DEMONSTRATIO MATHEMATICA
Vol. XLIII No 2 2010

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MEDIAL AXIS AND A MOMENT MAP

Abstract. We use Maslov dequantization to compute the medial axis of an embedded hypersurface.

1. Introduction

It is well known that the medial axis of a compact embedded submanifold M in $\mathbb{R}^n$ can be calculated as an approximation of a Voronoi diagram, see for instance [ABK98]. In this paper we use tropical geometry to give a new interpretation of that fact.

Let M be a compact connected smooth manifold of dimension $n-1$, embedded by $\gamma\colon M \to \mathbb{R}^n$. For every x in $\mathbb{R}^n$ there exists a distance function $g_x : M \to \mathbb{R}$ defined by

$$g_x(s) = \|x - \gamma(s)\|^2.$$

The *medial axis* $\mathrm{Cut}(M)$ consists of the closure of the set of those x in $\mathbb{R}^n$ for which g_x has a non-unique global minimum. Recall that the *corner locus* $\mathrm{Corner}(f)$ of a convex function f is the set of points where f is not differentiable.

The main statement of this paper is:

THEOREM 1. *Let M be a compact connected smooth manifold of dimension $n-1$, embedded by $\gamma\colon M \to \mathbb{R}^n$. Write for $s \in M$:*

$$(1) \qquad f_s(x) = \langle x, \gamma(s)\rangle - \frac{1}{2}\|\gamma(s)\|^2; \qquad f(x) \stackrel{\text{def}}{=} \sup_{s\in M} f_s(x)$$

and

$$r_h(x) = \log_h \Big(\int_M h^{f_s(x)} ds \Big).$$

Then:

2000 *Mathematics Subject Classification*: Primary 52B55; Secondary 57Q99, 68U05.
Key words and phrases: Voronoi diagrams, medial axis.

(1) *The function f is a convex function, and satisfies the equality:*

$$f(x) = \lim_{h \to \infty} r_h(x). \tag{2}$$

(2) *The corner locus of f is the medial axis* $\mathrm{Cut}(M)$ *of* M.
(3) *The derivative of f outside the corner locus of f maps* $x \in \mathbb{R}^n \setminus \mathrm{Cut}(M)$ *to the nearest point on* M.
(4) *The derivative of the Legendre transform of f maps* $\mathrm{CH}(\gamma(M)) \setminus \gamma(M)$ *onto the medial axis.*
(5) *The sequence of function* $r_h(x)$ *is monotonely increasing.*
(6) *The gradient of* $r_h(x)$ *is a diffeomorphism from* $\mathbb{R}^n$ *to the interior of the convex hull of* $\gamma(M)$.

We know of no cases where the integrals $r_h(x)$ can be evaluated explicitly, but if one uses the integral formula and numerical integration for h sufficiently big one can plot f, as is illustrated in Figure 1.

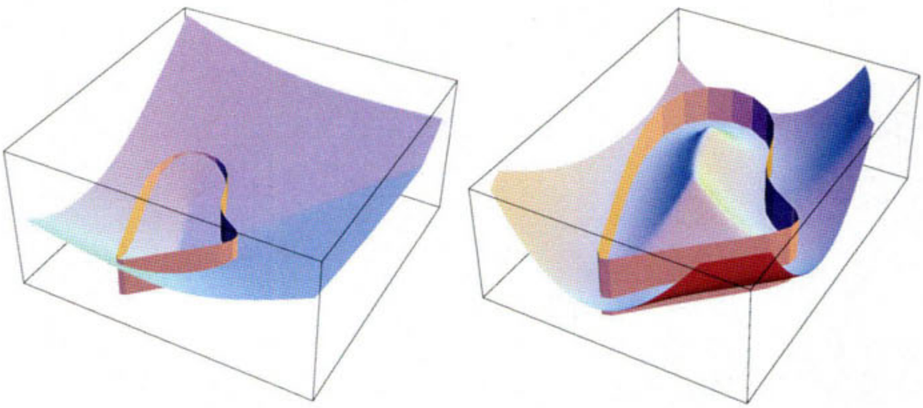

Fig. 1. On the left hand side the graph of $f(x) = \sup_{s \in M} f_s(x)$ for γ, on the right hand side the graph of $g(x) = \inf_{s \in M} \frac{1}{2}\|x - \gamma(s)\|^2$.

