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SPACELIKE SUBMANIFOLDS IN DE SITTER SPACE

Abstract. We investigate the differential geometry of spacelike submanifolds of codimension at least two in de Sitter space as an application of the theory of Legendrian singularities. We also discuss related geometric property of spacelike hypersurfaces in de Sitter space.

1. Introduction

It is known that de Sitter space is a Lorentzian space form with a positive curvature. Recently, Izumiya, Pei and Sano [3] investigated the extrinsic differential geometry of hypersurfaces in the hyperbolic space by applying the theory of Legendrian singularities. The main tool is a lightcone Gauss indicatrix, which is defined by a lightlike normal of hypersurface, and their singularity sets correspond to lightcone parabolic sets of hypersurfaces. For higher codimension case, the normal vector is not uniquely determined, however it is possible to construct hypersurfaces from normal unit vector fields of the subspace. Izumiya, Pei, Romero Fuster and Takahashi [6] introduced the notion of canal hypersurfaces and horospherical hypersurfaces from the normal frames of submanifolds in the hyperbolic space, and investigated submanifolds of higher codimension in the hyperbolic space from the viewpoint of singularity theory. On the other hand, the differential geometry of de Sitter space is also studied. In [7] we introduced the notion of lightcone Gauss image which is an analogous tool introduced in [3], and investigate the case of spacelike hypersurface in de Sitter space. For codimension two case, Fusho and Izumiya [2] firstly introduced the notion of lightlike surface of a spacelike curve in the de Sitter three-space. In [8] we investigated singularities of lightlike hypersurface of spacelike submanifold of codimension

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two in de Sitter n-space for $n \geq 3$ by using the lightlike normal direction, which is an analogous study in the Minkowski space [4, 5].

In this paper, we argue an analogous study of the submanifolds of higher codimension in hyperbolic space [6] and introduce the notions of horospherical hypersurfaces and spacelike canal hypersurfaces by using timelike unit normal vector fields. The singular point of horospherical surface corresponds to the parabolic point of spacelike canal hypersurface, which we call a horospherical point, and the spacelike submanifold is tangent to a de Sitter hyperhorosphere at the horospherical point. If we assume a hypothesis of Theorem 6.5, then a contact type of a de Sitter hyperhorosphere and a spacelike submanifold corresponds to a singular type of horospherical hypersurface, and also corresponds to a singular type of lightcone Gauss image of spacelike canal surface. In this paper we consider timelike normal direction of spacelike submanifolds, so that this study is not a generalization of [8, 9]. In §2 we review briefly the basic notions of differential geometry of spacelike hypersurfaces [7]. In §3, 4 we define a timelike normal vector field of spacelike submanifolds in de Sitter space and introduce a notion of horospherical height function and horospherical hypersurface. We also define a spacelike canal hypersurface, whose lightcone Gauss image is diffeomorphic to a horospherical hypersurface. In §5 we naturally interpret a horospherical hypersurfaces of a spacelike submanifold as a wave front set of horospherical height functions in the theory of Legendrian singularities. In §6 we use the theory of contacts between the submanifolds due to Montaldi [10], and we discuss geometric properties of singularities of horospherical hypersurfaces. We also consider generic properties of spacelike submanifolds.

2. Spacelike hypersurfaces in de Sitter space

In this section we review the extrinsic differential geometry of spacelike hypersurfaces in de Sitter space [7], which is an analogous study of [3]. Let $\mathbb{R}^{n+1} = \{\mathbf{x} = (x_0, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, \dots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\mathbf{x} = (x_0, \dots, x_n)$, $\mathbf{y} = (y_0, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$. We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ a *Minkowski* $(n+1)$*-space* and write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$.

We say that a vector $\mathbf{x} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For a vector $\mathbf{v} \in \mathbb{R}^{n+1}_1 \setminus \{\mathbf{0}\}$ and a real number c, we define a *hyperplane with pseudo normal* $\mathbf{v}$ in the Minkowski space by $\mathrm{HP}(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. We say that a hyperplane $HP(\mathbf{v}, c)$ is spacelike, timelike or lightlike if the vector $\mathbf{v}$ is timelike, spacelike or lightlike.

We now respectively define *hyperbolic n-space* and *de Sitter n-space* by

$$H^n_\pm(-1) = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, sgn(x_0) = \pm 1\},$$

$$S^n_1 = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\},$$

and we write $H^n(-1) = H^n_+(-1) \cup H^n_-(-1)$. For any $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}^{n+1}_1$, we can define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \dots \wedge \mathbf{x}_n$ with the property $\langle \mathbf{x}, \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_n \rangle = \det(\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_n)$, so that $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_n$ is pseudo-orthogonal to any $\mathbf{x}_i$ for $i = 1, \dots, n$. We also define *future* (resp. *past*) *lightcone* at the origin by

$$LC^*_+ = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0,\ x_0 > 0\ \},$$

$$LC^*_- = \{\mathbf{x} \in \mathbb{R}^{n+1}_1 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0,\ x_0 < 0\ \},$$

and we write $LC^* = LC^*_+ \cap LC^*_-$.

Let $\mathbf{X} : U \to S^n_1$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We say $\mathbf{X}$ is a *spacelike hypersurface* in S^n_1 if every non zero vector generated by $\{\mathbf{X}_{u_i}(\mathbf{u})\}_{i=1}^{n-1}$ is always spacelike, where $\mathbf{u} = (u_1, \dots, u_{n-1})$ is an element of U and $\mathbf{X}_{u_i}$ is a partial derivative of $\mathbf{X}$ with respect to u_i. We denote $M = \mathbf{X}(U)$ and identify M with U through the embedding $\mathbf{X}$. Since $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$, we have $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$ for $i = 1, \dots, n-1$. It follows that a hyperplane spanned by $\{\mathbf{X}, \mathbf{X}_{u_1}, \dots, \mathbf{X}_{u_{n-1}}\}$ is spacelike. We define a vector $\mathbf{e}(\mathbf{u}) = \mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \dots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u}) / \|\mathbf{X}(\mathbf{u}) \wedge \mathbf{X}_{u_1}(\mathbf{u}) \wedge \dots \wedge \mathbf{X}_{u_{n-1}}(\mathbf{u})\|$. Then $\mathbf{e}$ is pseudo orthogonal to $\mathbf{X}$ and $\mathbf{X}_{u_i}$ for $i = 1, \dots, n-1$. We define a map $\mathbb{L}^\pm : U \to LC^*_\pm$ by

$$\mathbb{L}^\pm(\mathbf{u}) = \mathbf{X}(\mathbf{u}) \pm \mathbf{e}(\mathbf{u}),$$

which is called a *positive* (resp. *negative*) *lightcone Gauss image* of $\mathbf{X}$.

We now consider a hypersurface defined by $HP(\mathbf{v}, c) \cap S^n_1$. We say that $HP(\mathbf{v}, c) \cap S^n_1$ is an *elliptic* hyperquadric or a *hyperbolic* hyperquadric if $HP(\mathbf{v}, c)$ is spacelike or timelike respectively. We say that $HP(\mathbf{v}, c) \cap S^n_1$ is a *de Sitter hyperhorosphere* if $c \neq 0$ and $HP(\mathbf{v}, c)$ is lightlike. We have the following proposition analogous to ([3], Proposition 2.2).

Proposition 2.1. ([7]) *Let $\mathbf{X} : U \to S^n_1$ be a spacelike hypersurface in S^n_1. The lightcone Gauss image $\mathbb{L}^\pm$ is constant if and only if the spacelike hypersurface $M = \mathbf{X}(U)$ is a part of a de Sitter hyperhorosphere.*

We now define the lightcone Gauss-Kronecker curvature and the lightcone mean curvature of the spacelike hypersurface $M = \mathbf{X}(U)$. For any $p \in M$ and $\mathbf{v} \in T_pM$, we can show $D_\mathbf{v}\mathbf{e}$ and $D_\mathbf{v}\mathbb{L}^\pm \in T_pM$, where $D_\mathbf{v}$ is the covariant derivative with respect to the tangent vector $\mathbf{v}$. Under the identification of U and M, $d\mathbb{L}^\pm(\mathbf{u})$ is a linear transformation on T_pM. We call $S^\pm_p = -d\mathbb{L}^\pm(\mathbf{u})$ a *lightcone shape operator* of $M = \mathbf{X}(U)$ of at $p = \mathbf{X}(\mathbf{u})$.

The *lightcone Gauss-Kronecker curvature* $K_\ell^\pm$ of $M = \mathbf{X}(U)$ at $p = \mathbf{X}(\mathbf{u})$ is defined to be the determinant of the lightcone shape operator.

