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THE SECOND JUMP OF MILNOR NUMBERS

Abstract. Let f_0 be a plane curve singularity. Let $(\mu_0, \mu_1, \ldots, \mu_k)$ be all possible Milnor numbers of non-degenerate deformations of f_0 (in decreasing order). We prove that $\mu_2 = \mu_1 - 1$ for f_0 with one segment Newton polygon (μ_1 is given by the Bodin formula).

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity (for short a singularity), i.e. f_0 is the germ of a holomorphic function having an isolated critical point at 0. A *deformation* of f_0 is a family $(f_s)_{s \in U}$ of isolated singularities (or smooth germs) analytically dependent on the parameter s in an open neighborhood $U \subset \mathbb{C}$ of $0 \in \mathbb{C}$. The *jump of Milnor numbers of the deformation* $(f_s)_{s \in U}$ is the number

$$\mu(f_0) - \mu(f_s) \quad s \in U \setminus \{0\},$$

where $\mu(f_s)$ is the Milnor number of f_s. This number is well defined because $\mu(f_s)$ is independent of $s \neq 0$ for sufficiently small s. We will denote it by $\lambda((f_s))$. Moreover, by the upper semi-continuity of μ (Proposition II.5.3 in [8], Theorem 2.6 in [2]) it is a non-negative integer. The *jump* $\lambda(f_0)$ (or the first jump) of f_0 is the minimum of the non-zero jumps of the (f_s) over all deformations of f_0. According to A. Bodin [1] N. A'Campo posed the problem to compute $\lambda(f_0)$. It is still an open problem. S. Gusein-Zade [3] proved that there are singularities f_0 such that $\lambda(f_0) > 1$ and that for irreducible f_0, $\lambda(f_0) = 1$.

Bodin in [1] considered the following weaker problem: to compute the jump $\lambda'(f_0)$ of f_0 over all non-degenerate deformations of f_0 (i.e. f_s are non-degenerate in the Kouchnirenko sense for $s \neq 0$). Of course, we have always $\lambda(f_0) \leq \lambda'(f_0)$. For $n = 2$ he gave a formula for $\lambda'(f_0)$ for f_0 with the

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Newton polygon reduced to one segment (in particular for f_0 irreducible; in this case $\lambda'(f_0) = 1$). Much more general problem is to compute all Milnor numbers arising from all deformations of f_0 or at least from all non-degenerate deformations of f_0. In the last case to each singularity f_0 we may associate a finite strictly decreasing sequence

$$\Lambda'(f_0) = (\mu_0, \mu_1, \dots, \mu_k),$$

of all possible Milnor numbers of non-degenerate deformations of f_0. We have $\mu_0 = \mu(f_0)$, $\mu_1 = \mu(f_0) - \lambda'(f_0)$, $\mu_k = 0$. This sequence may be curious. We easily check that

1. for $f_0(x,y) = x^8 - y^5$, we have $\Lambda'(f_0) = (28, 27, \dots, 0)$,
2. for $f_0(x,y) = x^8 - y^4$, we have $\Lambda'(f_0) = (21, 18, 17, \dots, 0)$,
3. for $f_0(x,y) = x^7 - y^5$, we have $\Lambda'(f_0) = (24, 23, \dots, 16, 15, 13, 12 \dots, 0)$.

The Bodin formula gives μ_1 for singularities with one segment Newton polygon. The main result of the paper is that for such singularities $\mu_2 = \mu_1 - 1$ i.e. the "second jump" of f_0 is always equal to 1.

2. Non-degenerate singularities

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $f_0(x,y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij}x^i y^j$, $f_0(0,0) = 0$ be a singularity. Let $\mathrm{supp}(f_0) := \{(i,j) \in \mathbb{N}^2 : a_{ij} \neq 0\}$. The *Newton diagram* of f_0 is the convex closure of $\bigcup_{(i,j)\in\mathrm{supp}(f_0)}((i,j) + \mathbb{R}^2_+)$, ($\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x \geq 0 \wedge y \geq 0\}$). We denote it by $\Gamma_+(f_0)$.