In the next section we recall the definition of Voronoi diagrams and relate this to affine linear functions. In the third section we proove the main theorem.

2. Voronoi diagrams and piecewise affine convex functions

2.1. The Voronoi diagram

Take a point set $\{P_1, \ldots, P_N\} \subset \mathbb{R}^n$. Throughout this article we assume that $\dim(\mathrm{CH}(\{P_1, \ldots, P_N\})) = n$. We want to divide $\mathbb{R}^n$ according to the nearest distance principle. Define the functions:

$$g_i \colon \mathbb{R}^n \to \mathbb{R}, \quad g_i(x) = \frac{1}{2}\|x - P_i\|^2 \quad g(x) = \min_{1 \le i \le N} g_i(x). \tag{3}$$

Here we have used the following notation:

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2.$$

DEFINITION 1. For a non-void subset $\alpha \subset \{P_1, \dots, P_N\}$ the set $\mathrm{Vor}(\alpha)$ is the closure of

$$\{x \in \mathbb{R}^n \mid g_i(x) = g(x)\ P_i \in \alpha \text{ and } g_j(x) > g(x)\ P_j \notin \alpha\}.$$

We put $\mathrm{Vor}(\emptyset) = \mathbb{R}^n$.

For a subset $\alpha \subset \{P_1, \dots, P_N\}$ we have

$$\mathrm{Vor}(\alpha) \subset \cap_{P_i \in \alpha} \mathrm{Vor}(P_i).$$

If the intersection of $\mathrm{Vor}(\alpha)$ and $\mathrm{Vor}(\alpha')$ is not empty then there is a $\beta \subset \{P_1, \dots, P_N\}$ such that $\mathrm{Vor}(\alpha) \cap \mathrm{Vor}(\alpha') = \mathrm{Vor}(\beta)$. The union over non-void α of all $\mathrm{Vor}(\alpha)$ is $\mathbb{R}^n$. In this situation we can also consider the affine functions and their maximum

$$f_i(x) = \langle x, P_i \rangle - \frac{1}{2}\|P_i\|^2; \quad f(x) = \max_i f_i \tag{4}$$

which are related to the distance function by

$$g_i(x) = \frac{1}{2}\|x\|^2 - f_i(x); \quad g(x) = \frac{1}{2}\|x\|^2 - f(x).$$

Note that we use here the same notation in the finite case, as we did in the continuous case, as stated in Theorem 1, formula (1). Obviously we might have defined the sets $\mathrm{Vor}(\alpha)$ using the functions f_i, i.e.: for a subset $\alpha \subset \{P_1, \dots, P_N\}$ the set $\mathrm{Vor}(\alpha)$ is

$$\{x \in \mathbb{R}^n \mid f_i(x) = f(x)\ P_i \in \alpha \text{ and } f_j(x) < f(x)\ P_j \notin \alpha\}.$$

The affine definition of Voronoi diagrams is a bit more elegant.

The union of all $\mathrm{Vor}(\alpha)$, where α contains at least two points is a closed set of codimension 1, and is called *the Voronoi diagram.* It is easy to see that the function f is convex and that its corner locus is equal to the Voronoi diagram. We have thus defined the Voronoi diagram as a corner locus of a convex function. Below we will define the medial axis as a corner locus of a convex function.

It follows that the non-differentiability locus of the functions g and f are the same and that the Voronoi diagram can thus be defined using only affine functions.

2.2. The Legendre transform

Dual to the Voronoi diagram is the Delaunay triangulation.

DEFINITION 2.

$$\mathrm{Del}(\alpha) = \begin{cases} \mathrm{CH}(\alpha) & \text{if } \mathrm{Vor}(\alpha) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

We are also interested in defining the Delaunay triangulation in a very different way.

