a *curve* means the image of a generic immersion of an oriented circle into an oriented surface. The word "generic" means that the curve has only a finite set of self-intersections which are all double and transversal. A *k-component curve* is defined in the same way as a curve with the difference that they may be formed by k curves. These curves are called *components* of the k-component curve. A k-component curve are *pointed* if each component is endowed with a base point (the origin) distinct from the crossing points of the k-component curve. A k-component curve is *ordered* if its components are numerated.

Next we define equivalence relation which is called stable equivalence. First, we define stably homeomorphic of surface-curves. Two ordered, pointed curves are *stably homeomorphic* if there is an orientation preserving homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first multi-component curve onto the second one and preserving the order, the origins, and the orientations of the components.

Now two ordered, pointed multi-component curves are *stably equivalent* if they can be related by a finite sequence of the following transformations: (i) a move replacing a ordered, pointed multi-component curve with a stably homeomorphic one; (ii) the flattened Reidemeister moves away from the origin as in Fig. 1. See also [12].

We denote the set of stable equivalence classes of ordered, pointed k-component curves by $\mathcal{C}_k$.

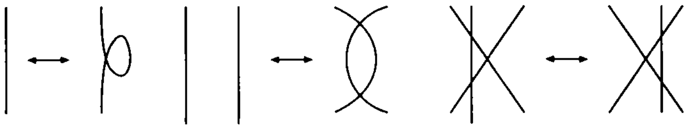

Fig. 1. The flattened Reidemeister moves.

5.2. Nanophrases versus multi-component surface-curves

In [16], Turaev proved the study of stable equivalence classes of ordered, pointed, k-component curves is equivalent to the study of homotopy of nanophrases of length k over an alphabet $\alpha_0 = \{a, b\}$ with an involution $\tau : \alpha_0 \to \alpha_0$ permuting a and b. More precisely, Turaev showed a following theorem.

THEOREM 5.1. (V.Turaev [16]) *Let α_0 be the set $\{a, b\}$ with an involution $\tau : \alpha_0 \to \alpha_0$ permuting a and b. Then there is a canonical bijection C_k to $\mathcal{P}_k(\alpha_0)$.*

We explain the method of making a nanophrase $P(C)$ over α_0 from an ordered, pointed k-component curve C. Let us label the double points of C by distinct letters $A_1, \dots, A_n$. Starting at the base point of the first component of the curve C and following along C in the direction of C, we write down the labels of double points which we passes until return to the base point. Then we obtain a word w_1 on an alphabet $\mathcal{A} = \{A_1, \dots, A_n\}$. Similarly we obtain words $w_2, \dots, w_k$ on $\mathcal{A}$ from the second component, ..., the k-th component. Let t_i^1 (respectively, t_i^2) be the tangent vector to C at the double point which is labeled A_i appearing at the first (respectively, the second) passage through this point. Set $|A_i|$ is equal to a, if the pair (t_i^1, t_i^2) is positively oriented, and $|A_i|$ is equal to b otherwise. Then we obtain an α_0-alphabet $\mathcal{A}$. Finally we obtain a required nanophrase $P(C) := (\mathcal{A}, (w_1 | \cdots | w_k))$.

EXAMPLE 5.2. Consider a two-component pointed ordered curve showed in Fig. 2. Assume that a left circle is the first component of this curve and a right circle is the second component of this curve. Then a nanophrase which corresponds to this curve is $(\{A, B\}, (AB|AB))$ with $|A|$ is equal to b and $|B|$ is equal to a.

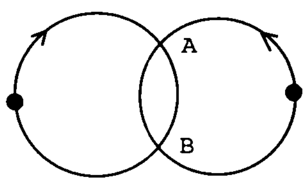

Fig. 2. Example

By the above theorem, if we classify nanophrases of length k up to homotopy, then we obtain the classification of the stable equivalence classes of ordered, pointed k-component curves as a corollary.

REMARK 5.3. Note that the homotopy theory of nanowords over α_0 can be identified to the theory of open virtual strings which was defined by Turaev in [18]. See also [7] and [14].

5.3. Classification of stable equivalence classes of multi-component surface-curves

In this subsection we classify stable equivalence classes of ordered, pointed, multi-component surface-curves with minimal crossing number less than or equal to two using the theory of words. By Theorem 5.1, stable equivalence classes of ordered, pointed, 2 (respectively 3, 4) component surface-curves is one to one corresponds to nanophrases of length 2 (respectively 3, 4) over α_0 with less than or equal to 4 letters. So if we apply Theorem 3.4 for the case of α is equal to α_0, then we obtain following corollaries of the classification theorems of nanophrases.

COROLLARY 5.4. ([2]) *There are exactly* 19 *stable equivalence classes of ordered, pointed,* 2 *component surface curves with minimal crossing number less than or equal to* 2.

COROLLARY 5.5. *There are exactly* 73 *stable equivalence classes of ordered, pointed,* 3 *component surface curves with minimal crossing number less than or equal to* 2.

