DEMONSTRATIO MATHEMATICA
Vol. XLIII No 2 2010

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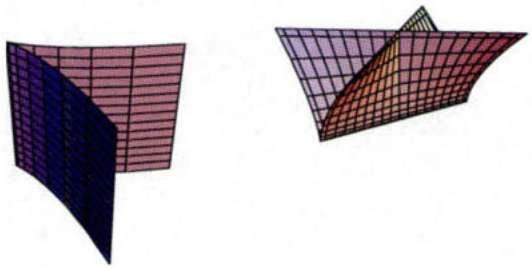
LIGHTLIKE DEVELOPABLES IN MINKOWSKI 3-SPACE

Abstract. We say that a surface in Minkowski 3-space is a *lightlike developable* if all pseudo-normal vectors of the regular part of the surface are lightlike. The tangent surface of a lightlike curve is one of the lightlike developables. We give a generic classification of such surfaces. The all arguments in this paper are elementary. However, we discovered the H_3 type singularity appears in generic for such a class of surfaces. Since the H_3 type singularity usually appears in non-generic situation, this is a quite interesting phenomenon.

1. Introduction

A surface in Euclidean space whose Gauss curvature vanishes on the regular part is called a *developable surface*. It has been known that a developable surface is a part of a conical surface, a cylindrical surface, the tangent surface of a space curve or the glue of such surfaces. Developable surfaces have singularities in general. The tangent surface of a space curve has the most interesting singularities in the above three kinds of surfaces. In fact Cleave [2] showed that the germ of the tangent surface of a generic



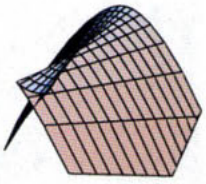
cuspdialedge cuspidal cross cap

2000 *Mathematics Subject Classification*: 53A35, 58C27, 58C28.
Key words and phrases: lightlike developables, Minkowski 3-space, singularities.
*Work partially supported by Grant-in-Aid for Scientific Research (B) (18340013) and Grant-in-Aid for challenging Exploratory Research (21654007).

space curve is locally diffeomorphic to the cuspidal edge $C\times\mathbb{R}$ or the cuspidal cross cap CCR. Here, $C \times \mathbb{R} = \{(x_0, x_1) \in \mathbb{R}^3 \mid x_1^2 = x_1^3\}$ and $CCR = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 = u,\ x_1 = uv^3,\ x_2 = v^2,\ (u, v) \in \mathbb{R}^2\}$.

In this paper we consider the developable surfaces in Minkowski 3-space. In [11] Pei introduced the $\mathbb{R}P^2$-valued Gauss map for the study of Lorentzian geometric properties of surfaces in Minkowski 3-space. We say that a surface is a developable surface in the Minkowski sense if the $\mathbb{R}P^2$-valued Gauss map is singular at each point analogous to the definition of developable surfaces in the Euclidean sense. We can show that the developable surfaces in the Minkowski sense are nothing but the developable surfaces as in the Euclidean sense (cf., Theorem 3.1). Of course the notion of the developable surfaces is independent of the Euclidean structure. However it might be specially interesting subject if we assume that any pseudo-normal is lightlike. We call such the developable surface a *lightlike developable*. This class of surfaces is one of the important subjects in the theory of relativity because they are models of different types of horizons [1, 8]. As an application of the classical classification theorem of developable surfaces, we show that a lightlike developable is a part of a lightlike plane, a part of a lightcone, a part of the tangent surface of a spacelike curve in a lightlike plane, a part of the tangent surface of a lightlike curve or the glue of such four kinds of surfaces (Theorem 5.1). The most interesting case is the tangent surface of a lightlike space curve. We can show that the germ of the tangent surface of a generic lightlike curve at a singular point is locally diffeomorphic to the cuspdialedge $C\times\mathbb{R}$, the Scherbak surface SB or the swallowtail SW (Theorems 5.2, 5.3). Here, $SB = \{(x_1, x_2, x_3) \mid x_1 = u, x_2 = v^3 + uv^2, x_3 = 12v^5 + 10uv^4\}$ and $SW = \{(x_1, x_2, x_3) \mid = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

Scherbak surface



swallowtail

The space of lightlike curves will be described in §5, so that the exact meaning of genericity of the lightlike curve will be established in the same section. We remark that Scherbak [12] showed that SB is given as the

irregular orbit of the finite reflection group H_3 on $\mathbb{C}^3$. We also remark that any lightlike developable is obtained (at least locally) as a one parameter family of lightlike lines along a spacelike curve. In [6] we gave a classification of singularities of the lightlike developable along a generic spacelike curve. As a consequence in [6], only $C \times \mathbb{R}$ or SW appear as generic singularities. The results in [6] is different from the result in this paper, because the space of spacelike curves is different from the space of lightlike curves. The classification of the singularities in this paper is generic for lightlike curves (Theorems 5.2 and 5.3).