Since $\mathbf{X}_{u_i}$ (for $i = 1, \ldots, n-1$) are spacelike vectors, we have the Riemannian metric (*first fundamental form*) $ds^2 = \sum_{i,j=1}^{n-1} g_{ij} du_i du_j$ on $M = \mathbf{X}(U)$, where $g_{ij}(\mathbf{u}) = \langle \mathbf{X}_{u_i}(\mathbf{u}), \mathbf{X}_{u_j}(\mathbf{u})\rangle$ for any $\mathbf{u} \in U$. We define a *lightcone second fundamental invariants* by $\bar{h}_{ij}^\pm(\mathbf{u}) = \langle -\mathbb{L}_{u_i}^\pm(\mathbf{u}), \mathbf{X}_{u_j}(\mathbf{u})\rangle$ for any $\mathbf{u} \in U$. In [7] we obtained explicit expression for the lightcone Gauss-Kronecker curvature:

$$K_\ell^\pm = \det(\bar{h}_{ij}^\pm)/\det(g_{\alpha\beta}).$$

We say that $p = \mathbf{X}(\mathbf{u})$ is a *lightcone parabolic point* of $\mathbf{X}$ if $K_\ell^\pm(\mathbf{u}) = 0$.

We define a family of functions $H : U \times LC^* \to \mathbb{R}$ by

$$H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v}\rangle - 1,$$

which we call a *lightcone height function* of M. We have the following proposition analogous to ([3], Proposition 3.1).

PROPOSITION 2.2. ([7]) *Let H be a lightcone height function, then $H(\mathbf{u}, \mathbf{v}) = \partial H(\mathbf{u}, \mathbf{v})/\partial u_i = 0$ for $i = 1, \ldots, n-1$ if and only if $\mathbf{v} = \mathbb{L}^\pm(\mathbf{u})$.*

We also naturally interpreted the lightcone Gauss image of a spacelike hypersurface as a wave front set in the frame work of contact geometry in [7]. This is the analogous way to the differential geometry of hypersurfaces in hyperbolic space [3].

Let $\pi^\pm : PT^*(LC^*_\pm) \to LC^*_\pm$ be the projective cotangent bundles with canonical contact structures. Consider the tangent bundle $\tau^\pm : TPT^*(LC^*_\pm) \to PT^*(LC^*_\pm)$ and the differential map $d\pi^\pm : TPT^*(LC^*_\pm) \to T(LC^*_\pm)$ of $\pi^\pm$. For any $X \in TPT^*(LC^*_\pm)$, there exists an element $\alpha \in T^*(LC^*_\pm)$ such that $\tau^\pm(X) = [\alpha]$. For $\mathbf{v} \in LC^*_\pm$ and $V \in T_\mathbf{v}(LC^*_\pm)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus, we can define the canonical contact structure on $PT^*(LC^*_\pm)$ by

$$K^\pm = \{X \in TPT^*(LC^*_\pm) \mid \tau^\pm(X)(d\pi^\pm(X)) = 0\}.$$

On the other hand, we consider a point $\mathbf{v} = (v_0, v_1, \ldots, v_n) \in LC^*_\pm$, then we have the relation $v_0 = \pm\sqrt{v_1^2 + \cdots + v_n^2}$. So we adopt the coordinate system $(v_1, \ldots, v_n)$ of the manifold $LC^*_\pm$. Then we have the trivialization $PT^*(LC^*_\pm) \equiv LC^*_\pm \times P^*\mathbb{R}^{n-1}$, and call $((v_0, \ldots, v_n), [\xi_1 : \ldots : \xi_n])$ homogeneous coordinates of $PT^*(LC^*_\pm)$, where $[\xi_1 : \ldots : \xi_n]$ are the homogeneous coordinates of the dual projective space $P^*\mathbb{R}^{n-1}$.

It is easy to show that $X_\bullet \in K_\bullet^\pm$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $\bullet = (\mathbf{v}, [\xi])$ and $d\pi_\bullet^\pm(X_\bullet) = \sum_{i=1}^n \mu_i \partial/\partial v_i \in T_\bullet LC^*_\pm$. An immersion $i : L \to PT^*(LC^*_\pm)$ is said to be a *Legendrian immersion* if $\dim L = n-1$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called the *Legendrian* *map* and the image $W(i) = \text{image}(\pi \circ i)$, the *wave front* of i. Moreover, i (or the image of i) is called the *Legendrian lift* of $W(i)$.

Let $F : (\mathbb{R}^s \times \mathbb{R}^k, (\mathbf{u}_0, \mathbf{v}_0)) \to (\mathbb{R}, 0)$ be a function germ. We say that F is a *Morse family* of hypersurfaces if the map germ $\Delta^* F : (\mathbb{R}^s \times \mathbb{R}^k, (\mathbf{u}_0, \mathbf{v}_0)) \to (\mathbb{R}^{s+1}, \mathbf{0})$ defined by

$$\Delta^* F = \left(F, \frac{\partial F}{\partial u_1}, \ldots, \frac{\partial F}{\partial u_s} \right)$$

is non singular. In this case, we have a smooth $(k-1)$-dimensional smooth submanifold,

$$\Sigma_*(F) = \left\{ (\mathbf{u}, \mathbf{v}) \in \left(\mathbb{R}^{n-r} \times \mathbb{R}^k, (\mathbf{u}_0, \mathbf{v}_0) \right) \,\middle|\, F(\mathbf{u}, \mathbf{v}) = \frac{\partial F}{\partial u_1}(\mathbf{u}, \mathbf{v}) = \ldots = \frac{\partial F}{\partial u_{n-r}}(\mathbf{u}, \mathbf{v}) = 0 \right\},$$

and the map germ $\mathcal{L}_F : (\Sigma_*(F), (\mathbf{u}_0, \mathbf{v}_0)) \to PT^*\mathbb{R}^k$ defined by

$$\mathcal{L}_F(\mathbf{u}, \mathbf{v}) = \left(\mathbf{v}, \left[\frac{\partial F}{\partial v_1}(\mathbf{u}, \mathbf{v}) : \ldots : \frac{\partial F}{\partial v_k}(\mathbf{u}, \mathbf{v}) \right] \right)$$

is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd and Zakalyukin [1, 12].

PROPOSITION 2.3. *All Legendrian submanifold germs in $PT^*\mathbb{R}^k$ are constructed by the above method.*

We call F a generating family of $\mathcal{L}_F(\Sigma_*(F))$. Therefore the wave front is

$$W(\mathcal{L}_F) = \left\{ \mathbf{v} \in \mathbb{R}^k \,\middle|\, \exists \mathbf{u} \in \mathbb{R}^{n-r} \text{ such that } F(\mathbf{u}, \mathbf{v}) = \frac{\partial F}{\partial u_1}(\mathbf{u}, \mathbf{v}) = \cdots = \frac{\partial F}{\partial u_{n-r}}(\mathbf{u}, \mathbf{v}) = 0 \right\}.$$

We call it the *discriminant set* of F. In [9] we showed that the lightcone height function H is a Morse family of hypersurface and its discriminant set is the image of lightcone Gauss images $\mathbb{L}^{\pm}(U)$. Therefore we have a immersion germ $\mathcal{L}^{\pm} : (\Sigma_*^{\pm}(H), (\mathbf{u}_0, \mathbf{v}_0^{\pm})) \to PT^*(LC_{\pm}^*)$ defined by

$$\mathcal{L}^{\pm}(\mathbf{u}) = \left(\mathbf{v}^{\pm}, \left[\frac{\partial H}{\partial v_1}(\mathbf{u}, \mathbf{v}^{\pm}) : \ldots : \frac{\partial H}{\partial v_n}(\mathbf{u}, \mathbf{v}^{\pm}) \right] \right),$$

where $\mathbf{v}^{\pm} = \mathbb{L}^{\pm}(\mathbf{u})$ and $\Sigma_*^{\pm}(H)$ is a singular set of H.

3. Spacelike submanifolds in de Sitter space

In this section, we consider the differential geometry of spacelike submanifolds in de Sitter space, which is analogous to [6].

Let $r \geq 2$ be an integer and $\mathbf{X} : U \to S^n_1$ be an embedding from an open set $U \subset \mathbb{R}^{n-r}$. We say that $\mathbf{X}$ is *spacelike* in S^n_1 if every non zero vector generated by $\{\mathbf{X}_{u_i}(\mathbf{u})\}_{i=1}^{n-r}$ is spacelike, where $\mathbf{u} \in U$ and $\mathbf{X}_{u_i} = \partial \mathbf{X}/\partial u_i$. We identify $M = \mathbf{X}(U)$ with U through the embedding $\mathbf{X}$ and call M a *spacelike submanifold of codimension* r in de Sitter space. Since $\langle \mathbf{X}, \mathbf{X} \rangle \equiv 1$, so that $\langle \mathbf{X}_{u_i}, \mathbf{X} \rangle \equiv 0$ for $i = 1, \dots, n-r$. The tangent space of M at $p = \mathbf{X}(\mathbf{u})$ is spanned by the vectors $\mathbf{X}_{u_i}(\mathbf{u})$ for $i = 1, \dots, n-r$.