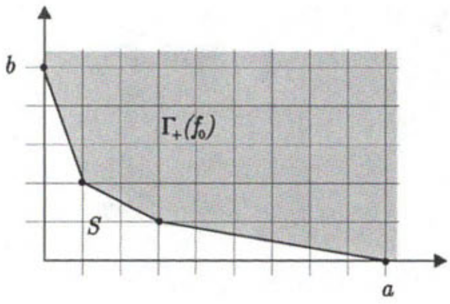
The boundary of the Newton diagram $\Gamma_+(f_0)$ is the union of two semi-lines and a finite number of compact, non-parallel segments, which are not contained in these semi-lines. These segments constitute the *Newton polygon of singularity* f_0, which we will denote by $\Gamma(f_0)$. Often we will identify pairs $(i,j) \in \mathbb{N}^2$ with monomials $x^i y^j$. We will call singularity f_0 *convenient* if $\Gamma(f_0)$ has common points with OX and OY axes.

For segment $\gamma \in \Gamma(f_0)$ we define $(f_0)_\gamma := \sum_{(i,j)\in\gamma} a_{ij}x^i y^j$. We call a singularity f_0 *non-degenerate on* $\gamma \in \Gamma(f_0)$ (in the Kouchnirenko sense), when the system of equations

$$\frac{\partial (f_0)_\gamma}{\partial x}(x,y) = 0, \qquad \frac{\partial (f_0)_\gamma}{\partial y}(x,y) = 0$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. We call a singularity f_0 *non-degenerate*, when f_0 is non-degenerate on every segment $\gamma \in \Gamma(f_0)$. We notice that if (f_s) is a deformation of f_0, then for sufficiently small $s \neq 0$, Newton's diagram $\Gamma_+(f_s)$ doesn't depend on s.

Let f_0 be a convenient singularity. By S we denote area of the set bounded by OX and OY axes and the polygon $\Gamma(f_0)$. By a and b we denote distance between the origin $(0,0)$ and the common part of Newton polygon $\Gamma_+(f_0)$ with OX and OY axes.



For a convenient singularity f_0 we define its *Newton's number* by

$$\nu(f_0) := 2S - a - b + 1.$$

It is easy to check that $\nu(f_0) \geq 0$.

We will remind a known theorem about non-degenerate singularities that we will use further.

THEOREM 1. (Kouchnirenko [4]) *Assume that a singularity f_0 is convenient. Then*

- $\mu(f_0) \geq \nu(f_0)$,
- *if f_0 is non-degenerate, then $\mu(f_0) = \nu(f_0)$.*

3. Non-degenerate jump of Milnor numbers of a singularity

Let f_0 be a singularity. A deformation $(f_s)_{s\in U}$ of f_0 is called *non-degenerate* if f_s is non-degenerate for $s \neq 0$. The set of all non-degenerate deformations of the singularity f_0 we will denote $\mathcal{D}^{nd}(f_0)$. *Non-degenerate jump $\lambda'(f_0)$ of the singularity f_0* is the minimal of non-zero jumps over all non-degenerate deformations f_0, what means

$$\lambda'(f_0) := \min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} \lambda((f_s)),$$

where by $\mathcal{D}_0^{nd}(f_0)$ we denote the all non-degenerate deformations (f_s) of f_0 for which $\lambda((f_s)) \neq 0$.

Obviously

PROPOSITION 1. *For each singularity f_0 we have the inequality*

$$\lambda(f_0) \leq \lambda'(f_0).$$

The above inequality may be strict.