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2. Developable surfaces in Euclidean space

In this section we briefly review the results on developable surfaces in Euclidean space. Let $\boldsymbol{x} : U \to \mathbb{R}^3$ be an embedding from an open region $U \subset \mathbb{R}^2$. We call $\boldsymbol{x}$ or the image $S = \boldsymbol{x}(U)$ a *regular surface* in $\mathbb{R}^3$. For any regular surface $\boldsymbol{x} : U \to \mathbb{R}^3$, we define the *first fundamental invariants*:

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v,$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the Euclidean scaler product of $\boldsymbol{a}, \boldsymbol{b}$. We define the unit normal vector

$$\boldsymbol{n} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\sqrt{EG - F^2}},$$

where $\boldsymbol{a} \times \boldsymbol{b}$ is the vector product of $\boldsymbol{a}, \boldsymbol{b}$. Then we define the *second fundamental invariants* by

$$\begin{aligned} L &= \boldsymbol{x}_{uu} \cdot \boldsymbol{n} = -\boldsymbol{x}_u \cdot \boldsymbol{n}_u, \\ M &= \boldsymbol{x}_{uv} \cdot \boldsymbol{n} = -\boldsymbol{x}_u \cdot \boldsymbol{n}_v = -\boldsymbol{x}_v \cdot \boldsymbol{n}_u, \\ N &= \boldsymbol{x}_{vv} \cdot \boldsymbol{n} = -\boldsymbol{x}_v \cdot \boldsymbol{n}_v. \end{aligned}$$

The *Gauss curvature* $K(u, v)$ is defined by

$$K(u, v) = \frac{LN - M^2}{EG - F^2}.$$

We say that a surface $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *developable surface* if $K(u, v) = 0$ at any point $(u, v) \in U$. If the surface has singularities, we say that it is a developable surfaces if the Gauss curvature of the regular part of the surface vanishes. Since the Gauss curvature is the determinant of the differential of the Gauss map, $S = \boldsymbol{x}(U)$ is a developable surface if and only if the Gauss map of the surface is singular at any point of S. It has been known that a developable surface is a ruled surface [14]. A *ruled surface* in $\mathbb{R}^3$ is a surface given by a one-parameter family of lines [7, 14]. It is locally defined as a mapping $F_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{R}^3$ by $F_{(\gamma,\delta)}(t, u) = \boldsymbol{\gamma}(t) + u\boldsymbol{\delta}(t)$, where $\boldsymbol{\gamma} : I \to \mathbb{R}^3$,

$\boldsymbol{\delta} : I \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ are smooth mappings. By a straight forward calculation, we can show that $(t_0, u_0) \in I \times \mathbb{R}$ is a singular point of $F_{(\gamma,\delta)}$ if and only if $\dot{\boldsymbol{\gamma}}(t_0) \times \boldsymbol{\delta}(t_0) + u_0\dot{\boldsymbol{\delta}}(t_0) \times \boldsymbol{\delta}(t_0) = \mathbf{0}$. If we calculate the Gauss curvature of $F_{(\gamma,\delta)}$, then $K \equiv 0$ if and only if $\det(\dot{\boldsymbol{\gamma}}, \boldsymbol{\delta}, \dot{\boldsymbol{\delta}}) = 0$. We say that $F_{(\gamma,\delta)}$ is a *cylindrical surface* if $\boldsymbol{\delta}$ is a constant vector, $F_{(\gamma,\delta)}$ is a *conical surface* if $\boldsymbol{\gamma}$ is a constant vector and $F_{(\gamma,\delta)}$ is a *tangent surface* if $\boldsymbol{\delta}$ is tangent to $\boldsymbol{\gamma}$. Then we have the following well-known classification theorem of developable surfaces [14].

THEOREM 2.1. *A developable surface is one of the following:*

(1) *A part of a cylindrical surface.*
(2) *A part of a conical surface.*
(3) *A part of a tangent developable surface.*
(4) *A glue of the above three surfaces.*

We remark that once we have the above classification theorem, the notion of the developable surfaces is independent of the metric structure of $\mathbb{R}^3$. We only need the affine structure on $\mathbb{R}^3$ for defining the developable surfaces. In the reminder of the paper, we say that a surface is a developable surface if it is one of the four surfaces in the above theorem. In general, developable surfaces have singularities. The tangent surface has the most interesting singularities of the surfaces in the above theorem. Therefore there are many articles concerning the singularities of tangent surfaces. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a smooth curve and denote that $\boldsymbol{\gamma}(t) = (x_1(t), x_2(t), x_3(t))$. We consider the germ of $\boldsymbol{\gamma}$ at $t_0 \in I$. We say that $\boldsymbol{\gamma}$ at t_0 is a *finite type* if there exist natural numbers a_i $(i = 1, 2, 3)$ with $1 \leq a_1 \leq a_2 \leq a_3$ such that $x_i(t) = t^{a_i} + o(t^{a_i})$ $(i = 1, 2, 3)$ under a suitable Affine coordinate transformation of $\mathbb{R}^3$ around $\boldsymbol{\gamma}(t_0)$ and a parameter transformation. In this case we say that $\mathbb{A} = (a_1, a_2, a_3)$ is the *type* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(t_0)$ and denote that $\mathbb{A}(\boldsymbol{\gamma}_{t_0})$. We say that a type $\mathbb{A}$ is *deterministic* if $\mathbb{A}(\boldsymbol{\gamma}_{t_0}) = \mathbb{A}(\widetilde{\boldsymbol{\gamma}}_{\tilde{t_0}}) = \mathbb{A}$ then the map germs $(F_{(\gamma,\dot{\gamma})}, (t_0, 0))$ and $(F_{(\tilde{\gamma},\dot{\tilde{\gamma}})}, (\tilde{t_0}, 0))$ are $\mathcal{A}$-equivalent. Here two map germs $f : (N, x) \to (P, y)$, $g : (N', x') \to (P', y')$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (N', x') \to (N, x)$ and $\psi : (P', y') \to (P, y)$ such that $f \circ \phi = \psi \circ g$. We have the following theorem [2, 4, 5, 9, 10, 13].