DEFINITION 3. The Legendre transform of a convex function f, with domain $D \subset \mathbb{R}^n$ is

$$\hat{f}(\xi) = \sup_{x \in D} \left(\langle \xi, x \rangle - f(x) \right).$$

When the supremum does not exists, we put $\hat{f}(\xi) = \infty$. The domain $\mathrm{Dom}(\hat{f})$ of $\hat{f}$ are those ξ for which $\hat{f}(\xi) < \infty$.

Let us proceed to give a few examples. The Legendre transform of $h(x) = \frac{1}{2}\|x\|^2$ is the function h itself. The Legendre transform of a linear function $f_i = \langle x, P_i \rangle - \frac{1}{2}\|P_i\|^2$ is $< \infty$ only when $\xi = P_i$.

Let us now calculate the Legendre transform of f defined by (4). The domain where $\hat{f}(\xi) < \infty$ is the convex hull of the points $\{P_1, \ldots, P_N\}$. In fact, we have

$$\hat{f}_i(P_i) = \frac{1}{2}\|P_i\|^2 \Rightarrow \hat{f}(\xi) = \inf \left(\sum_i \frac{\lambda_i}{2} \|P_i\|^2 \right)$$

where the infimum is taken over all λ_i such that

$$\sum_{1 \leq i \leq N} \lambda_i P_i = \xi \quad \text{and} \quad \forall i \colon \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = 1.$$

We can be more precise: for each ξ in the convex hull $\mathrm{CH}(\{P_1, \ldots, P_N\})$ there exists a subset $\alpha = \alpha(\xi)$, of $\{P_1, \ldots, P_N\}$ such that $f(x) = f_i(x)$ for exactly those $i \in \alpha$. This requires $x \in \mathrm{Vor}(\alpha)$, which should be non-void in this case. This means $\xi \in \mathrm{Del}(\alpha)$. It follows

$$\hat{f}(\xi) = \left(\sum_{i \in \alpha} \frac{\lambda_i}{2} \|P_i\|^2 \right)$$

and $\hat{f}$ is an affine function on each of the cells in the Delaunay triangulation. Note that the domain of definition of $\max_{i \in \alpha} f_i$ is exactly $\mathrm{Del}(\alpha)$.

One can combine f and $\hat{f}$ in the following function

$$(\xi, x) \mapsto F(x, \xi) = f(x) + \hat{f}(\xi) - \langle x, \xi \rangle.$$

This function hides the *Gateau differential.* Namely, let $h \colon \mathbb{R}^n \to \mathbb{R}$ be a function. If the limit

$$h'(x;\xi) = \lim_{t\downarrow 0} \frac{h(x+t\xi)-h(x)}{t}$$

exists, it is called the Gateau differential. See [Hör94, Theorem 2.1.22]. For a convex set $K \subset \mathbb{R}^n$ the *supporting function* of K is the function

$$\xi \mapsto \sup_{x\in K}\langle x, \xi\rangle.$$

Theorem 2.2.11 in [Hör94] says that the Gateau differential $\xi \to f'(x;\xi)$ is the supporting function of

$$\{\mu \mid F(x,\mu) = 0\}.$$

The Gateau differential is called Clarke's generalized derivative in [APS97]. The set of which it is the support function is called $\partial f(x)$. It is stated in that article that $\partial f(x)$ is the convex hull of the gradients of the functions f_i for which $f_i(x) = f(x)$.

That last statement and the Theorem 2.1.22 are all equivalent to what is neatly formulated in Proposition 1 of [PR04]:

THEOREM 2. *We have*

$$\mathrm{Del}(\alpha) = \{\xi \mid F(x,\xi) = 0 \quad \forall x \in \mathrm{Vor}(\alpha)\}.$$

Reversely

$$\mathrm{Vor}(\alpha) = \{x \mid F(x,\xi) = 0 \quad \forall \xi \in \mathrm{Del}(\alpha)\}.$$

In other words, if x is a point in the relative interior of $\mathrm{Vor}(\alpha)$ then $\partial f(x) = \mathrm{Del}(\alpha)$, and reversely, if ξ is a point in the relative interior of $\mathrm{Del}(\alpha)$ then $\partial \hat{f}(\xi) = \mathrm{Vor}(\alpha)$. We see that there are two ways of looking at the Delaunay triangulation. One simply through the duality, and the other through the generalized derivative of f.