Example 1. Let $f_0(x,y) = x^4 - y^4$. From Gusein-Zade [3] we have $\lambda(f_0) > 1$. It is easy to check, that $\lambda((f_s)) = 2$ for $f_s(x,y) = x^4 - (y^2 + sx)^2$. Therefore $\lambda(f_0) = 2$. From the next part of the article (see Theorem 3 and Example 2) we have $\lambda'(f_0) = 3$. It realizes non-degenerate deformation $f_s(x,y) = x^4 - y^4 + sx^3$, $s \in \mathbb{C}$. Therefore, in this case $\lambda(f_0) < \lambda'(f_0)$.

4. Formula for non-degenerate jump of a non-degenerate singularity

First we recall definitions and some well known facts about quasi-homogeneous polynomials. Let $f \in \mathbb{C}[X,Y]$ be a non-constant polynomial. We call f *quasi-homogeneous polynomial of degree* d, when there exists $(m,n) \in \mathbb{N}_+^2$ such that, $\mathrm{GCD}(m,n) = 1$ and

$$f(\lambda^m x, \lambda^n y) = \lambda^d f(x,y).$$

We call numbers m and n *weights of variables* x and y. f is a *homogeneous polynomial*, when $m = n = 1$.

Property 1. *Let f be a quasi-homogeneous polynomial with weights m and n. Then there exists a homogeneous polynomial (a form) ν and numbers r, $s \in \mathbb{N}$ such that*

$$f(x,y) = x^r y^s \nu(x^n, y^m), \quad \nu(0,y) \neq 0, \quad \nu(x,0) \neq 0.$$

The form ν is called *corresponding to* f_0. Before we give Bodin formula for non-degenerate jump we will recall known properties about non-degenerate singularities. Let f_0 be a singularity and $\Gamma(f_0)$ its Newton polygon.

Property 2. *For any $\gamma \in \Gamma(f_0)$ polynomial $(f_0)_\gamma$ is quasi-homogeneous.*

Property 3. *For any $\gamma \in \Gamma(f_0)$ the ends of γ belong to* $\mathrm{supp} f_0$. *If γ doesn't contain any other point of* $\mathrm{supp} f_0$ *besides ends, then f_0 is non-degenerate on γ.*

Property 4. *f_0 is non-degenerate on $\gamma \in \Gamma(f_0)$ $\Leftrightarrow$ the form ν corresponding to $(f_0)_\gamma$ has no multiple factors $\Leftrightarrow$ discriminant $\Delta(\nu)$ of the form ν is not zero.*

One can find Property 3 in [7] (the proof of Property 2.6) and Property 4 in [5].

Let f_0 be a non-degenerate and convenient singularity. We will denote by J the set of all monomials $x^p y^q$, where $p + q \geq 1$, lying in closed domain bounded by axes and Newton diagram $\Gamma_+(f_0)$. Obviously J is a finite set.

Lemma 1. *For any $x^p y^q \in J$ the deformation $f_s = f_0 + s x^p y^q$, $s \in U$, is non-degenerate.*

Proof. Because $x^p y^q \in J$, so for $s \neq 0$, $\mathrm{supp}(f_s) = \{(p,q)\} \cup \mathrm{supp} f_0$. Therefore Newton diagram f_s is constant for sufficiently small $s \neq 0$. Let γ be a segment of the Newton polygon of f_s, for $s \neq 0$. We will consider cases:

1. $(p,q) \notin \gamma$. Then ends γ lie in $\mathrm{supp} f_0$, γ is segment of Newton polygon of singularity f_0 and $(f_s)_\gamma = (f_0)_\gamma$. Because f_0 is non-degenerate, so f_s is non-degenerate over γ.

2. $(p,q) \in \gamma$ and besides (p,q) there exists the only one point from $\mathrm{supp} f_0$, (which we denote by (k,l)) lying in γ. Then (k,l) and (p,q) are the ends of γ. From Property 3 f_s is non-degenerate on γ.