THEOREM 2.2. *The type $\mathbb{A}$ of a smooth curve germ in $\mathbb{R}^3$ is deterministic if and only if $\mathbb{A}$ is one of the following types:*

(1) $\mathbb{A} = (1, 2, 2 + r)$, $r = 1, 2, 3 \ldots$,
(2) $\mathbb{A} = (1, 3, 4)$,
(3) $\mathbb{A} = (1, 3, 5)$,
(4) $\mathbb{A} = (2, 3, 4)$,
(5) $\mathbb{A} = (3, 4, 5)$.

We can recognize the type of a smooth curve germ by using the following simple calculations.

PROPOSITION 2.3. *Let a_1, a_2, a_3 be natural numbers with $a_1 < a_2 < a_3$. For a smooth curve germ $\gamma : (I, t_0) \to \mathbb{R}^3$, $\mathbb{A}(\gamma(t_0)) = (a_1, a_2, a_3)$ if and only if*

$$\det(\gamma^{(a_1)}(t_0), \gamma^{(a_2)}(t_0), \gamma^{(a_3)}(t_0)) \neq 0$$

and for any natural numbers b_1, b_2, b_3 with $b_1 < b_2 < b_3$ such that $b_1 < a_1, b_2 \leq a_2, b_3 \leq a_3$, $b_1 = a_1, b_2 < a_2, b_3 \leq a_3$ or $b_1 = a_1, b_2 = a_2, b_3 < a_3$, we have

$$\det(\gamma^{(b_1)}(t_0), \gamma^{(b_2)}(t_0), \gamma^{(b_3)}(t_0)) = 0.$$

Let $\gamma : I \to \mathbb{R}^3$ be a space like curve with type $\mathbb{A}$ at t_0. Then we can show the following assertions ([13, 9, 10, 5, 7]).

(1) If $\mathbb{A} = (1, 2, 3)$, then the germ of the tangent surface $F_{(\gamma,\delta)}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to $C \times \mathbb{R}$.
(2) If $\mathbb{A} = (1, 3, 5)$, then the germ of the tangent surface $F_{(\gamma,\delta)}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to SB.
(3) If $\mathbb{A} = (2, 3, 4)$, then the germ of the tangent surface $F_{(\gamma,\delta)}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to SW.

3. Developable surfaces in Minkowski 3-space

We now prepare basic notions on Minkowski space. Let $\mathbb{R}^3 = \{(x_0, x_1, x_2) \mid x_i \in \mathbb{R}, i = 0, 1, 2\}$ be a 3-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, x_2), \boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2$. The space $(\mathbb{R}^3, \langle\ ,\ \rangle)$ is called *Minkowski 3-space* (or, *Lorentz-Minkowski 3-space*) and denoted by $\mathbb{R}^3_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $= 0$ or < 0 respectively. We remark that the zero vector is considered to be lightlike in this paper. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{n} \in \mathbb{R}^3_1$ and a real number c, the plane with pseudo normal $\boldsymbol{n}$ is given by

$$P(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}.$$

We say that $P(\boldsymbol{n}, c)$ is a *spacelike* , *timelike* or *lightlike hyperplane* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. For any point $\boldsymbol{p} \in \mathbb{R}^3_1$, the *lightcone* with the vertex $\boldsymbol{p}$ is defined by

$$LC_p = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x} - \boldsymbol{p}, \boldsymbol{x} - \boldsymbol{p} \rangle = 0\}.$$

We also define the *lightcone circle* by

$$S^1_+ = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \boldsymbol{x} = (1, x_2, x_3), \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

For any non zero lightlike vector $\boldsymbol{x} = (x_0, x_1, x_2)$, we denote that

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in S^1_+.$$

Moreover, the following hypersurface is called *de Sitter space*:

$$S^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

For any $\boldsymbol{x} = (x_0, x_1, x_2)$, $\boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3_1$, the *pseudo vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = (-(x_1 y_2 - x_2 y_1), x_2 y_0 - x_0 y_2, x_0 y_1 - x_1 y_0).$$

In [11] D. Pei studied Lorentzian geometric properties of surfaces by defining the $\mathbb{R}P^2$-valued Gauss map, where $\mathbb{R}P^2$ is the real projective plane. Let $\boldsymbol{x} : U \to \mathbb{R}^3_1$ be an immersion from an open region $U \subset \mathbb{R}^2$. We define a map $G_{\mathbb{M}} : U \to \mathbb{R}P^2$ by $G_{\mathbb{M}}(u, v) = \langle \boldsymbol{x}_u(u, v) \wedge \boldsymbol{x}_v(u, v) \rangle_{\mathbb{R}}$. We call $G_{\mathbb{M}}$ the *Minkowski Gauss map* of $S = \boldsymbol{x}(U)$. We consider a surface in Minkowski 3-space such that the Minkowski Gauss map is singular at any point of the surface. We can show that such surfaces are developable surfaces.