2.3. Maslov dequantization and the moment map

In this section we approximate the function f with a parameterized smooth family of convex functions. The function $f = \max f_i$ is convex, it can be approximated by convex functions $x \to f(h,x) = \log_h(\sum_{i=1}^N h^{f_i(x)})$, whose properties are similar to those of f.

Note that

$$f(x) = \lim_{h\to\infty} f(h,x)$$

due to Maslov dequantization, which is a fancy name for the identity:

$$\lim_{h\to\infty} \log_h(h^a + h^b) = \max(a,b) \tag{5}$$

that holds for any two real numbers a and b.

PROPOSITION 1. *Let $0 < h' < h$, then*

- *the functions $f(x)$ and $f(h,x)$ are convex,*
- *$\forall x \in \mathbb{R}^n$ $f(h,x) > f(h',x) > f(x)$,*
- *the gradient of $f(h,x)$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathrm{CH}(\{P_1,\dots,P_N\})$.*

Proof. We need to show that

$$f\left(h, \frac{1}{2}x + \frac{1}{2}y\right) \leq \frac{1}{2}(f(h,x) + f(h,y)),$$

or

$$\sum_{i=1}^{N} h^{f_i(\frac{1}{2}x+\frac{1}{2}y)} \leq \Big(\sum_{i=1}^{N} h^{f_i(x)}\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{N} h^{f_i(x)}\Big)^{\frac{1}{2}}. \tag{6}$$

Put $a_i = h^{\frac{1}{2}f_i(x)} > 0$ and $b_i = h^{\frac{1}{2}f_i(y)} > 0$ then $a_i b_i = h^{f_i(\frac{1}{2}x+\frac{1}{2}y)}$, $a_i^2 = h^{f_i(x)}$ and $b_i^2 = h^{f_i(y)}$. Hölder's inequality says that

$$\sum_{i=1}^{N} a_i b_i \leq \Big(\sum_{i=1}^{N} a_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{N} b_i^2\Big)^{\frac{1}{2}}$$

which is equivalent to (6).

The second part is proven using Jensen's inequality, which asserts that for a convex function q:

$$q\left(\frac{1}{N}\sum_{i=1}^{N} a_1\right) \leq \frac{1}{N}\sum_{i=1}^{N} q(a_i).$$

We need to show that

$$0 < h' < h \quad \Rightarrow \quad f(h',x) < f(h,x).$$

We put

$$q(y) = h^{\log_{h'}(y)} = y^{\log_{h'}(h)} \text{ and } a_i = (h')^{f_i(x)}.$$

The function $q(y) = y^a$, for some $a > 1$, thus it is convex. Use Jensen's inequality and take $\log_h$ at both sides to get the desired result.

For the third statement consider the moment map from toric geometry:

$$H_e\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad H_e(x) = \frac{\sum_{i=1}^{N} P_i e^{f_i(x)}}{\sum_{i=1}^{N} e^{f_i(x)}}.$$

It is proved in [Ful93] that H_e is a real analytic diffeomorphism from $\mathbb{R}^n$ to the interior of $\mathrm{CH}(\{P_1,\dots,P_N\})$, when that interior is not empty. One recognizes H_e as a gradient:

$$H(x) = \nabla_x\Big(\log\Big(\sum_{i=1}^{N} e^{f_i(x)}\Big)\Big) = \nabla_x(f(e,x)).$$

Now consider the gradient of $f(h,x)$:

$$H_h(x) = \nabla_x\Big(\log_h\Big(\sum_{i=1}^N h^{f_i(x)}\Big)\Big) = \frac{\sum_{i=1}^N P_i h^{f_i(x)}}{\sum_{i=1}^N h^{f_i(x)}}.$$