3. $(p,q) \in \gamma$ and besides (p,q) there exist more than one point of $\mathrm{supp}(f_s)$ lying on γ. We will consider subcases:

(i) $(p,q) \in \Gamma(f_0)$. Consider the discriminant $\Delta(s)$ of the form ν_s corresponding to $(f_s)_\gamma$. The value $\Delta(0)$ is equal to the discriminant of the form corresponding to $(f_0)_\gamma$, so $\Delta(0) \neq 0$ (because f_0 is non-degenerate on γ). Therefore $\Delta(s) \neq 0$ for s from sufficiently small neighborhood of zero. From Property 4, f_s is non-degenerate on γ.

(ii) $(p,q) \notin \Gamma(f_0)$. Then (p,q) is an end of γ. In this case γ is a continuation of a certain segment $\gamma_0 \in \Gamma(f_0)$. Without loss of generality we may assume, that (p,q) is the left end of γ. Let $(f_s)_\gamma(x,y) = (f_0)_\gamma(x,y) + s x^p y^q$. From Property 2 the polynomial $(f_s)_\gamma$ is quasi-homogeneous. We denote by m, n weights of variables x and y and d degree of this polynomial. From Property 1 there exists homogeneous polynomial ν_s and numbers r, $t \in \mathbb{N}_+$ such that

$$(f_s)_\gamma(x,y) = x^r y^t \nu_s(x^n, y^m) \quad \nu_s(0,y) \neq 0, \quad \nu_s(x,0) \neq 0.$$

Hence and from assumption $\nu_s(x,y)$ has the form

$$\nu_s(x,y) = s y^d + a_1 y^{d-1} x + \ldots + a_d x^d, \quad \text{where} \quad a_d \neq 0.$$

Consider the discriminant $\Delta(s)$ of the form ν_s corresponding to $(f_s)_\gamma$. It is easy to check that $\Delta(s) = (d^d a_d^{d-1}) \cdot s^d +$ terms of a degrees less than d. Because $a_d \neq 0$, so $\deg_s \Delta(s) > 0$. It means, that $\Delta(s) \neq 0$ for $s \neq 0$ in a certain neighborhood of zero. From Property 4 f_s is non-degenerate on γ. ∎

Lemma 1 says, that for each convenient and non-degenerate singularity f_0 the deformation $f_s = f_0 + s x^p y^q$, $x^p y^q \in J$, $s \in U$, is a non-degenerate deformation of f_0. We will denote it by $(f_s^{(p,q)})$. In [1] Bodin gave the formula for $\lambda'(f_0)$ in terms of the deformations $(f_s^{(p,q)})$. Since [1] has been published only in preprint form with sketchy proofs, we will give a full proof of the Bodin formula.

THEOREM 2. (Bodin [1]) *If f_0 is a non-degenerate and convenient singularity, then*

$$\lambda'(f_0) = \min_{x^p y^q \in J_0} \lambda((f_s^{(p,q)})),$$

where $J_0 \subset J$ is the set of monomials $x^p y^q$ such that $\lambda((f_s^{(p,q)})) \neq 0$.

Proof. By the definition of $\lambda'(f_0)$ we have to prove the equality

$$\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0) - \mu(f_s)) = \min_{x^p y^q \in J_0} (\mu(f_0) - \mu(f_s^{(p,q)})).$$

The inequality "$\leq$" is obvious. We will prove the opposite inequality "$\geq$". Take any non-degenerate deformation $(f_s) \in \mathcal{D}_0^{nd}(f_0)$ of f_0.