THEOREM 3.1. *Let $\boldsymbol{x} : U \to \mathbb{R}^3_1$ be a surface. If the Minkowski Gauss map $G_{\mathbb{M}}$ is singular at any point of $S = \boldsymbol{X}(U)$, then $S = \boldsymbol{x}(U)$ is a developable surface.*

Proof. We consider the canonical Euclidean scalar product on $\mathbb{R}^3_1$:

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_0 y_0 + x_1 y_1 + x_2 y_2.$$

For any $\boldsymbol{x} = (x_0, x_1, x_2) \in \mathbb{R}^3_1$, we denote that $\overline{\boldsymbol{x}} = (-x_0, x_1, x_2)$. It follows that $\boldsymbol{x}$ and $\boldsymbol{y}$ are pseudo-orthogonal by the Minkowski scalar product if and only if $\overline{\boldsymbol{x}}$ and $\boldsymbol{y}$ are orthogonal by the canonical Euclidean scalar product. We define a map $G_{\mathbb{E}} : U \to S^2$ by

$$G_{\mathbb{E}}(u, v) = \frac{\overline{\boldsymbol{x}_u(u, v) \wedge \boldsymbol{x}_v(u, v)}}{\|\overline{\boldsymbol{x}_u(u, v) \wedge \boldsymbol{x}_v(u, v)}\|_E},$$

where $\|\boldsymbol{a}\|_E$ is the Euclidean norm of $\boldsymbol{a}$. Then $G_{\mathbb{E}}$ is the Gauss map of $S = \boldsymbol{x}(U)$ in the Euclidean sense.

On the other hand we consider a mapping $C : \mathbb{R}P^2 \to \mathbb{R}P^2$ defined by $C(\langle \boldsymbol{x} \rangle_{\mathbb{R}}) = \langle \bar{\boldsymbol{x}} \rangle_{\mathbb{R}}$. Then C is a diffeomorphism such that $C \circ G_{\mathbb{M}} = \pi \circ G_{\mathbb{E}}$, where $\pi : S^2 \to \mathbb{R}P^2$ is the canonical double covering. It follows that $G_{\mathbb{M}}$ is singular at a point $p = \boldsymbol{x}(u, v)$ if and only if $G_{\mathbb{E}}$ is singular at p. This completes the proof. ∎

Since vectors in $\mathbb{R}^3_1$ are classified into three kinds of vectors, $\mathbb{R}P^2$ is a disjoint union of the disk D^2, the circle S^1 and the Moebius strip MB such that $\langle \boldsymbol{x} \rangle_{\mathbb{R}} \in D^2$ if $\boldsymbol{x}$ is timelike, $\langle \boldsymbol{x} \rangle_{\mathbb{R}} \in S^1$ if $\boldsymbol{x}$ is lightlike and $\langle \boldsymbol{x} \rangle_{\mathbb{R}} \in MB$ if $\boldsymbol{x}$ is spacelike.

4. Curves in Minkowski 3-space

In this section we consider the properties of curves in Minkowski 3-space which will be used in §5. Let $\gamma : I \to \mathbb{R}^3_1$ be a spacelike regular curve. We denote that $N(t) = \ddot{\gamma}(t)$ and $B(t) = \dot{\gamma}(t) \wedge N(t)$.

PROPOSITION 4.1. *For any unit speed spacelike curve $\gamma : I \to \mathbb{R}^3_1$, if $\gamma''(s)$ is lightlike for any $s \in I$, then $\gamma(I)$ is a curve in a lightlike plane. Here $\gamma'(s) = d\gamma/ds(s)$.*

Proof. Since $\langle \gamma'(s), \gamma'(s) \rangle = 1$, we have $\langle \gamma''(s), \gamma'(s) \rangle = 0$. Since $\widetilde{\gamma''(s)}$ is a tangent vector of S^2_1 and lightlike, $\widetilde{\gamma''(s)}$ is a lightlike line, so that $\widetilde{\gamma''(s)}$ is a constant vector. Therefore we have $\widetilde{\gamma''(s)} = \widetilde{\gamma''(s_0)}$. By the above relation, we have $\langle \gamma'(s), \gamma''(s_0) \rangle = 0$. It follows that

$$\frac{d}{ds}\langle \gamma(s), \gamma''(s_0) \rangle = 0.$$

If we put $c = \langle \gamma(s_0), \gamma''(s_0) \rangle$, then $\langle \gamma(s), \gamma''(s_0) \rangle = c$. The last equation means tat γ is a curve in the lightlike plane $LP(\gamma''(s_0), c)$. ∎

We say that a curve $\gamma : I \to \mathbb{R}^3_1$ is a *lightlike curve* if $\dot{\gamma}$ is lightlike.