Let N_pM be the normal space of M at p in $\mathbb{R}^{n+1}_1$ and we define $N^*_p(M) = N_pM \cap T_pS^n_1$. Let $\mathbf{n} : U \to N^*_p(M)$ be a timelike unit normal vector field on M. Since $\langle \mathbf{n}, \mathbf{n} \rangle \equiv -1$ and $\langle \mathbf{X}, \mathbf{n} \rangle \equiv 0$, $\mathbf{n}_{u_i}$ is pseudo orthogonal to both of $\mathbf{X}$ and $\mathbf{n}$ for $i = 1, \dots, n-r$. Therefore we have $\mathbf{n}_{u_i}(\mathbf{u}) \in T_pM \oplus N^*_p(M)$. Consider two pseudo orthonormal projections

$$\pi^t_p : T_p\mathbb{R}^{n+1}_1 \to T_pM, \qquad \pi^n_p : T_p\mathbb{R}^{n+1}_1 \to N_pM.$$

Let $d_{\mathbf{u}}\mathbf{n}$ be the derivative of $\mathbf{n}$ at $\mathbf{u}$, under the identification of M and U through $\mathbf{X}$, we have the linear transformations on T_pM

$$d_p\mathbf{n}^T = \pi^t_p \circ d_{\mathbf{u}}\mathbf{n}, \qquad d_p\mathbf{n}^N = \pi^n_p \circ d_{\mathbf{u}}\mathbf{n}.$$

We respectively call the linear transformation $A_p(\mathbf{n}) = -d_p\mathbf{n}^T$ and $S_p(\mathbf{n}) = -(\mathrm{id}_{T_pM} + d_p\mathbf{n}^T)$ an $\mathbf{n}$-*shape operator* and a *horospherical* $\mathbf{n}$-*shape operator* of M at $p = \mathbf{X}(\mathbf{u})$. We also call the linear map $d_{\mathbf{u}}\mathbf{n}^N$ a *normal connection* with respect to the timelike normal $\mathbf{n}$ of M.

We denote eigenvalues of $A_p(\mathbf{n})$ and $S_p(\mathbf{n})$ by $\kappa_p(\mathbf{n})$ and $\bar{\kappa}_p(\mathbf{n})$, which we respectively call an $\mathbf{n}$-*principal curvature* and a *horospherical* $\mathbf{n}$-*principal curvature*. The *horospherical Gauss-Kronecker curvature* with respect to $\mathbf{n}$ at $p = \mathbf{X}(\mathbf{u})$ is defined to be

$$K_h(\mathbf{n})(\mathbf{u}) = \det S_p(\mathbf{n}).$$

We say that a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $\mathbf{n}$-*umbilic point* if $S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$. Since the eigenvectors of $S_{p_0}(\mathbf{n})$ and $A_{p_0}(\mathbf{n})$ are the same, the above condition is equivalent to $A_{p_0}(\mathbf{n}) = \kappa_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$. We say that the spacelike submanifold M is *totally* $\mathbf{n}$-*umbilic* if every point on M is $\mathbf{n}$-umbilic. We also say that the timelike unit normal vector field $\mathbf{n}$ is *parallel at* p_0 if $d_{p_0}\mathbf{n}^N = 0_{T_{p_0}M}$. The timelike unit normal field $\mathbf{n}$ is *parallel* if $\mathbf{n}$ is parallel at any points on M. Then we have the following result which is analogous to ([6], Proposition 3.1).

Proposition 3.1. *Let* $\mathbf{X} : U \to S^n_1$ *be a spacelike submanifold of codimension* $r \geq 2$. *Suppose that* $M = \mathbf{X}(U)$ *is totally* $\mathbf{n}$-*umbilic, where* $\mathbf{n}$ *is a timelike unit normal parallel vector field. Then* $\kappa_p(\mathbf{n})$ *and* $\bar{\kappa}_p(\mathbf{n})$ *are constant*

$\kappa(\mathbf{n})$ and $\bar{\kappa}(\mathbf{n})$, and there exists a vector $\mathbf{v} \in \mathbb{R}_1^{n+1}$ and real number c such that M is a part of a hyperquadric $HP(\mathbf{v}, c) \cap S_1^n$ in de Sitter space. Under this condition we have following cases:

(1) *If $1 < |\bar{\kappa}(\mathbf{n}) + 1| = |\kappa(\mathbf{n})|$ then M is a part of a hyperbolic hyperquadric $HP(\mathbf{v}, +1)$.*
(2) *If $0 < |\bar{\kappa}(\mathbf{n}) + 1| = |\kappa(\mathbf{n})| < 1$ then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, +1)$.*
(3) *If $\bar{\kappa}(\mathbf{n}) + 1 = \kappa(\mathbf{n}) = 0$ then M is a part of an elliptic hyperquadric $HP(\mathbf{v}, 0)$.*
(4) *If $\kappa(\mathbf{n}) = 1$ (namely $\bar{\kappa}(\mathbf{n}) = 0$) then M is a part of a de Sitter hyperhorosphere $HP(\mathbf{v}, +1)$.*

Proof. By the assumption, we have $A_p(\mathbf{n}) \equiv \kappa_p \mathrm{id}_{T_pM}$. This means that $\pi_p^T \circ \mathbf{n}_{u_i}(\mathbf{u}) \equiv \kappa_p \mathbf{X}_{u_i}(\mathbf{u})$. Since $\mathbf{n}$ is parallel, we have $\mathbf{n}_{u_i}(\mathbf{u}) = \kappa_p \mathbf{X}_{u_i}(\mathbf{u})$. So that $\mathbf{n}_{u_i u_j}(\mathbf{u}) = \kappa_{u_j,p} \mathbf{X}_{u_i}(\mathbf{u}) + \kappa_p \mathbf{X}_{u_i u_j}(\mathbf{u})$ and $\mathbf{n}_{u_j u_i}(\mathbf{u}) = \kappa_{u_i,p} \mathbf{X}_{u_j}(\mathbf{u}) + \kappa_p \mathbf{X}_{u_j u_i}(\mathbf{u})$. It follows that $\mathbf{X}_{u_i u_j} \equiv \mathbf{X}_{u_j u_i}$ and $\mathbf{n}_{u_i u_j} \equiv \mathbf{n}_{u_j u_i}$, then we have $\kappa_{u_j,p} \mathbf{X}_{u_i}(\mathbf{u}) = \kappa_{u_i,p} \mathbf{X}_{u_j}(\mathbf{u})$. Since $\mathbf{X}_i(\mathbf{u})$ and $\mathbf{X}_j(\mathbf{u})$ are linearly independent, $\kappa_{u_i,p} = \kappa_{u_j,p} = 0$. This means that κ_p and $\bar{\kappa}_p$ are constant κ and $\bar{\kappa}$.

We now assume that $\bar{\kappa} + 1 = \kappa \neq 0$. By the assumption, we have $\mathbf{n}_{u_i}(\mathbf{u}) = -\kappa \mathbf{X}_{u_i}(\mathbf{u})$, so that there exists a constant vector $\mathbf{v}$ such that $\mathbf{X}(\mathbf{u}) = \mathbf{v} - (1/\kappa)\mathbf{n}(\mathbf{u})$. Then the vector $\mathbf{v}$ satisfies $\langle \mathbf{v}, \mathbf{v} \rangle = 1 - 1/\kappa^2$ and $\langle \mathbf{X}(\mathbf{u}) - \mathbf{v}, \mathbf{X}(\mathbf{u}) - \mathbf{v} \rangle = -1/\kappa^2$, so that $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = 1$ for any $\mathbf{u} \in U$. This means that M is a part of a hyperquadric in de Sitter space $HP(\mathbf{v}, +1)$. Therefore we have (1), (2) and (4).