Rearranging terms in f_s, $s \neq 0$, we can rewrite it as follows

$$f_s(x,y) = f_0(x,y) + c_1(s)x^{p_1}y^{q_1} + \ldots + c_k(s)x^{p_k}y^{q_k} + R(s,x,y),$$

$c_i \neq 0$, $c_i(0) = 0$, $(p_i, q_i) \in \Gamma(f_s)$, $i = 1,\ldots,k$, and supp$R$ lie above the Newton polygon of f_s. Because $\lambda((f_s)) > 0$, it is easy to prove, that among points (p_i, q_i), $i = 1,\ldots,k$, there exists a point (p_j, q_j), such that $\lambda((f_0 + c_j(s)x^{p_j}y^{q_j})) > 0$. We will show, that for this j

$$\mu(f_0) - \mu(f_0 + c_j(s)x^{p_j}y^{q_j}) \leq \mu(f_0) - \mu(f_s).$$

It is enough to prove that

$$(\star) \qquad \mu(f_0 + c_j(s)x^{p_j}y^{q_j}) \geq \mu(f_s).$$

Let S_1, S_2 be areas corresponding to deformations $(f_0 + c_j(s)x^{p_j}y^{q_j})$ and (f_s), respectively. By a_1, b_1 and a_2, b_2 we denote the distance beetwen the origin $(0,0)$ and common part $\Gamma(f_0 + c_j(s)x^{p_j}y^{q_j})$ and $\Gamma(f_s)$ with axes OX and OY, respectively. Because deformations $(f_0 + c_j(s)x^{p_j}y^{q_j})$ and (f_s) are non-degenerate, so by the Kouchnirenko Theorem it is enough to prove, that $2(S_1 - S_2) - (a_1 - a_2) - (b_1 - b_2) \geq 0$.

We will consider possible cases:

1. $a_1 > a_2$, $b_1 > b_2$. We will denote by (m_l, n_l), $l = 1,\ldots,t$, consecutive vertices of the Newton polygon of $\Gamma(f_0 + c_j(s)x^{p_j}y^{q_j})$. From $a_1 > a_2$, $b_1 > b_2$ it follows, that $t \geq 3$. We have, that $(m_1, n_1) = (0, b_1)$ and $(m_t, n_t) = (a_1, 0)$. If we consider now triangles with vertices: $(0,b_1)$, $(0,b_2)$, (m_2,n_2) and $(a_1,0)$, $(a_2,0)$, (m_{t-1},n_{t-1}), then denoting by h_1, h_2 (h_1, $h_2 \geq 1$) their heights to bases $\overline{(0,b_1),(0,b_2)}$ and $\overline{(a_1,0),(a_2,0)}$, respectively, we have

$$\begin{aligned} &2(S_1 - S_2) - (a_1 - a_2) - (b_1 - b_2) \\ &\quad \geq 2(\frac{1}{2}(a_1 - a_2)\cdot h_2 + \frac{1}{2}(b_1 - b_2)\cdot h_1) - (a_1 - a_2) - (b_1 - b_2) \\ &\quad = (a_1 - a_2)\cdot(h_2 - 1) + (b_1 - b_2)\cdot(h_1 - 1) \geq 0. \end{aligned}$$

2. $a_1 > a_2$, $b_1 = b_2$. With the same notations as in the first case we have that

$$2(S_1 - S_2) - (a_1 - a_2) \geq 2 \cdot \frac{1}{2}(a_1 - a_2) \cdot h_2 - (a_1 - a_2) = (a_1 - a_2) \cdot (h_2 - 1) \geq 0.$$

3. When $a_1 = a_2$ and $b_1 > b_2$, we have situation analogical to the second case.

4. If $a_1 = a_2$, $b_1 = b_2$, then obviously $S_1 \geq S_2$ and then $2(S_1 - S_2) \geq 0$.

Therefore in every case we have $(\star)$. ∎

COROLLARY 1. *If f_0, $\tilde{f}_0$ are non-degenerate and convenient singularities and $\Gamma(f_0) = \Gamma(\tilde{f}_0)$, then $\lambda'(f_0) = \lambda'(\tilde{f}_0)$.*

5. The case of one segment Newton polygon

In some cases we can give exact value of the non-degenerate jump of a singularity. It happens when Newton polygon of f_0 consists of only one segment (particularly when f_0 is an irreducible singularity). We will begin with the simplest case.