PROPOSITION 4.2. *Let $\gamma : S^1 \to \mathbb{R}^3_1$ be a lightlike curve. Then there exists a point $t_0 \in S^1$ such that $\dot{\gamma}(t_0) = \mathbf{0}$ (i.e., singularities exist).*

Proof. We denote that $\gamma(t) = (x_0(t), x_1(0), x_2(0))$. Since γ is a lightlike curve, we have the relation $(\dot{x}_0(t))^2 = (\dot{x}_1(t))^2 + (\dot{x}_2(t))^2$, so that $\dot{\gamma}(t_0) = \mathbf{0}$ if and only if $\dot{x}_0(t) = 0$. However, S^1 is compact, then $\dot{x}_0(t)$ has the maximum and the minimum points. At such points, we have $\dot{x}_0(t) = 0$. ∎

5. Lightlike developables in Minkowski 3-space

In this section we study a special class of developable surfaces in Minkowski 3-space. By Theorem 3.1, if the Minkowski Gauss map is singular at any point of a surface, then the surface is a developable surface. The most interesting developable surfaces in Minkowski 3-space are surfaces whose pseudo normal field $\boldsymbol{x}_u \wedge \boldsymbol{x}_v$ is always lightlike. We call such a surface a *lightlike developable surface*. Of course the lightlike developable surface is a developable surface, so that we can apply the classification theorem.

THEOREM 5.1. *A lightlike developable surface is one of the following surfaces:*

(1) *A part of a lightlike plane.*
(2) *A part of the lightcone.*
(3) *A part of the tangent surface of a curve in a lightlike plane.*
(4) *A part of the tangent surface of a lightlike curve.*
(5) *A glue of the above four surfaces.*

Proof. Let $\boldsymbol{x} : U \to \mathbb{R}^3_1$ be a lightlike developable surface. If the Minkowskian Gauss map $G_{\mathbb{M}}$ is a point, then $\boldsymbol{x}(U)$ is a part of a lightlike plane. We now assume that the image of the Minkowski Gauss map $G_{\mathbb{M}}$ is a non-singular curve. By Theorem 2.1, a developable surface is a conical surface, a cylindrical surface, a tangent surface of a space curve or a glue of these three surfaces. We distinguish three cases.

(1) Suppose that a surface is a cylindrical surface $\boldsymbol{x}(t,u) = \boldsymbol{\gamma}(t) + u\boldsymbol{e}$, where $\boldsymbol{e}$ is a constant vector. The pseudo normal vector is given by

$$\boldsymbol{x}_t(t,u) \wedge \boldsymbol{x}_u(t,u) = \dot{\boldsymbol{\gamma}}(t) \wedge \boldsymbol{e}.$$

Suppose that $\dot{\boldsymbol{\gamma}}(t) \wedge \boldsymbol{e}$ is lightlike. If the smooth curve $\boldsymbol{\gamma}(t) \wedge \boldsymbol{e}$ is not a line, there exist three points $t_0, t_1, t_2 \in \mathbb{R}$ such that two pairs $\dot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{e}, \dot{\boldsymbol{\gamma}}(t_1) \wedge \boldsymbol{e}$ $\dot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{e}, \dot{\boldsymbol{\gamma}}(t_2) \wedge \boldsymbol{e}$ are consist of linearly independent vectors. Therefore we have two different lines

$$LP(\dot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{e}, 0) \cap LP(\dot{\boldsymbol{\gamma}}(t_1), 0)\ LP(\dot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{e}, 0) \cap LP(\dot{\boldsymbol{\gamma}}(t_2), 0).$$

Since $\langle \dot{\boldsymbol{\gamma}}(t) \wedge \boldsymbol{e}, \boldsymbol{e} \rangle = 0$, we have $\boldsymbol{e} \in LP(\dot{\boldsymbol{\gamma}}(t) \wedge \boldsymbol{e}, 0)$ for any t. However,we have

$$\boldsymbol{e} \in LP(\dot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{e}, 0) \cap LP(\dot{\boldsymbol{\gamma}}(t_1), 0)$$
$$\boldsymbol{e} \in LP(\dot{\boldsymbol{\gamma}}(t_0) \wedge \boldsymbol{e}, 0) \cap LP(\dot{\boldsymbol{\gamma}}(t_2), 0).$$

This is a contradiction. Therefore $\dot{\boldsymbol{\gamma}}(t) \wedge \boldsymbol{e}$ has a constant direction $\boldsymbol{v}$. Since $\dot{\boldsymbol{\gamma}}(t) \in LP(\boldsymbol{v}, 0)$, we have $\langle \boldsymbol{\gamma}(t), \boldsymbol{v} \rangle = c$. It follows from the fact $\boldsymbol{e} \in LP(\boldsymbol{v}, 0)$ that $\boldsymbol{x}(t,u) \in LP(\boldsymbol{v}, c)$, so that a lightlike cylindrical surface is a part of a lightlike plane.