On the other hand, if $\bar{\kappa} + 1 = \kappa = 0$ then there exists a constant timelike vector $\mathbf{v}$ such that $\mathbf{n}(\mathbf{u}) = \mathbf{v}$ for any $\mathbf{u} \in U$. So that $\langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{X}(\mathbf{u}), \mathbf{n}(\mathbf{u}) \rangle = 0$ for any $\mathbf{u} \in U$. This means that $M \subset HP(\mathbf{v}, 0)$ Therefore (3) holds. This completes the proof. ∎

We now consider the following Weingarten type formula. Since $\{\mathbf{X}_{u_i}\}_{i=1}^{n-r}$ spans a spacelike vector subspace, we induce a Riemannian metric (the *horospherical first fundamental form*) by $ds^2 = \sum_{i,j=1}^{n-r} g_{ij} du_i du_j$ on $M = \mathbf{X}(U)$, where $g_{ij} = \langle \mathbf{X}_{u_i}, \mathbf{X}_{u_j} \rangle$. We respectively define the *second fundamental invariant* and *horospherical second fundamental invariant* with respect to the timelike unit normal vector field $\mathbf{n}$ by $h_{ij}(\mathbf{n}) = -\langle \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$ and $\bar{h}_{ij}(\mathbf{n}) = -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j} \rangle$. We have the relation

$$\bar{h}_{ij}(\mathbf{n}) = -g_{ij} + h_{ij}(\mathbf{n}) \qquad (\text{for } i, j = 1, \ldots, n-r).$$

Under the above notations, we have the following *Weingarten type formula* with respect to the timelike unit normal vector field $\mathbf{n}$, which is analogous to ([6], Proposition 3.2)

$$\pi^T \circ (\mathbf{X}+\mathbf{n})_{u_i} = -\sum_{k=1}^{n-r} \bar{h}_i^j(\mathbf{n})\mathbf{X}_{u_j},$$

where $(\bar{h}_i^j(\mathbf{n}))_{ij} = (\bar{h}_{ik}(\mathbf{n}))_{ik}(g^{kj})_{kj}$ and $(g^{kj}) = (g_{kj})^{-1}$. Therefore, the Gauss-Kronecker curvature with respect to $\mathbf{n}$ is given by

$$K_h(\mathbf{n}) = \det(\bar{h}_{ik}(\mathbf{n}))/\det(g_{kj}).$$

Since $\langle \mathbf{X}+\mathbf{n}, \mathbf{X}_{u_j}\rangle \equiv 0$, the coefficients of the second fundamental invariant with respect to the timelike parallel unit normal vector field $\mathbf{n}$ are expressed by

$$\begin{aligned}\bar{h}_{ij}(\mathbf{n}) &= -\langle \mathbf{X}_{u_i} + \mathbf{n}_{u_i}, \mathbf{X}_{u_j}\rangle \\ &= -\partial\langle \mathbf{X}+\mathbf{n}, \mathbf{X}_{u_j}\rangle/\partial u_i + \langle \mathbf{X}+\mathbf{n}, \mathbf{X}_{u_iu_j}\rangle \\ &= \langle \mathbf{X}+\mathbf{n}, \mathbf{X}_{u_iu_j}\rangle.\end{aligned}$$

Therefore the horospherical second fundamental invariant at a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ depends only on the timelike vector $\mathbf{n}_0 = \mathbf{n}(\mathbf{u}_0)$. It is independent of the choice of timelike parallel unit normal vector field $\mathbf{n}$ with $\mathbf{n}_0 = \mathbf{n}(\mathbf{u}_0)$.

Let $\mathbf{n}_0$ be a timelike unit normal vector. We say that a point $p_0 = \mathbf{X}(\mathbf{u}_0)$ is an $\mathbf{n}_0$-*parabolic point* (resp. $\mathbf{n}_0$-*umbilic point*) of M if $K_h(\mathbf{n})(\mathbf{u}_0) = 0$ ($S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})\mathrm{id}_{T_{p_0}M}$) for some timelike parallel unit normal vector field $\mathbf{n}$ with $\mathbf{n}(\mathbf{u}_0) = \mathbf{n}_0$. We also say that p_0 is an $\mathbf{n}_0$-*horospherical point* if it is an $\mathbf{n}_0$-parabolic point and an $\mathbf{n}_0$-umbilic point.

4. Horospherical hypersurfaces and horospherical height functions

In this section we introduce the notions of horospherical height function and horospherical hypersurface.

Let $\mathbf{X}: U \to S_1^n$ be a spacelike submanifolds of codimension $r \geq 2$ in de Sitter space and $p = \mathbf{X}(\mathbf{u})$. We choose unit orthonormal sections

$$N_p(M) = \langle \mathbf{X}(\mathbf{u}), \mathbf{n}_0(\mathbf{u}), \mathbf{n}_1(\mathbf{u}), \ldots, \mathbf{n}_{r-1}(\mathbf{u})\rangle_{\mathbb{R}},$$

where $\mathbf{n}_0(\mathbf{u})$ is a timelike unit normal vector and $\mathbf{n}_i(\mathbf{u})$ for $i = 1, \ldots, r-1$ are spacelike unit normal vectors. We define a map $\mathbf{e}: U \times H^{r-1}(-1) \to H^{n-1}(-1)$ by

$$\mathbf{e}(\mathbf{u}, \bar{\mu}) = \mu_0\mathbf{n}_0(\mathbf{u}) + \sum_{i=1}^{r-1} \mu_i \mathbf{n}_i(\mathbf{u}),$$

where $\bar{\mu} = (\mu_0, \ldots, \mu_{r-1})$. Let θ be a fixed real number, we also define a map $\bar{\mathbf{X}}_\theta : U \times H^{r-1}(-1) \to S_1^n$ by

$$\bar{\mathbf{X}}_\theta(\mathbf{u}, \bar{\mu}) = \cosh\theta\mathbf{X}(\mathbf{u}) + \sinh\theta\mathbf{e}(\mathbf{u}, \bar{\mu}).$$

We remark that for any spacelike submanifold $\mathbf{X}$ and point $(\mathbf{u}_0, \bar{\mu}_0) \in U \times H^{r-1}(-1)$, there are a real number $\theta \neq 0$ and an open neighborhood V of $(\mathbf{u}_0, \bar{\mu}_0)$ such that $\bar{\mathbf{X}}_\theta$ is spacelike embedding on V. We assume that for any $(\mathbf{u}, \bar{\mu}) \in V$ then $(\mathbf{u}, -\bar{\mu}) \in V$. We write CM as an image $\bar{\mathbf{X}}_\theta(V)$ and call it a *spacelike canal hypersurface* of $M = \mathbf{X}(U)$. Izumiya, Pei, Romero Fuster and Takahashi [6] introduced the notion of canal surfaces of submanifolds in the hyperbolic space.

We now consider the horospherical height function on a spacelike submanifold. For a spacelike submanifolds $\mathbf{X}$ of codimension r, we define the family of functions

$$H : U \times LC^* \to \mathbb{R}$$

by $H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v} \rangle - 1$, and we call H a *horospherical height function on M*. For $\mathbf{v}_0 \in LC^*$ we denote $h_{\mathbf{v}_0}(\mathbf{u}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v}_0 \rangle - 1$. We have the following proposition which is analogous to ([6], Proposition 3.4).

PROPOSITION 4.1. *Let $H : U \times LC^* \to \mathbb{R}$ be a horospherical height function of a spacelike submanifold $X : U \to S^n_1$ of codimension r. Then $H(\mathbf{u}, \mathbf{v}) = \partial H(\mathbf{u}, \mathbf{v})/\partial u_i = 0$ for $i = 1, \ldots, n-r$ if and only if $\mathbf{v} = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu})$ for some $\bar{\mu} \in H^{r-1}(-1)$.*

The proof of the above proposition is similar to that of Proposition 3.4 in [6], so it is omitted. The discriminant set of the horospherical height function H is

$$D_H = \{\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu}) \mid (\mathbf{u}, \bar{\mu}) \in U \times H^{r-1}(-1)\}.$$

We define a map $HS_{\mathbf{X}} : U \times H^{r-1}(-1) \to LC^*$ by

$$HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u}, \bar{\mu}),$$

which we call a *horospherical hypersurface of M*. We remark that $HS_{\mathbf{X}}$ depends on the choice of the orthonormal frames of $N(M)$.

Let $\{\mathbf{X}, \mathbf{n}_0, \ldots, \mathbf{n}_{r-1}\}$ and $\{\mathbf{X}, \mathbf{n}'_0, \ldots, \mathbf{n}'_{r-1}\}$ be two orthonormal frames of $N(M)$ with $\mathbf{n}_0, \mathbf{n}'_0 \in H^{n-1}_+(-1)$. Then we have $\mathbf{n}_i = \sum_{j=0}^{r-1} \lambda_i^j \mathbf{n}'_j$, where

$$\lambda_i^j(\mathbf{u}) = \begin{cases} -\langle \mathbf{n}_i, \mathbf{n}'_j \rangle & \text{if } j = 0, \\ \langle \mathbf{n}_i, \mathbf{n}'_j \rangle & \text{if } j = 1, \ldots, r-1. \end{cases}$$

Then we have a diffeomorphism $\Phi : U \times H^{r-1}(-1) \to U \times H^{r-1}(-1)$ defined by

$$\Phi(\mathbf{u}, \bar{\mu}) = \left(\mathbf{u}, \left(\sum_{i=0}^{r-1} \lambda_i^0(\mathbf{u})\mu_i, \ldots, \sum_{i=0}^{r-1} \lambda_i^{r-1}(\mathbf{u})\mu_i\right)\right).$$

We also define $\mathbf{e}'(\mathbf{u},\bar{\mu}) = \sum_{i=0}^{r-1} \mu_i \mathbf{n}'_i(\mathbf{u})$. It follows from the above that $\mathbf{e}(\mathbf{u},\bar{\mu}) = \mathbf{e}' \circ \Phi(\mathbf{u},\bar{\mu})$. Therefore we have

$$HS_{\mathbf{X}}(\mathbf{u},\bar{\mu}) = HS'_{\mathbf{X}} \circ \Phi(\mathbf{u},\bar{\mu}),$$

where $HS'_{\mathbf{X}} = \mathbf{X}(\mathbf{u}) + \mathbf{e}'(\mathbf{u},\bar{\mu})$. This means that $HS_{\mathbf{X}}$ is independent to the choice of orthonormal frames of $N(M)$ up to the diffeomorphic parametrization. We have a following proposition which is analogous to ([6], Proposition 3.5).