COROLLARY 5.6. *There are exactly* 201 *stable equivalence classes of ordered, pointed,* 4 *component surface curves with minimal crossing number less than or equal to* 2.

More generally we can prove a following statement.

COROLLARY 5.7. *Let k be an positive integer. Then there are exactly*

$$1 + \frac{1}{2}k^2 + k^3 + \frac{1}{2}k^4$$

stable equivalence classes of ordered, pointed, k-component surface curves with minimal crossing number less than or equal to two.

Proof. In this proof, for $X \in \{(1,1), 4, (3,1), \dots, (1,1,1,1III)\}$, *an ordered, pointed surface-curves C is type P^X* means a nanophrase arise from C is homotopic to $P_a^{X;Y}$ or $P_{a,b}^{X;Y}$ for some $a, b \in \alpha_0$, $Y \in \{1, \dots, k, (1,2), \dots, (k-3, k-2, k-1, k)\}$. We denote the number of stable equivalence classes of k-component ordered, pointed surface-curves of type P^X by $N(P^X)$. By Theorems 3.3, 3.4 and 5.1, $N(P^{1,1}) = N(P^{2,2I}) = N(P^{2,2II}) = k(k-1)$, $N(P^4) = 2k$, $N(P^{3,1}) = N(P^{1,3}) = 2k(k-1)$, $N(P^{2,1,1I}) = N(P^{2,1,1II}) =$

$N(P^{1,2,1I}) = N(P^{1,2,1II}) = N(P^{1,1,2I}) = N(P^{1,1,2II}) = \frac{2}{3}k(k-1)(k-2)$,
$N(P^{1,1,1,1I}) = N(P^{1,1,1,1II}) = N(P^{1,1,1,1III}) = \frac{1}{6}k(k-1)(k-2)(k-3)$.
Moreover, the number of stable equivalence classes of ordered, pointed, k-component surface curves with minimal crossing number less than or equal to two is equal to $1 + \sum_X N(P^X)$ where X runs over

$$\{(1,1), 4, (3,1), \ldots, (1,1,1,1III)\}.$$

So we obtain the claim of the corollary. ∎

5.4. Classification of stable equivalence classes of irreducible surface-curves

In this paper, an ordered pointed multi-component surface-curve is *irreducible* if it is not stably equivalent to a surface-curve with a simple closed component. In this subsection, we give the classification of irreducible ordered, pointed surface-curves with minimal crossing number less than or equal to two up to stably equivalent using Turaev's theory of words and phrases. First we prepare a following lemma.

LEMMA 5.8. ([3]) *The nanophrases over* α, $(A|A)$, $(AB|AB)$ *with* $|A| \neq \tau(|B|)$, $(AB|BA)$ *with* $|A| \neq \tau(|B|)$, $(ABA|B)$, $(A|BAB)$, $(AB|A|B)$, $(BA|A|B)$, $(A|AB|B)$, $(A|BA|B)$, $(A|B|AB)$, $(A|B|BA)$, $(A|A|B|B)$, $(A|B|A|B)$ *and* $(A|B|B|A)$ *are not homotopic to nanophrases over* α *which have the empty words in its components.*

Now by Theorem 3.4 and Lemma 5.8, we obtain a following corollary.

COROLLARY 5.9. ([3]) *Any irreducible ordered, pointed multi-component surface-curve with minimal crossing number less than or equal to two is stably equivalent to one of the ordered, pointed multi-component curves arise from the following list (see also Remark* 5.10*). Moreover two different pictures from Fig.* 3 *never produce equivalent ordered, pointed multi-component surface-curves. There are* 2 *(respectably* 2, 8, 4, 24, 12*) different ordered, pointed multi-component surface-curves arise from upper left (respectably upper middle, upper right, lower left, lower middle, lower right) picture. So there are exactly* 52 *stable equivalence classes of irreducible ordered, pointed, multi-component surface-curves.*

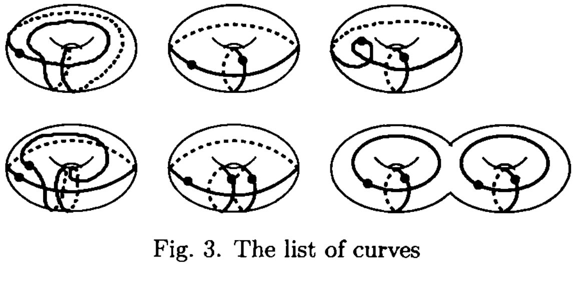

Fig. 3. The list of curves

Remark 5.10. We want to list up the stable equivalence classes of irreducible ordered, pointed multi-component surface-curves with minimal crossing number less than or equal to two. However there are too many curves to list up. So in Fig. 3 we make just the list of multi-component curves without orders and orientations of the components. If we choose an order and an orientations, then we obtain an ordered, pointed multi-component curve.

Remark 5.11. To find application of the classification of êtale words and êtale phrases are problem in future.

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