(2) Suppose that a surface is a conical surface $\boldsymbol{x}(t,u) = \boldsymbol{a} + u\boldsymbol{e}(t)$, where $\boldsymbol{a}$ is a constant vector. The pseudo normal vector is given by

$$\boldsymbol{x}_t(t,u) \wedge \boldsymbol{x}_u(t,u) = \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t).$$

Suppose that $\dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t)$ is lightlike. We remark that the surface $\boldsymbol{x}(t,u)$ is the envelope of the family of tangent planes

$$LP(\dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t), c(t)) = \{\boldsymbol{X} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{X}, \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t) \rangle = c(t)\ \},$$

where

$$c(t) = \langle \boldsymbol{a} + u\boldsymbol{e}(t), \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t) \rangle = \langle \boldsymbol{a}, \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t) \rangle.$$

On the other hand, we consider a lightcone defined by

$$\overline{\boldsymbol{x}}(t,v) = \boldsymbol{a} + v\dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t).$$

We also consider a function $F(\boldsymbol{X},t) = \langle \boldsymbol{X} - \boldsymbol{a}, \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t)\rangle$. Then we have

$$\frac{\partial F}{\partial t}(\boldsymbol{X},t) = \langle \boldsymbol{X} - \boldsymbol{a}, \ddot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t)\rangle.$$

Since $\dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t)$ is lightlike, we have

$$F(\overline{\boldsymbol{x}}(t,v),t) = \langle v\dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t), \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t)\rangle = 0.$$

If we have derivative with respect to t, we have

$$\frac{\partial F}{\partial t}(\overline{\boldsymbol{x}}(t,v),t) = 2v\langle \ddot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t), \dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t)\rangle = 0.$$

Therefore the lightcone $\overline{\boldsymbol{x}}(t,v)$ is also the envelope of the same families of lightlike planes $LP(\dot{\boldsymbol{e}}(t) \wedge \boldsymbol{e}(t), c(t))$, so that the surface $\boldsymbol{x}(t,u)$ is a part of a lightcone.

(3) Suppose that a surface is a tangent surface $\boldsymbol{x}(t,u) = \boldsymbol{\gamma}(t) + u\dot{\boldsymbol{\gamma}}(t)$. Since $\boldsymbol{x}_t(t,u) = \dot{\boldsymbol{\gamma}}(t) + u\ddot{\boldsymbol{\gamma}}(t)$, $\boldsymbol{x}_u(t,u) = \dot{\boldsymbol{\gamma}}(t)$, the tangent space is $\langle \dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t)\rangle_{\mathbb{R}}$. By the assumption, this space is lightlike. If $\dot{\boldsymbol{\gamma}}(t)$ is spacelike, we may assume that $\dot{\boldsymbol{\gamma}}(t)$ has unit length (i.e., $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle = 1$). Therefore we have $2\langle \dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t)\rangle = 0$. If we also suppose that $\ddot{\boldsymbol{\gamma}}(t)$ is spacelike, then the pseudo-normal vector $\dot{\boldsymbol{\gamma}}(t) \wedge \ddot{\boldsymbol{\gamma}}(t)$ is timelike, so that it contradicts to the assumption that $\langle \dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t)\rangle_{\mathbb{R}}$ is a lightlike plane. It follows that $\dot{\boldsymbol{\gamma}}(t)$ is spacelike and $\ddot{\boldsymbol{\gamma}}(t)$ is lightlike or $\dot{\boldsymbol{\gamma}}(t)$ is lightlike and $\ddot{\boldsymbol{\gamma}}(t)$ is spacelike. By Proposition 4.1, if $\ddot{\boldsymbol{\gamma}}(t)$ is lightlike for any t, then $\boldsymbol{\gamma}$ is a curve in a lightlike plane. In this case the surface is the tangent surface of a curve in a lightlike plane. If $\dot{\boldsymbol{\gamma}}(t)$ is lightlike for any t, the surface is the tangent surface of a lightlike curve. ∎

In the above list of lightlike developable surfaces the most interesting surface might be the tangent surface of a lightlike curve. We call such the surface a *lightlike tangent surface*. We stick to lightlike tangent surfaces. We now consider the space of lightlike curves in $\mathbb{R}^3_1$. If $\boldsymbol{\sigma}(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t))$ is a smooth mapping, then

$$\boldsymbol{\gamma}(t) = \int_{t_0}^{t} \sigma(\tau)d\tau$$

is a lightlike curve. On the other hand, let $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ be a lightlike curve. If there exists an interval $J \subset I$ such that $\dot{\boldsymbol{\gamma}}(t) = 0$ for any $t \in J$, then $\boldsymbol{\gamma}(t)$ is constant on J. Therefore, we assume that $\boldsymbol{\gamma}$ has only isolated singular points. In this case, we denote that $\boldsymbol{\gamma}(t) = (x_0(t), x_1(t), x_2(t))$ and $\dot{\boldsymbol{\gamma}}(t) = (\dot{x}_0(t), \dot{x}_1(t), \dot{x}_2(t))$. Since $\dot{\boldsymbol{\gamma}}(t)$ is lightlike, $\dot{x}_0(t_0) = 0$ if and only if

$\dot{\boldsymbol{\gamma}}(t_0) = \mathbf{0}$. If $\dot{x}_0(t) \neq 0$, then there exists a smooth function $\theta(t)$ that

$$\frac{\dot{x}_1(t)}{\dot{x}_0(t)} = \cos\theta(t), \quad \frac{\dot{x}_2(t)}{\dot{x}_0(t)} = \sin\theta(t).$$