Proposition 4.2. *Let $\mathbf{X} : U \to S^n_1$ be a spacelike hypersurface of codimension $r \geq 2$ in de Sitter space, then $HS_{\mathbf{X}}(\mathbf{u},\bar{\mu}) = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu})$ is constant map for some smooth map $\bar{\mu} : U \to H^{r-1}(-1)$ if and only if M is a part of de Sitter hyperhorosphere $HP(\mathbf{v},1) \cap S^n_1$. By Proposition 3.1, if M is totally $\mathbf{e}(\mathbf{u},\bar{\mu}(\mathbf{u}))$-umbilic for some parallel normal vector field $\mathbf{e}(\mathbf{u},\bar{\mu}(\mathbf{u}))$ and $K_h(\mathbf{e}(\mathbf{u},\bar{\mu}(\mathbf{u})))(\mathbf{u}) = 0$, then the above assertion holds.*

Proof. Suppose that $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu})$ is a constant vector. Since $\mathbf{e}(\mathbf{u},\bar{\mu})$ is pseudo orthogonal to $\mathbf{X}(\mathbf{u})$, then we have $\langle \mathbf{X}(\mathbf{u}), \mathbf{v}_0 \rangle = +1$ for any $\mathbf{u} \in U$. This means that M is a part of a de Sitter hyperhorosphere $HP(\mathbf{v}_0,1) \cap S^n_1$. On the other hand, if $M \subset HP(\mathbf{v}_0,1) \cap S^n_1$ for some lightlike vector, then $\langle \mathbf{v}_0 - \mathbf{X}(\mathbf{u}), \mathbf{X}(\mathbf{u}) \rangle = 0$ for any $\mathbf{u} \in U$. Since $\mathbf{X}(\mathbf{u})$ is pseudo orthogonal to $\mathbf{X}_{u_i}(\mathbf{u})$, it follows that $\langle \mathbf{v}_0 - \mathbf{X}(\mathbf{u}), \mathbf{X}_{u_i}(\mathbf{u}) \rangle = 0$. This means that $\mathbf{X}(\mathbf{u}) - \mathbf{v}_0$ is a normal vector of M at $p = \mathbf{X}(\mathbf{u})$. We define a function $\bar{\mu}(\mathbf{u})$ by

$$\bar{\mu}(\mathbf{u}) = -\langle \mathbf{X}(\mathbf{u}) - \mathbf{v}_0, \mathbf{n}_0(\mathbf{u}) \rangle \mathbf{n}_0(\mathbf{u}) + \sum_{i=1}^{r-1} \langle \mathbf{X}(\mathbf{u}) - \mathbf{v}_0, \mathbf{n}_i(\mathbf{u}) \rangle \mathbf{n}_i(\mathbf{u}).$$

Then we have $\mathbf{v}_0 - \mathbf{X}(\mathbf{u}) = \mathbf{e}(\mathbf{u},\bar{\mu})$. This completes the proof. ∎

Since the image of $HS_{\mathbf{X}}$ is the discriminant set of the horospherical height function H on M, the singular set of $HS_{\mathbf{X}}$ corresponds to the null set of the Hessian matrix of the horospherical height function with the fixed parameter $\mathbf{v}$ at each point. Therefore we have the following proposition which is analogous to ([6], Proposition 3.6).

Proposition 4.3. *The singular set of $HS_{\mathbf{X}}$ is given by*

$$\Sigma(HS_{\mathbf{X}}) = \left\{ (\mathbf{u},\bar{\mu}) \in U \times H^{r-1}(-1) \mid K_h(\mathbf{e}(\mathbf{u},\bar{\mu}))(\mathbf{u}) = 0 \right\}.$$

Proof. Let $h_{\mathbf{v}}(\mathbf{u})$ be a horospherical height function with $\mathbf{v} \in LC^*$, then we have $\operatorname{Hess} h_{\mathbf{v}}(\mathbf{u}) = \langle \mathbf{X}_{u_i u_j}(\mathbf{u}), \mathbf{v} \rangle$. Suppose that $(\mathbf{u},\mathbf{v}) \in \Sigma_*(H)$, then $\mathbf{v} = \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu})$ for some $\bar{\mu} \in H^{r-1}(-1)$. We recall that $\bar{h}_{ij}(\mathbf{v})(\mathbf{u}) = \langle \mathbf{X}_{u_i u_j}(\mathbf{u}), \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu}) \rangle$, where $(\bar{h}_{ij}(\mathbf{v})(\mathbf{u}))$ is the horospherical second fundamental invariant with respect to the timelike direction $\mathbf{e}(\mathbf{u},\bar{\mu})$. The horospherical Gauss-Kronecker curvature is

$$K_h(\mathbf{e}(\mathbf{u},\bar{\mu}))(\mathbf{u}) = \det(\langle \mathbf{X}_{u_iu_j}(\mathbf{u}), \mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u})\rangle)/\det(g_{ij}(\mathbf{u}))$$
$$= \det \operatorname{Hess} h_{\mathbf{v}}(\mathbf{u})/\det(g_{ij}(\mathbf{u})),$$

where $(g_{ij}(\mathbf{u}))$ is the first fundamental invariant of M. Therefore $\operatorname{Hess} h_{\mathbf{v}}(\mathbf{u}) = 0$ if and only if $K_h(\mathbf{e}(\mathbf{u},\bar{\mu}))(\mathbf{u}) = 0$. This completes the proof. ∎

The singular set of $HS_{\mathbf{X}}$ corresponds to the parabolic set of M with respect to some timelike parallel normal vector field $\mathbf{e}(\mathbf{u},\bar{\mu})$. By the proof of above proposition, we have $\operatorname{rank} \operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0) = \operatorname{rank}(\bar{h}_{ij}(\mathbf{v}_0)(\mathbf{u}_0))_{ij}$. Therefore we also have the following proposition which is analogous to ([6], Proposition 3.7).

Proposition 4.4. *For any spacelike submanifold* $\mathbf{X}$ *of codimension* $r \geq 2$ *and lightlike vector* $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0,\bar{\mu}_0)$*, we have the following assertions.*

(1) *A point* $p_0 = \mathbf{X}(\mathbf{u}_0)$ *is an* $\mathbf{e}(\mathbf{u}_0,\bar{\mu}_0)$*-parabolic point if and only if*

$$\det \operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0) = 0.$$

(2) *A point* p_0 *is an* $\mathbf{e}(\mathbf{u}_0,\bar{\mu}_0)$*-horospherical point if and only if*

$$\operatorname{rank} \operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0) = 0.$$

Here $\operatorname{Hess} h_{\mathbf{v}_0}(\mathbf{u}_0)$ *is a Hessian matrix of* $h_{\mathbf{v}_0}(\mathbf{u})$ *at* $\mathbf{u} = \mathbf{u}_0$.

We now consider the lightcone height function and the lightcone Gauss image of spacelike canal hypersurface $\bar{\mathbf{X}}_\theta : V \to S^n_1$ with $V \subset U \times H^{r-1}(-1)$. The lightcone height function $\bar{H} : V \times LC^* \to \mathbb{R}$ of the spacelike hypersurface $\bar{\mathbf{X}}_\theta$ is

$$\bar{H}((\mathbf{u},\bar{\mu}),\mathbf{v}) = \langle \bar{\mathbf{X}}_\theta(\mathbf{u},\bar{\mu}),\mathbf{v}\rangle - 1.$$

We denote $\bar{h}_{\mathbf{v}}(\mathbf{u}) = \bar{H}((\mathbf{u},\bar{\mu}),\mathbf{v})$ for any $\mathbf{v} \in LC^*$. Now we define a map $\bar{\mathbf{e}} : V \to H^{n-1}(-1)$ by $\bar{\mathbf{e}}(\mathbf{u},\bar{\mu}) = \sinh\theta\mathbf{X}(\mathbf{u}) + \cosh\theta\mathbf{e}(\mathbf{u},\bar{\mu})$. Then we have $\langle \bar{\mathbf{e}}(\mathbf{u},\bar{\mu}), \bar{\mathbf{X}}_\theta(\mathbf{u})\rangle = \langle \bar{\mathbf{e}}(\mathbf{u},\bar{\mu}), \bar{\mathbf{X}}_{\theta,u_i}(\mathbf{u})\rangle = 0$ for any $(\mathbf{u},\bar{\mu}) \in V$ and $i = 1,\dots,n-r$. Therefore $\bar{\mathbf{e}}$ is a timelike normal of CM. The positive lightcone Gauss image $\mathbb{L}_{CM} : V \to LC^*$ is defined by

$$\mathbb{L}_{CM}(\mathbf{u},\bar{\mu}) = \bar{\mathbf{X}}_\theta(\mathbf{u}) + \bar{\mathbf{e}}(\mathbf{u},\bar{\mu}) = (\cosh\theta + \sinh\theta)(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu})).$$