The f_i are of the form $\langle x, P_i\rangle - \frac{1}{2}\|P_i\|^2$. Put $\tilde{f}_i(u) = \langle u, P_i\rangle - \frac{\log(h)}{2}\|P_i\|^2$. We already know that

$$u \longrightarrow \tilde{H}(u) = \frac{\sum_{i=1}^N P_i e^{\tilde{f}_i(u)}}{\sum_{i=1}^N e^{\tilde{f}_i(u)}}$$

is a real analytic diffeomorphism from $\mathbb{R}^n$ to the interior of $\mathrm{CH}(\{P_1,\ldots,P_N\})$. Because we have $H_h(x) = \tilde{H}(x\log(h))$ the proof is complete. ∎

When x does not lie on the Voronoi diagram there is an i, with $1 \le i \le N$, $f_i(x) = f(x)$ and $f_i(x) > f_j(x)$ for all $j \neq i$. It follows that $\lim_{h\to\infty} H_h(x) = P_i$. Note that $\lim_{h\to\infty} H_h$ is no longer a diffeomorphism; full Voronoi cells collapse to a point!

We continue to describe the inverse of H_h. If t is smooth convex function that the Legendre transform $\hat{t}$ of t is

$$\hat{t}(\xi) = \langle(\nabla_x t)^{-1}(\xi), \xi\rangle - t\left((\nabla_x t)^{-1}(\xi)\right).$$

As a consequence

$$\begin{aligned}\nabla\hat{t}(\xi) &= (\nabla_x t)^{-1}(\xi) + \sum_{j=1}^n \frac{\partial(\nabla_x t)^{-1}}{\partial\xi_j}\xi_j \\ &\quad - \sum_{j=1}^n (\nabla_{x_j} t)\left((\nabla_x t)^{-1}(\xi)\right)\frac{\partial(\nabla_x t)^{-1}}{\partial\xi} \\ &= (\nabla t)^{-1}(\xi).\end{aligned} \tag{7}$$

Thus, the inverse of H_h is the gradient of Legendre transform $\hat{f}(h,x)$ of $f(h,x)$.

3. The medial axis and Maslov dequantization

We have seen in the above that the Voronoi diagram can also be calculated using the affine functions f_i, instead of the distance function. We have also seen that the corner locus of the function f defined in (4) is the Voronoi diagram. We will now follow the same procedure with the embedding $\gamma\colon M \to \mathbb{R}^n$. We will show next that the medial axis is a corner locus of a convex function. Let us prove Theorem 1.

Proof. We switch back from the function f as defined in (4) (the finite case), to the function f defined in equation (1) (the continuous case). The function $f(x)$ is convex, because it is the supremum of a number of convex functions.

Let us prove part 2: *the corner locus of f is the medial axis* $\mathrm{Cut}(M)$ *of* M. Because M is compact the supremum is attained for one or more points $s = s(x)$ in M. We see that $x \in \mathbb{R}^n$ and $s \in M$, for which $f(x) = f_s(x)$, are related by

$$(8) \qquad \langle x - \gamma(s), \dot\gamma(s) \rangle = 0.$$

Here $\dot\gamma(s)$ is the tangent vector to γ and the inner product is taken as vectors with source in $\gamma(s)$. So x lies on the normal of one or more $s \in M$. Of those that satisfy (8) we choose one that has the highest value of $f_{s_i}(x)$, and thus the lowest value of

$$\frac{1}{2}\|x\|^2 - f_{s_i}(x) = \frac{1}{2}\|x - \gamma(s_i)\|^2.$$

The s we get from $\sup_{s \in M} f_s$ corresponds to the point closest to x on the submanifold M, because it is the s we get from the minimal distance

$$x \longrightarrow \inf_{s \in M} \frac{1}{2}\|x - \gamma(s)\|^2.$$

If x does not lie on the medial axis there is a unique closest point s on M and it follows that f is smooth in some neighborhood of x. If x lies on the medial axis there are either two or more points that realize the supremum $f(x)$ or x lies on the caustic. In both cases f is not smooth. Hence the points where $f(x)$ is not differentiable form exactly the medial axis of M.