On the other hand, suppose that $\dot{x}_0(t_0) = 0$. By Taylor's theorem, we have

$$\begin{aligned}
\dot{x}_0(t) &= a_0(t-t_0)^{r_0} + o(r_0) \quad (a_0 \neq 0),\\
\dot{x}_1(t) &= a_1(t-t_0)^{r_1} + o(r_1) \quad (a_1 \neq 0),\\
\dot{x}_2(t) &= a_0(t-t_0)^{r_2} + o(r_2) \quad (a_2 \neq 0)
\end{aligned}$$

at any $t \in (t_0-\varepsilon, t_0+\varepsilon)$ for sufficiently small $\varepsilon > 0$. Since $\dot{x}_0(t)^2 = \dot{x}_1(t)^2 + \dot{x}_2(t)^2$, we have

$$a_0^2(t-t_0)^{2r_0} + o(2r_0) = a_1^2(t-t_0)^{2r_1} + o(2r_1) + a_2^2(t-t_0)^{2r_2} + o(2r_2),$$

so that we have $r_0 = \min(r_1, r_2)$. Therefore,

$$\frac{\dot{x}_1(t)}{\dot{x}_0(t)}, \quad \frac{\dot{x}_2(t)}{\dot{x}_0(t)}$$

are smooth functions at t_0. This means that there exists smooth functions $r(t)$ and $\theta(t)$ we have

$$\dot{\boldsymbol{\gamma}}(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)).$$

We now consider the following set of curves in $\mathbb{R}^3_1$:

$L(I, \mathbb{R}^3_1) = \{\boldsymbol{\sigma}(t) = (r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)) \mid t \in I$

$r(t), \theta(t)$ are smooth functions such that $r(t)$ has isolated zero points.$\}$.

We now regard $L(I, \mathbb{R}^3_1)$ as the space of lightlike curves equipped with the Whitney C^∞-topology. From now on, we say that $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ is a lightlike curve if $\dot{\boldsymbol{\gamma}} \in L(I, \mathbb{R}^3_1)$. Moreover, we define a smooth mapping $\psi : \mathbb{R} \times S^1 \to LC_0$ by

$$\psi(r, \cos\theta, \sin\theta) = (r, r\cos\theta, r\sin\theta).$$

Then $\psi|_{(\mathbb{R}\setminus\{0\})\times S^1} : (\mathbb{R} \setminus \{0\}) \times S^1 \to LC_0 \setminus \{\mathbf{0}\}$ is a diffeomorphism. We define that

$C^\infty_*(I, \mathbb{R} \times S^1) = \{(r(t), \cos\theta(t), \sin\theta(t)) \mid t \in I$

$r(t), \theta(t)$ are smooth functions such that $r(t)$ has isolated zero points.$\}$.

We define a mapping $\psi_* : C^\infty_*(I, \mathbb{R} \times S^1) \to L(I, \mathbb{R}^3_1)$ by $\psi_*(\delta) = \psi \circ \delta$. Then ψ_* is continuous with respect to the Whitney C^∞-topology. We can also define a well-defined continuous mapping $\iota : L(I, \mathbb{R}^3_1) \to C^\infty_*(I, \mathbb{R} \times S^1)$ by $\iota(r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)) = (r(t), \cos\theta(t), \sin\theta(t))$. Then we have $\psi_* \circ \iota = 1_{L(I,\mathbb{R}^3_1)}$, $\iota \circ \psi_* = 1_{C^\infty_*(I,\mathbb{R}\times S^1)}$. Therefore ψ_* is a homeomorphism. This means that we can consider that $C^\infty_*(I, \mathbb{R} \times S^1)$ is the space of lightlike curves. For any $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ with $\dot{\boldsymbol{\gamma}} \in L(I, \mathbb{R}^3_1)$, we have $\dot{\boldsymbol{\gamma}}(t) =$

$(r(t), r(t)\cos\theta(t), r(t)\sin\theta(t))$. In this case $\widetilde{\boldsymbol{\gamma}}(t_0) = (1, \cos\theta(t_0), \sin\theta(t_0))$ determines the tangent direction of $\boldsymbol{\gamma}$ at t_0 even if $\dot{\boldsymbol{\gamma}}(t_0) = \mathbf{0}$. Therefore, we can define the tangent surface of $\boldsymbol{\gamma}$ by

$$F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\gamma}})}(t,u) = \boldsymbol{\gamma}(t) + u\widetilde{\boldsymbol{\gamma}}(t).$$

We have the following classification of singularities.

THEOREM 5.2. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ be a smooth curve such that $\dot{\boldsymbol{\gamma}} \in L(I, \mathbb{R}^3_1)$. Then we have the followings:*

(1) *The tangent surface germ $F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\gamma}})}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ if $r(t_0) \neq 0$ and $\dot{\theta}(t_0) \neq 0$.*

(2) *The tangent surface germ $F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\gamma}})}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the Scherbak surface SB if $r(t_0) \neq 0$, $\dot{\theta}(t_0) = 0$ and $\ddot{\theta}(t_0) \neq 0$.*

(3) *The tangent surface germ $F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\gamma}})}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the swallowtail SW if $r(t_0) = 0$, $\dot{r}(t_0) \neq 0$ and $\dot{\theta}(t_0) \neq 0$.*

Proof. We now calculate the type $\mathbb{A}$ of $\boldsymbol{\gamma}$ at t_0 under the above three conditions.

(1) By a straight forward calculation, we have

$$\det(\dot{\boldsymbol{\gamma}}(t), \ddot{\boldsymbol{\gamma}}(t), \dddot{\boldsymbol{\gamma}}(t)) = \dddot{r}(t)(\dot{\theta}(t))^3.$$

If $r(t_0) \neq 0$ and $\dot{\theta}(t_0) \neq 0$, then the type of $\boldsymbol{\gamma}$ at t_0 is $(1,2,3)$. This means that the tangent surface germ $F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\gamma}})}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the cuspidal edge.