By Proposition 2.2, $\bar{H}((\mathbf{u},\bar{\mu}),\mathbf{v}) = \bar{H}_{u_i}((\mathbf{u},\bar{\mu}),\mathbf{v}) = \bar{H}_{\mu_j}((\mathbf{u},\bar{\mu}),\mathbf{v}) = 0$ for $i = 1,\dots,n-r$ and $j = 0,\dots,r-1$ if and only if $\mathbf{v} = \bar{\mathbf{X}}_\theta(\mathbf{u}) \pm \bar{\mathbf{e}}(\mathbf{u},\bar{\mu}) = e^{\pm\theta}(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\pm\bar{\mu}))$. By assumption, $(\mathbf{u},-\bar{\mu})$ is also an element of V. Therefore the discriminant set of the lightcone height function $\bar{H}$ is

$$D(\bar{H}) = \{e^{\pm\theta}(\mathbf{X}(\mathbf{u}) + \mathbf{e}(\mathbf{u},\bar{\mu})) \mid (\mathbf{u},\bar{\mu}) \in V\}.$$

We now define a diffeomorphism

$$\mathcal{M}_c : LC^* \to LC^*$$

given by $M_c(\mathbf{v}) = c\mathbf{v}$ for a fixed positive real number c. Then we have the following lemma, which is analogous to ([6], Proposition 3.9).

Lemma 4.5. *Under the above notations, we have*

$$\mathcal{M}_c \circ HS_{\mathbf{X}}(\mathbf{u}, \bar{\mu}) = \mathbb{L}_{CM}(\mathbf{u}, \bar{\mu})$$

on $V \subset U \times H^{r-1}(-1)$, *where* $c = e^{\pm\theta}$.

By the above lemma, the horospherical hypersurface $HS_{\mathbf{X}}$ is locally diffeomorphic to the lightcone Gauss image of the spacelike canal hypersurface $\bar{\mathbf{X}}_\theta$.

5. Horospherical hypersurfaces as wave fronts

In this section we naturally interpret the horospherical hypersurfaces of M as a wave front set of the horospherical height functions in the theory of Legendrian singularities.

By proceeding arguments in §2, the horospherical hypersurface $HS_{\mathbf{X}}$ is the discriminant set of the horospherical height function H, and the singular point set of the horospherical hypersurface is the horospherical point set. We have the following proposition which is analogous to ([6], Proposition 4.1).

Proposition 5.1. *Let* $\mathbf{X} : U \to S_1^n$ *be a spacelike submanifold of codimension* $r \geq 2$ *and* $H : U \times LC^* \to \mathbb{R}$ *be a horospherical height function of* M. *Then* H *is a Morse family.*

Proof. We denote

$$\mathbf{X}(\mathbf{u}) = (X_0(\mathbf{u}), \dots, X_n(\mathbf{u})) \quad \text{and}$$
$$\mathbf{X}_{u_i}(\mathbf{u}) = (X_{0,u_i}(\mathbf{u}), \dots, X_{n,u_i}(\mathbf{u})).$$

For any $\mathbf{v} = (v_0, \dots, v_n) \in LC^*$, we have $v_0 \neq 0$. Without loss of generality, we assume that $v_0 = \sqrt{v_1^2 + \cdots + v_n^2} > 0$, so that we have

$$H(\mathbf{u}, \mathbf{v}) = \langle \mathbf{X}(\mathbf{u}), \mathbf{v}\rangle - 1 = -X_0 v_0 + \sum_{i=1}^{n} X_n v_n - 1.$$

We now prove a map

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_{n-r}}\right)$$

is non singular at any $(\mathbf{u}, \mathbf{v}) \in \Sigma_*(H)$. The Jacobian matrix of $\Delta^* H$ is

$$J\Delta^* H(\mathbf{u}, \mathbf{v}) = \left(\begin{array}{c|c} * & \dfrac{\partial H}{\partial v_j}(\mathbf{u}, \mathbf{v})_{j=1,\dots,n} \\ \hline * & \left(\dfrac{\partial^2 H}{\partial u_i \partial v_j}(\mathbf{u}, \mathbf{v})\right)_{\substack{j=1,\dots,n \\ i=1,\dots,n-r}} \end{array}\right).$$

We denote an $(n-r+1)\times n$ matrix B by $J\Delta^*H = (*\mid B)$. It is sufficient to show that $\operatorname{rank} B = n-r+1$ at $(\mathbf{u},\mathbf{v})\in\Sigma_*(H)$. We also denote an $(n-r+3)\times(n+1)$ matrix C by

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_0 & v_1 & \cdots & v_n \\ X_0 & X_1 & \cdots & X_n \\ X_{0,u_1} & X_{1,u_1} & \cdots & X_{n,u_1} \\ \vdots & \vdots & \ddots & \vdots \\ X_{0,u_{n-r}} & X_{1,u_{n-r}} & \cdots & X_{n,u_{n-r}} \end{pmatrix}.$$

We now show that the rank of the matrix C is equal to $n-r+3$. Since $\mathbf{v},\mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$ are linearly independent for all $(\mathbf{u},\mathbf{v})\in\Sigma_*(H)$, it is sufficient to show that timelike unit vector $e=(1,0,\dots,0)$ can not be written by a linear combination of $\mathbf{v},\mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$. If that is not so, there exists some real numbers η,μ,ξ_i such that $e=\eta\mathbf{v}+\mu\mathbf{X}(\mathbf{u})+w$ and $w=\sum_{i=1}^{n-r}\xi_i\mathbf{X}_{u_i}(\mathbf{u})$. Then we have $\langle e,e\rangle = \mu^2+\langle w,w\rangle$. However, w is a spacelike vector, so that $\langle e,e\rangle$ would not be negative, which contradicts our assumption. This means that $e,\mathbf{v},\mathbf{X}(\mathbf{u})$ and $\mathbf{X}_{u_i}(\mathbf{u})$ are linearly independent, therefore we have $\operatorname{rank} C = n-r+3$.

We now show $\operatorname{rank} B = \operatorname{rank} C' - 2$. We subtract the second row multiplied by X_0/v_0 from the third row of the matrix C, and add the second row multiplied by $X_{0,u_k}(\mathbf{u})/v_0$ from the $(3+k)$-th row for $k=1,\dots,n-r$. Then we have a matrix

$$C' = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ v_0 & v_1 \cdots v_n \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right).$$

Therefore we have $\operatorname{rank} B = \operatorname{rank} C' - 2 = n-r+1$. This completes the proof. ∎

Since H is a Morse family of hypersurfaces, we have the Legendrian immersion germ $\mathcal{L}_H : (\Sigma_*(H),(\mathbf{u}_0,\mathbf{v}_0))\to PT^*(LC^*)$ defined by

$$\mathcal{L}_H(\mathbf{u},\mathbf{v}) = \left(\mathbf{v},\left[\frac{\partial H}{\partial v_1}(\mathbf{u},\mathbf{v}):\dots:\frac{\partial H}{\partial v_n}(\mathbf{u},\mathbf{v})\right]\right),$$

where $(v_1,\dots,v_n)$ is the coordinate system of LC^*.

We remark that the wave front set of the Legendrian immersion germ $\mathcal{L}_H$ is the horospherical hypersurfaces $HS_{\mathbf{X}}$ of M. On the other hand, we define a contact diffeomorphism $\tilde{\mathcal{M}}_c : PT^*(LC^*) \to PT^*(LC^*)$ by

$$\tilde{\mathcal{M}}_c(\mathbf{v}, [\xi]) = (c\mathbf{v}, [\xi]),$$

where c is a fixed real parameter with $c \neq 0$. By definition, we have the following theorem.

THEOREM 5.2. *For a spacelike submanifold $\mathbf{X} : U \to S^n_1$, we have*

$$\tilde{\mathcal{M}}_c \circ \mathcal{L}_H = \mathcal{L}_{\bar{H}},$$

where $c = e^{\pm\theta}$ and $\mathcal{L}_{\bar{H}}$ is a Legendrian lift of the lightcone Gauss image $\mathbb{L}_{CM}$ of the spacelike canal hypersurface of M.

By the above theorem, the Legendrian lift of the lightcone Gauss image $\mathbb{L}_{CM}$ is $\mathcal{A}$-equivalent to the Legendrian lift of the horospherical hypersurface $HS_{\mathbf{X}}$ of M.