THEOREM 3. (Bodin [1]) *Let $f_0(x,y) = x^p - y^q$, $p, q \geq 2$ and $d := \mathrm{GCD}(p,q)$. Without loss of generality we may assume, that $p \geq q$.*

1. *If $1 \leq d < q \leq p$, then $\lambda'(f_0) = d$.*
2. *If $d = q$, then $\lambda'(f_0) = q - 1$.*

Proof. 1. There exist integers a, b such that

$$ap + bq = d.$$

We may assume, that $a > 0$, $b < 0$ and $a < q$. Let's take monomial $x^{-b}y^{q-a}$. This monomial belongs to J, because $-b > 0$, $q - a > 0$ and the point $(-b, q-a)$ lies under line $\frac{x}{p} + \frac{y}{q} = 1$ defined by the only segment of the Newton polygon f_0. Moreover it is an element of J_0, because the area of the triangle with vertices $(p,0)$, $(0,q)$ and $(-b, q-a)$ is equal to $\frac{d}{2}$ which implies $\lambda((f_s^{(-b,q-a)})) = d > 0$. Hence

$$\lambda'(f_0) \leq d.$$

To prove the opposite inequality we will take any monomial $x^r y^{q-s} \in J_0$, $r \geq 0$, $q - s \geq 0$ and $r + (q - s) > 0$. Then the area of triangle with vertices $(p,0)$, $(0,q)$, $(r, q-s)$ is equal to $\frac{|-sp+rq|}{2}$. Since $x^r y^{q-s} \in J_0$ then $|-sp+rq| > 0$.

Consider cases:

1^0 $r > 0$ and $q - s > 0$. Then by the property of greatest common divisor

$$|-sp+rq| \geq d$$

and hence

$$\lambda((f_s^{(r,q-s)})) \geq d.$$

2^0 $r = 0$. Then $q - s > 0$ and

$$\lambda((f_s^{(0,q-s)})) = sp - s = s(p-1) \geq sd \geq d.$$

3^0 $q - s = 0$. Then $r > 0$ and

$$\lambda((f_s^{(r,0)})) = (p-r)q - (p-r) = (p-r)(q-1) \geq (p-r)d \geq d.$$

Hence by Theorem 2

$$\lambda'(f_0) \geq d.$$

Together

$$\lambda'(f_0) = d.$$

2. Observe first, that for the point $(p-1,0)$ we have

$$\lambda((f_s^{(p-1,0)})) = 2\left(\frac{1}{2} \cdot 1 \cdot q\right) - 1 = q - 1.$$

Therefore $\lambda'(f_0) \leq q - 1$. On the other hand, taking any point of the form $(p-m, 0)$, $m = 2, \dots, p-1$ we get

$$\lambda((f_s^{(p-m,0)})) = 2\left(\frac{1}{2} \cdot m \cdot q\right) - m = m(q-1) > q - 1.$$

Similarly for any point of the form $(0, q-m)$, $m = 1, \dots, q-1$ we get $\lambda((f_s^{(0,q-m)})) = 2(\frac{1}{2} \cdot m \cdot p) - m = m(p-1) \geq q - 1$. Consider now a point $(-u, q-w) \in J_0$ such that $-u > 0$, $q - w > 0$. Then $\lambda((f_s^{(-u,q-w)})) = |-up - wq| = q\left|\frac{up}{q} + w\right| \geq q > q - 1$. ∎

Example 2. Let $f_0(x,y) = x^4 - y^4$. From the above theorem $\lambda'(f_0) = 3$. The jump is realized by the deformation $f_s(x,y) = x^4 - y^4 + sx^3$.

Consider now a general case of a singularity which Newton polygon consists of only one segment.