(2) Suppose that $r(t_0) \neq 0$ and $\theta(t_0) = 0$. Then we have

$$\ddot{\boldsymbol{\gamma}}(t_0) = (\dot{r}(t_0), \dot{r}(t_0)\cos\theta(t_0), \dot{r}(t_0)\sin\theta(t_0)).$$

It follows that $\dot{\boldsymbol{\gamma}}(t_0) \wedge \ddot{\boldsymbol{\gamma}}(t_0) = \mathbf{0}$. This means that $\det(\dot{\boldsymbol{\gamma}}(t_0), \ddot{\boldsymbol{\gamma}}(t_0), \boldsymbol{\gamma}^{(2+n)}(t_0)) = 0$ for any natural number n. Under the above assumption, we have

$$\begin{aligned}
&\det(\dot{\boldsymbol{\gamma}}(t_0), \dddot{\boldsymbol{\gamma}}(t_0), \boldsymbol{\gamma}^{(4)}(t_0)) \\
&= \begin{vmatrix} r & r\cos\theta & r\sin\theta \\ \ddot{r} & \ddot{r}\cos\theta - r\sin\theta\ddot{\theta} & \ddot{r}\sin\theta + r\cos\theta\ddot{\theta} \\ \dddot{r} & \dddot{r}\cos\theta - 3r\sin\theta\ddot{\theta} - r\sin\theta\dddot{\theta} & \dddot{r}\sin\theta + 3r\cos\theta\ddot{\theta} + r\cos\theta\dddot{\theta} \end{vmatrix}(t_0) \\
&= 0.
\end{aligned}$$

Hence the type of $\boldsymbol{\gamma}$ at t_0 is different from $(1,3,4)$. We can also calculate that

$$\det(\dot{\boldsymbol{\gamma}}(t), \dddot{\boldsymbol{\gamma}}(t), \boldsymbol{\gamma}^{(5)}(t)) = 3(r(t_0))^3(\ddot{\theta}(t_0))^3.$$

Therefore $\det(\dot{\boldsymbol{\gamma}}(t_0), \dddot{\boldsymbol{\gamma}}(t_0), \boldsymbol{\gamma}^{(5)}(t_0)) \neq 0$ under the assumption that $r(t_0) \neq 0$, $\dot{\theta}(t_0) = 0$ and $\ddot{\theta}(t_0) \neq 0$, so that the tangent surface germ $F_{(\boldsymbol{\gamma},\widetilde{\boldsymbol{\gamma}})}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the Scherbak surface.

(3) Suppose that $r(t_0) = 0$. This means that $\dot{\boldsymbol{\gamma}}(t_0) = \mathbf{0}$, so that the type of $\boldsymbol{\gamma}$ at t_0 is different from $(1, n, m)$ for any real number $1 < n < m$. By the direct calculation like as the case (2), we have

$$\det(\ddot{\boldsymbol{\gamma}}(t_0), \dddot{\boldsymbol{\gamma}}(t_0), \boldsymbol{\gamma}^{(4)}(t_0)) = 6(\dot{\boldsymbol{\gamma}}(t_0))^3(\dot{\theta}(t_0))^3 \neq 0.$$

Therefore, the type of $\boldsymbol{\gamma}$ at t_0 is $(2, 3, 4)$, so that the tangent surface germ $F_{(\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}})}(I \times \mathbb{R})$ at $(t_0, 0)$ is diffeomorphic to the swallowtail. ∎

By the standard jet transversality theorem (cf., [3], Theorem 4.9) we have the following proposition.

PROPOSITION 5.3. *Let $C^\infty(I, \mathbb{R}^2)$ be the space of smooth mappings with the Whitney C^∞-topology. Then the set*

$$\mathcal{O} = \{(r, \theta) \in C^\infty(I, \mathbb{R}^2) \mid (r, \theta) \text{ satisfies the conditon } (*)\}$$

is open and dense in $C^\infty(I, \mathbb{R}^2)$.

$(*)$
- (1) *$r(t)$ has isolated zero points.*
- (2) *If $r(t_0) = 0$, then $\dot{r}(t_0) \neq 0$ and $\dot{\theta}(t_0) \neq 0$.*
- (3) *The points t_0 with $r(t_0) \neq 0$ and $\dot{\theta}(t_0) = 0$ are isolated.*
- (4) *If $r(t_0) \neq 0$ and $\dot{\theta}(t_0) = 0$, then $\ddot{\theta}(t_0) \neq 0$.*

Since ψ_* is a homeomorphism, it follows from Proposition 5.3 that

$$\mathcal{O} = \{(r(t), r(t)\cos\theta(t), r(t)\sin\theta(t)) \in L(I, \mathbb{R}^3_1) \mid (r, \theta) \text{ satisfies (1), (2), (3) in Theorem 5.2}\}$$

is open and dense in $L(I, \mathbb{R}^3_1)$. Therefore Theorem 5.2 gives a classification of singularities of the tangent surface of a generic lightlike curve.

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Received December 12, 2009.