6. Contact with de Sitter hyperhorospheres

In this section we use the theory of contacts between the spacelike submanifolds and the de Sitter hyperhorospheres, following Montaldi [10].

Let X_i and Y_i $(i = 1,2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$ and $y_i \in X_i \cap Y_i$ for $i = 1,2$. We say that *the contact* of X_1 and Y_1 at y_1 is *the same type* as *the contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ such that $\Phi((X_1, y_1)) = (X_2, y_2)$ and $\Phi((Y_1, y_1)) = (Y_2, y_2)$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. Two function germs $g_1, g_2 : (\mathbb{R}^n, a_i) \to (\mathbb{R}, 0)$ $(i = 1,2)$ are *$\mathcal{K}$-equivalent* if there are a diffeomorphism germ $\Phi : (\mathbb{R}^n, a_1) \to (\mathbb{R}^n, a_2)$ and a function germ $\lambda : (\mathbb{R}^n, a_1) \to \mathbb{R}$ with $\lambda(a_1) \neq 0$ such that $f_1 = \lambda \cdot (g_2 \circ \Phi)$. In [10] Montaldi has shown the following theorem.

THEOREM 6.1. ([10]) *Let X_i and Y_i $(i = 1,2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, $\dim Y_1 = \dim Y_2$ and $y_i = X_i \cap Y_i$ for $i = 1,2$. Let $g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \to (\mathbb{R}, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

We now consider the function $\mathcal{H} : S^n_1 \times LC^* \to \mathbb{R}$ defined by $\mathcal{H}(x, \mathbf{v}) = \langle x, \mathbf{v} \rangle - 1$. Given $\mathbf{v}_0 \in LC^*$, we denote $\mathfrak{h}_{\mathbf{v}_0}(x) = \mathcal{H}(x, \mathbf{v}_0)$, so that we have $\mathfrak{h}^{-1}_{\mathbf{v}_0}(0) = HP(\mathbf{v}_0, +1) \cap S^n_1$. Let $\mathbf{X} : U \to \mathbb{R}^n_1$ be a spacelike submanifold of codimension $r \geq 2$. For any $\mathbf{u}_0 \in U$ and $\bar{\mu}_0 \in H^{r-1}(-1)$, we take a point $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$. By Proposition 4.1, we have

$$(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})(\mathbf{u}_0) = \mathcal{H} \circ (\mathbf{X} \times \mathrm{id}_{LC^*})(\mathbf{u}_0, \mathbf{v}_0) = H(\mathbf{u}_0, \mathbf{v}_0) = 0,$$

$$\frac{\partial(\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{X})}{\partial u_i}(\mathbf{u}_0) = \frac{\partial H}{\partial u_i}(\mathbf{u}_0, \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)) = 0.$$

It follows that the de Sitter hyperhorosphere $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, +1) \cap S_1^n$ is tangent to M at $p_0 = \mathbf{X}(\mathbf{u}_0)$. In this case we call $HP(\mathbf{v}_0, +1) \cap S_1^n$ a *tangent de Sitter hyperhorosphere* (briefly, *tangent hyperhorosphere*) with respect to $\mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$. We may also consider the contacts of the spacelike canal surface $CM = \bar{\mathbf{X}}(V)$ and the de Sitter hyperhorospheres. (see [7])

We now review some notions of Legendrian singularity theory to study the contact between hypersurfaces and de Sitter hyperhorospheres. We say that Legendrian immersion germs $\iota_i : (U_i, \mathbf{u}_i) \to (PT^*\mathbb{R}^n, p_i)$ $(i = 1, 2)$ are *Legendrian equivalent* if there are a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p_1) \to (PT^*\mathbb{R}^n, p_2)$ and a diffeomorphic germ $\tau : (U_1, \mathbf{u}_1) \to (U_2, \mathbf{u}_2)$ such that H preserves fibers of π and $H \circ \iota_1 = \iota_2 \circ \tau$. A Legendrian immersion germ at a point is said to be *Legendrian stable* if for every map with the given germ there are a neighborhood in the space of Legendrian immersions (in the Whitney C^∞-topology) and a neighborhood of the original point such that each Legendrian map belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ. (see [1]).

PROPOSITION 6.2. ([13]) *Let i_1, i_2 be Legendrian immersion germs such that regular sets of $\pi \circ i_1$ and $\pi \circ i_2$ are respectively dense. Then i_1, i_2 are Legendrian equivalent if and only if corresponding wave front sets $W(i_1)$ and $W(i_2)$ are diffeomorphic as set germs.*

Let $F_i : (\mathbb{R}^n \times \mathbb{R}^k, (a_i, b_i)) \to (\mathbb{R}, c)$ $(k = 1, 2)$ be k-parameter unfoldings of function germs f_i. We say that F_1 and F_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \to (\mathbb{R}^n \times \mathbb{R}^k, (a_2, b_2))$ of the form $\Phi(\mathbf{u}, x) = (\phi_1(\mathbf{u}, x), \phi_2(x))$ for $(\mathbf{u}, x) \in \mathbb{R}^n \times \mathbb{R}^k$ and a function germ $\lambda : (\mathbb{R}^n \times \mathbb{R}^k, (a_1, b_1)) \to \mathbb{R}$ such that $\lambda(a_1, b_1) \neq 0$ and $F_1(\mathbf{u}, x) = \lambda(\mathbf{u}, x) \cdot (F_2 \circ \Phi)(\mathbf{u}, x)$.

THEOREM 6.3. ([1, 12]) *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, \mathbf{0})$ be Morse families and denote the corresponding Legendrian immersion germs by $\mathcal{L}_F, \mathcal{L}_G$. Then*

(1) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are $\mathcal{P}$-$\mathcal{K}$-equivalent.*
(2) *$\mathcal{L}_F$ is Legendrian stable if and only if F is $\mathcal{K}$-versal deformation of f.*

Let $G_i : (\mathbb{R}^m, a_i) \to (\mathbb{R}^n, b_i)$ (for $i = 1, 2$) be map germs. We say that G_1 and G_2 are $\mathcal{A}$-equivalent if and only if there exist diffeomorphism germs $\phi : (\mathbb{R}^m, a_1) \to (\mathbb{R}^m, a_2)$ and $\Phi : (\mathbb{R}^n, b_1) \to (\mathbb{R}^n, b_2)$ such that $\Phi \circ G_1 = G_2 \circ \phi$.

We denote $h_{i,\mathbf{v}_i} : (U, \mathbf{u}_i) \to (\mathbb{R}, \mathbf{0})$ $(i = 1, 2)$ by $h_{i,\mathbf{v}_i}(\mathbf{u}) = H_i(\mathbf{u}, \mathbf{v}_i)$. Then we have $h_{i,\mathbf{v}_i}(\mathbf{u}) = (\mathfrak{h}_{i,\mathbf{v}_i} \circ \mathbf{X}_i)(\mathbf{u})$. By Theorem 6.1,

$$K(\mathbf{X_1}(U), HP(\mathbf{v}_1, 1) \cap S^n_1; p_1) = K(\mathbf{X_2}(U), HP(\mathbf{v}_2, 1) \cap S^n_1; p_2)$$

if and only if $h_{1,\mathbf{v}_1}$ and $h_{2,\mathbf{v}_2}$ are $\mathcal{K}$-equivalent.

Let $Q(\mathbf{X}, \mathbf{u}_0)$ be the local ring of the horospherical height function germ $h_{\mathbf{v}_0} : (U, \mathbf{u}_0) \to \mathbb{R}$ defined by

$$Q(\mathbf{X}, \mathbf{u}_0; \bar{\mu}_0) = C^\infty_{\mathbf{u}_0}(U)/\langle h_{\mathbf{v}_0}\rangle_{C^\infty_{\mathbf{u}_0}(U)},$$

where $\mathbf{v}_0 = \mathbf{X}(\mathbf{u}_0) + \mathbf{e}(\mathbf{u}_0, \bar{\mu}_0)$, $\bar{\mu}_0 \in H^{r-1}(-1)$ and $C^\infty_{\mathbf{u}_0}(U)$ is the local ring of function germs at $\mathbf{u}_0$ with the unique maximal ideal $\mathfrak{M}$. We also denote $Q(\bar{\mathbf{X}}_\theta, (\mathbf{u}_0, \bar{\mu}_0))$ as the local ring of the lightcone height function germ $\bar{h}_{\mathbf{v}'_0} : (U \times H^{r-1}(-1), (\mathbf{u}_0, \bar{\mu}_0)) \to (\mathbb{R}, 0)$ of the canal hypersurface $\bar{\mathbf{X}}_\theta$, where $\mathbf{v}'_0 = \mathbb{L}_{CM}(\mathbf{u}_0, \bar{\mu}_0)$.