Next we prove part 1: *the integral formula for f in* (2). The core of the argument is: approximate the integral on the right hand side by a point set on M. In the Riemann sum only the highest value of $f_s(x)$ counts as $h \to \infty$.

More precisely, let $\iota\colon \Sigma \longrightarrow M$ be a triangulation of M whose top-dimensional simplices are Γ (assume all of the same volume). Compute the lower and upper Riemann sums

$$\log_h \Big(\sum_{\Gamma \in \Sigma} \inf_{s \in \iota(\Gamma)} \mathrm{vol}(\Gamma) h^{f_s(x)} \Big) \le \log_h \Big(\int_M h^{f_s(x)} ds \Big)$$
$$\le \log_h \Big(\sum_{\Gamma \in \Sigma} \sup_{s \in \iota(\Gamma)} \mathrm{vol}(\Gamma) h^{f_s(x)} \Big).$$

Evaluate the lower Riemann sum

$$(9) \qquad \log_h \Big(\sum_{\Gamma \in \Sigma} \inf_{s \in \iota(\Gamma)} \mathrm{vol}(\Gamma) h^{f_s(x)} \Big)$$
$$= \log_h \left((\#\Gamma)\,\mathrm{vol}(\Gamma) \right) + \log_h \Big(\sum_{\Gamma \in \Sigma} \inf_{s \in \iota(\Gamma)} h^{f_s(x)} \Big)$$
$$= \log_h \left(\mathrm{vol}(M) \right) + \log_h \Big(\sum_{\Gamma \in \Sigma} h^{\inf_{s \in \iota(\Gamma)} f_s(x)} \Big).$$

From the Maslov dequantization identity (5) we know that the last expression tends to as $h \to \infty$

$$\max_{\Gamma \in \Sigma}(\inf_{s \in \iota(\Gamma)} f_s(x)). \tag{10}$$

The upper Riemann sum tends to

$$\max_{\Gamma \in \Sigma}(\sup_{s \in \iota(\Gamma)} f_s(x)). \tag{11}$$

Sub-dividing the simplices Γ further both (10) and (11) tend to $\sup_{s \in M} f_s(x)$, which is what we needed to prove.

We will now prove part 5, namely that *the functions* $r_h(x)$ *are convex and monotonic in* h: $r_{h'}(x) \leq r_h(x)$ if $h' < h$. The proof is completely analogous to the proof of Proposition 1.

To prove the convexity of $r_h(x)$ put

$$p(s) = h^{\frac{1}{2} f_s(x)} \quad \text{and} \quad q(s) = h^{\frac{1}{2} f_s(y)}$$

and apply the continuous version of the Hölder inequality. The monotonicity of the sequence $r_h(x)$ follows from the continuous version of Jensen's inequality, just as the monotonicity of the sequence $f(h, x)$ in the proof of Proposition 1 followed from the discrete version of Jensen's inequality.

Next we show part 6 that *the differential of* r_h *is diffeomorphism from* $\mathbb{R}^n$ *to the interior of* $\mathrm{CH}(\gamma(M))$. The gradient of r_h is

$$\nabla r_h(x) = \frac{\int_M \gamma(s) h^{f_s(x)} ds}{\int_M h^{f_s(x)} ds}. \tag{12}$$

Consider again a triangulation $\iota \colon \Sigma \to M$ of M, where each top-dimensional simplex Γ has the same volume. In each simplex Γ take a point P_Γ. Consider the quotient

$$\frac{\sum_{\Gamma \in \Sigma} \mathrm{vol}(\Gamma) P_\Gamma h^{\langle x, P_\Gamma \rangle - \frac{1}{2}\|P_\Gamma\|^2}}{\sum_{\Gamma \in \Sigma} \mathrm{vol}(\Gamma) h^{\langle x, P_\Gamma \rangle - \frac{1}{2}\|P_\Gamma\|^2}}. \tag{13}$$

From the remarks on Riemann sums above it follows that taking barycentric subdivisions the quotient (13) tends to (12). From Proposition 1 we know that the image of (13) is the convex hull $\mathrm{CH}(\{P_\Gamma \mid \Gamma \in \Sigma\})$. Hence the image of $\nabla r_h(x)$ is $\mathrm{CH}(\gamma(M))$.