Corollary 2. *Let f_0 be a non-degenerate and convenient singularity, with the Newton polygon reduced to only one segment. Then this segment connects points $(p,0)$ and $(0,q)$ for some $p, q \in \mathbb{N}$, $p, q \geq 2$. Moreover, if $d = \mathrm{GCD}(p,q)$, then:*

1. *If $1 \leq d < \min(p,q)$, then $\lambda'(f_0) = d$.*
2. *If $d = \min(p,q)$, then $\lambda'(f_0) = \min(p,q) - 1$.*

Proof. The first part of the corollary is obvious, the second follows from Corollary 1 and Theorem 3. ∎

6. The Milnor numbers of non-degenerate deformations of singularities

Let f_0 be a non-degenerate and convenient singularity. Let

$$\Lambda'(f_0) = (\mu_0, \mu_1, \mu_2, \dots, \mu_k)$$

be the strictly decreasing sequence of all possible Milnor numbers of all non-degenerate deformations (f_s) of f_0. In particular, $\mu_0 = \mu(f_0)$, $\mu_1 = \mu(f_0) - \lambda'(f_0)$, $\mu_k = 0$. From Corollary 2, we have a formula for μ_1 in the case f_0 is a singularity with one segment Newton polygon (in particular for f_0 irreducible). Namely, if the ends of this segment are $(p, 0)$ and $(0, q)$, $p, q \geq 2$, then for $d := \mathrm{GCD}(p, q)$

1. $\mu_1 = \mu_0(f) - d$ if $1 \leq d < \min(p, q)$,
2. $\mu_1 = \mu_0(f) - d + 1$ if $d = \min(p, q)$.

The main theorem of the paper is a formula for μ_2 in the same class of singularities.

We consider first simple singularities of the form $x^p - y^q$. Since the case $p = q = 2$ is trivial (we have $\Lambda'(x^2 - y^2) = (1, 0)$), we will confine to the cases $p > 2$ or $q > 2$.

Theorem 4. *Let $f_0(x, y) = x^p - y^q$, $p \geq q \geq 2$, $p + q \geq 5$. Then*

$$\mu_2 = \mu_1 - 1.$$

In the proof we will use the following elementary lemma.

Lemma 2. *Let $p, q \in \mathbb{N}$, $p > q \geq 1$ and $d = \mathrm{GCD}(p, q)$. Assume $d < q$ i.e. $q \nmid p$. Then there exist $a, b \in \mathbb{Z}$ such that*

$$ap + bq = d, \qquad 0 < a < \frac{q}{d}.$$

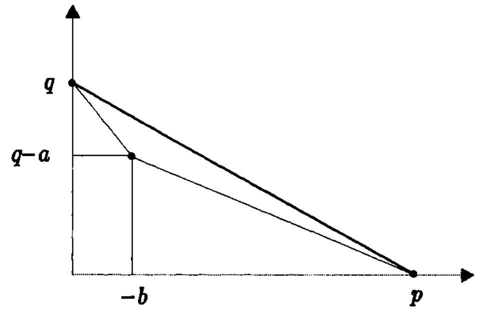
Moreover, a and b are unique and

$$\mathrm{GCD}(a, b) = 1.$$

The proof of Theorem 4. Let's consider cases:

1. $q \mid p$. There exists $n \in \mathbb{N}$ such that $p = nq$. From Corollary 2 $\lambda'(f_0) = q - 1$ and the jump is realized by the point $(p - 1, 0)$ (precisely by the deformation $(f_s^{(p-1,0)})$). Since $p \geq q \geq 2$, $p + q \geq 5$, then $p - 1 > 1$ i.e. $f(x, y) = x^{p-1} - y^q$ is an isolated singularity. The assumption $q \mid p$ implies $\mathrm{GCD}(p - 1, q) = 1$. Then there exists a, b such that $a(p - 1) + bq = 1$, $0 < a < q$. Then the point $(-b, q - a)$ realizes the jump equal to 1 for the function $f(x, y) = x^{p-1} - y^q$. So, the deformation composed of two points $(p - 1, 0)$ and $(-b, q - a)$ (i.e. $f_s(x, y) = f_0(x, y) + sx^{p-1} + sx^{-b}y^{q-a}$) realizes the jump for f_0 equal to $\lambda'(f_0) + 1$.