Proposition 6.4. ([3], Proposition A.4) *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, \mathbf{0})$ be Morse families. Suppose that Legendrian immersion germs $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable, then the following conditions are equivalent:*

(1) *$(W(\mathcal{L}_F), \lambda)$ and $(W(\mathcal{L}_G), \lambda')$ are diffeomorphic as set germs.*
(2) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent.*
(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ and $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$.*

We have following theorem.

Theorem 6.5. *Let $\mathbf{X}_i : (U_i, \mathbf{u}_i) \to (S^n_1, p_i)$ $(i = 1, 2)$ be spacelike submanifold germs of codimension at least two in de Sitter space. For $\bar{\mu}_i \in H^{r-1}(-1)$ $(i = 1, 2)$, we denote $\mathbf{v}_i = HS_i(\mathbf{u}_i, \bar{\mu}_i)$, $\mathbf{v}'_i = \mathbb{L}_{CM_i}(\mathbf{u}_i, \bar{\mu}_i)$, $h_{i,\mathbf{v}_i} = H_i|_{U \times \{\mathbf{v}_i\}}$, $\bar{h}_{i,\mathbf{v}'_i} = \bar{H}_i|_{U \times \{\mathbf{v}'_i\}}$ and $p'_i = \bar{\mathbf{X}}_{i,\theta_i}(\mathbf{u}_i, \bar{\mu}_i)$. If the corresponding Legendrian immersion germs $\mathcal{L}_{H_i}$ are Legendrian stable, then the following conditions are equivalent.*

(1) *Horospherical hypersurface germs $HS_{\mathbf{X}_1}$ and $HS_{\mathbf{X}_2}$ are $\mathcal{A}$-equivalent.*
(2) *Legendrian immersion germs $\mathcal{L}_{H_1}$ and $\mathcal{L}_{H_2}$ are Legendrian equivalent.*
(3) *Horospherical height function germs H_1 and H_2 are $\mathcal{P}$-$\mathcal{K}$-equivalent.*
(4) *$h_{1,\mathbf{v}_1}$ and $h_{2,\mathbf{v}_2}$ are $\mathcal{K}$-equivalent.*
(5) *$K(\mathbf{X_1}(U), HP(\mathbf{v}_1, 1) \cap S^n_1; p_1) = K(\mathbf{X_2}(U), HP(\mathbf{v}_2, 1) \cap S^n_1; p_2)$.*
(6) *$Q(\mathbf{X}_1, \mathbf{u_1})$ and $Q(\mathbf{X}_2, \mathbf{u_2})$ are isomorphic as $\mathbb{R}$-algebras.*
(7) *Lightcone Gauss image germs $\mathbb{L}_{CM_1}$ and $\mathbb{L}_{CM_2}$ are $\mathcal{A}$-equivalent.*
(8) *Legendrian immersion germs $\mathcal{L}_{\bar{H}_1}$ and $\mathcal{L}_{\bar{H}_2}$ are Legendrian equivalent.*
(9) *Lightcone height function germs $\bar{H}_1$ and $\bar{H}_2$ are $\mathcal{P}$-$\mathcal{K}$-equivalent.*
(10) *$\bar{h}_{1,\mathbf{v}'_1}$ and $\bar{h}_{2,\mathbf{v}'_2}$ are $\mathcal{K}$-equivalent.*
(11) *$K(CM_1, HP(\mathbf{v}'_1, +1) \cap S^n_1; p'_1) = K(CM_2, HP(\mathbf{v}'_2, +1) \cap S^n_1; p'_2)$.*
(12) *$Q(\bar{\mathbf{X}}_{\theta_1}, (\mathbf{u_1}, \bar{\mu}_1))$ and $Q(\bar{\mathbf{X}}_{\theta_2}, (\mathbf{u_2}, \bar{\mu}_2))$ are isomorphic as $\mathbb{R}$-algebras.*

In this case $(\mathbf{X}_1{}^{-1}(HP(\mathbf{v}_1,1)\cap S^n_1),\mathbf{u}_1)$ *and* $(\mathbf{X}_2{}^{-1}(HP(\mathbf{v}_2,1)\cap S^n_1),\mathbf{u}_2)$ *are diffeomorphic as set germs.*

Proof. Since $\mathcal{L}_{H_1}$ and $\mathcal{L}_{H_2}$ are Legendrian stable, regular sets of $HS_{\mathbf{X}_1}$ and $HS_{\mathbf{X}_2}$ are respectively dense, by applying Proposition 6.2, the conditions (1) and (2) are equivalent. By Theorem 6.3, the conditions (2) and (3) are equivalent. By the arguments in Theorem 6.1, the conditions (4) and (5) are equivalent. If we assume the condition (3), then the $\mathcal{P}$-$\mathcal{K}$-equivalence of H_i $(i=1,2)$ preserves the $\mathcal{K}$-equivalence of $h_{i,\mathbf{v}_i}$, so that the condition (4) holds. Since the local rings $Q(\mathbf{X}_i,\mathbf{u}_i)$ are $\mathcal{K}$-invariant, this means that the condition (6) holds. By Proposition 6.4, the condition (6) implies the condition (2). Therefore the statements from (1) to (6) are equivalent.

By Theorem 5.2, (2) and (8) are equivalent. Since $\mathcal{L}_{H_i}$ are Legendrian stable, $\mathcal{L}_{\bar{H}_i}$ are also Legendrian stable. So that we may similarly show the equivalence of the conditions from (7) to (12). On the other hand, $h^{-1}_{i,\mathbf{v}_i}(0)=(\mathbf{X}_i{}^{-1}(HP(\mathbf{v}_i,1)\cap S^n_1),\mathbf{u}_i)$ and $\mathcal{K}$-equivalence preserves the zero level sets, so that $(\mathbf{X}_i{}^{-1}(HP(\mathbf{v}_i,1)\cap S^n_1),\mathbf{u}_i)$ $(i=1,2)$ are diffeomorphic as set germs. This completes the proof. ∎

We consider generic properties of spacelike submanifolds of codimension $r\geq 2$ in S^n_1. Let U be an open subset of $\mathbb{R}^{n-r}$. We consider the space of spacelike embeddings $\text{Sp-Emb}(U,S^n_1)$ with Whitney C^∞-topology. We define a function $\mathcal{H}: S^n_1\times LC^*\to\mathbb{R}$ by $\mathcal{H}(x,\mathbf{v})=\langle x,\mathbf{v}\rangle$, and denote $\mathfrak{h}_{\mathbf{v}}(x)=\mathcal{H}(x,\mathbf{v})$. Then $\mathfrak{h}_{\mathbf{v}}$ is a submersion for any $\mathbf{v}\in LC^*$. For spacelike submanifolds $\mathbf{X}\in\text{Sp-Emb}(U,S^n_1)$, we have $H=\mathcal{H}\circ(\mathbf{X}\times\text{id}_{LC^*})$. We also have the ℓ-jet extension $j^\ell_1H: U\times S^n_1\to J^\ell(U,\mathbb{R})$ defined by $j^\ell_1H(x,\mathbf{v})=j^\ell\mathfrak{h}_{\mathbf{v}}(\mathbf{u})$. We consider the trivialization $J^\ell(U,\mathbb{R})\equiv U\times\mathbb{R}\times J^\ell(n-r,1)$. For any submanifold $Q\subset J^\ell(n-r,1)$, we denote $\widetilde{Q}=U\times\{0\}\times Q$. Then we have the following proposition as a corollary of Lemma 6 of Wassermann [11].

PROPOSITION 6.6. *Let Q be a submanifold of $J^\ell(n-1,1)$. Then the set*

$$T_Q=\{x\in \textit{Sp-Emb}(U,S^n_1)\mid j^\ell_1H \textit{ is transversal to } \widetilde{Q}\}$$

is a residual subset of $\textit{Sp-Emb}(U,S^n_1)$. If Q is a closed subset, then T_Q is open.

We remark that if the corresponding horospherical height function $h_{\mathbf{v}_0}$ is ℓ-$\mathcal{K}$-determined, then H is a $\mathcal{K}$-versal deformation if and only if j^ℓ_1H is transversal to $\mathcal{K}^\ell_{h,\mathbf{v}_0}$, where $\mathcal{K}^\ell_{h,\mathbf{v}_0}$ is the $\mathcal{K}$-orbit through $j^\ell h_{\mathbf{v}_0}(\mathbf{0})\in J^\ell(n-r,1)$. Applying Theorem 6.3, this condition is equivalent to the condition that the corresponding Legendrian immersion germ is Legendrian stable. From the previous arguments and §5 in [6], we have the following theorem. (See also [1].)

Theorem 6.7. *If $n \leq 6$, there exists an open subset $\mathcal{O} \subset Sp\text{-}Emb(U, S^n_1)$ such that for any $\mathbf{X} \in \mathcal{O}$, the corresponding Legendrian immersion germ $\mathcal{L}$ is Legendrian stable.*

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