Part 3. We show that *the derivative of* f *outside the corner locus of* f *maps* $x \in \mathbb{R}^n \setminus \mathrm{Cut}(M)$ *to the nearest point on* M. If x lies outside the medial axis then there is a point s_0 on x such $f(x) = f_{s_0}(x)$. The $\gamma(s_0)$ is the unique closest to x on M. For all $\mathbb{R}^n \ni y \neq x$ we have $f(y) \geq f_{s_0}(x)$, so that certainly $\gamma(s_0) \in \partial f(x)$. We are done if we show that f is smooth at x. Again, because x does not lie on the medial axis there is a neighborhood U of x in $\mathbb{R}^n \setminus \mathrm{Cut}(M)$ there is a smooth map $h \colon U \to M$ that assigns to y the closest point $h(y)$ on M. Hence, on U the f can be written $f(y) = f_{h(y)}(y)$, which is a smooth function.

Before we prove part 4 we discuss the situation at the medial axis. At the medial axis the gradient of f is not defined. The generalized derivative can be calculated for points on the medial axis. For each point x on the medial axis there is a subset $A(x) \subset M$ defined by

$$A(x) = \{s \in M \mid f(x) = f_s(x)\}.$$

The point x is the center of a sphere of radius $\sqrt{\|x\|^2 - 2f(x)}$ that is tangent to $\gamma(M)$ at all points of $A(x)$. The generalized derivative $\partial f(x)$ is $\mathrm{CH}(\gamma(A(x)))$. One can think of it as the Delaunay cell dual to a point on the medial axis. The union of all the $A(x)$ for all x on medial axis $Cut(M)$ is M itself. The sets $\partial f(x)$ fill $\mathrm{CH}(\gamma(M))$. For each $\xi \in \mathrm{CH}(\gamma(M)) \setminus \gamma(M)$ there is a unique $x \in \mathrm{Cut}(M)$.

Part 4: *The derivative of the Legendre transform of f maps* $\mathrm{CH}(\gamma(M)) \setminus \gamma(M)$ *onto the medial axis.* The convex functions f and $\hat{f}$ fulfil the duality relation:

$$F(x, \xi) = f(x) + \hat{f}(\xi) - \langle x, \xi \rangle = 0.$$

Standard theory of convex functions tells us that this is equivalent with each of the conditions:

$$\partial f(x) = \xi, \quad \partial \hat{f}(\xi) = x.$$

As we have seen above $\partial f(x) = \mathrm{CH}(\gamma(A(x)))$. And x is unique for each ξ. So $\partial \hat{f}(\xi)$ contains a single point. Now we are ready. ∎

The union of all the $A(x)$ for all x on the interior medial axis $\mathrm{IntCut}(M)$ is M itself. The convex structure of the Legendre transform gives is very intriguing, since it shows the similariry with the Delaunay triangulation. The cells dual to points on the medial axis can be seen if we take a lot of points on M and calculate the Delaunay triangulation of this point set. We get Figure 2. This is another way of saying that the dual of the interior medial

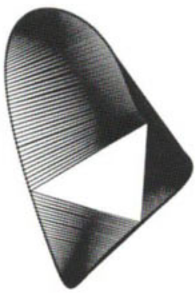

Fig. 2. Delaunay triangulation for a lot of points on the curve γ from Figure 1.

axis in the sense of the Theorem of Passare and Rullgård is the region in $\mathbb{R}^n$ bounded by manifold $\mathrm{CH}(M)$ itself.

A direct computation of the Legendre transform $\hat{f}$ of f shows, that its domain is $\mathrm{CH}(M)$ and that it is affine on each of the simplices $\partial f(x) = \mathrm{CH}(\gamma(A(x)))$. For a finite set $A(x)$ there is the formula $\hat{f}(\xi) = \sum_{A(x)} \frac{1}{2}\lambda_i \|y\|^2$, where λ_i are coordinates in the hull.

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Received November 28, 2009.