2. $q \nmid p$. Then $\mathrm{GCD}(p,q) =: d < q$. Let $ap + bq = d$, $0 < a < \frac{q}{d}$, $b < 0$, $\mathrm{GCD}(a,b) = 1$. Then the point $(-b, q-a)$ realizes the jump $\lambda'(f_0) = d$.



Observe that
1^0 $\mathrm{GCD}(a,-b) = 1$
2^0 $\mathrm{GCD}(p+b, q-a) = 1$
1^0 Follows from Lemma 2. For 2^0 let $\mathrm{GCD}(p+b, q-a) = r$. We have

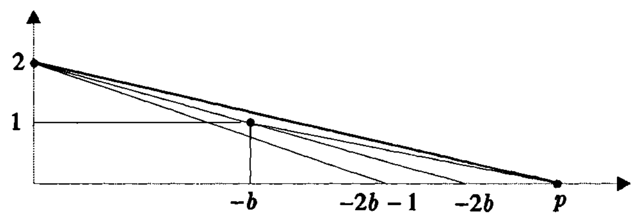
$$a(p+b) + b(q-a) = ap + ab + bq - ab = ap + bq = d.$$

Because $r \mid (p+b)$ and $r \mid (q-a)$, so $r \mid d$. Then $r \mid p$ and $r \mid q$. Since $r \mid (p+b)$, then $r \mid b$ and analogously $r \mid (q-a)$ implies $r \mid a$. Because $\mathrm{GCD}(a,b) = 1$, we obtain $r = 1$.

Consider subcases:

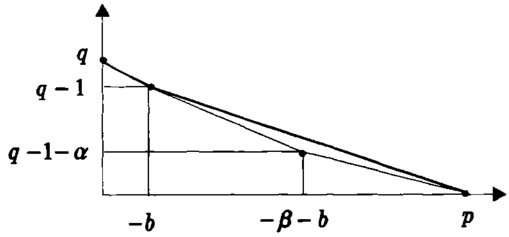
(i) $a = 1$.

• $q = 2$. Since $q \nmid p$ then p is odd. Then $-b = \frac{p-1}{2}$. Moreover, the point $(-b, 1)$ realizes the jump equal to 1 for the function $f_0(x,y) = x^p - y^2$. Hence $\lambda'(f_0) = 1$.



We easily check that the point $(-2b-1, 0)$ i.e. the deformation $f_s(x,y) = f_0(x,y) + sx^{-2b-1}$ realizes the jump equal to $2 = \lambda'(f_0) + 1$.

• $q > 2$. Because $\mathrm{GCD}(p+b, q-1) = 1$, so there exist integers α, β such that $\alpha(p+b) + \beta(q-1) = 1$, $0 < \alpha < q-1$, $\beta < 0$. The point $(-\beta, q-1-\alpha)$ realizes the jump equal to 1 for the function $f(x,y) = x^{p+b} - y^{q-1}$. Since the point $(-b, q-1)$ gives the first jump equal to d, then two points $(-b, q-1)$ and $(-\beta-b, q-1-\alpha)$ realize the jump $d+1$ for $f_0(x,y) = x^p - y^q$. In fact it suffices to show that the following broken line $\overline{(0,q)(-b,q-1)(-\beta-b,q-1-\alpha)(p,0)}$ is convex (as a graph of a function).



We must show that

$$\frac{-1}{b} \geq \frac{(q-1)-(q-1-\alpha)}{(-b-\beta)-(-b)}$$

i.e.

$$b\alpha - \beta \geq 0.$$

But, we have $p + bq = d$ and $\alpha(p+b) + \beta(q-1) = 1$. Calculating from the first equality $p + b = d - b(q-1)$ and substituting to the second, we get

$$\alpha(d - b(q-1)) + \beta(q-1) = 1.$$

Hence after simple calculations we obtain

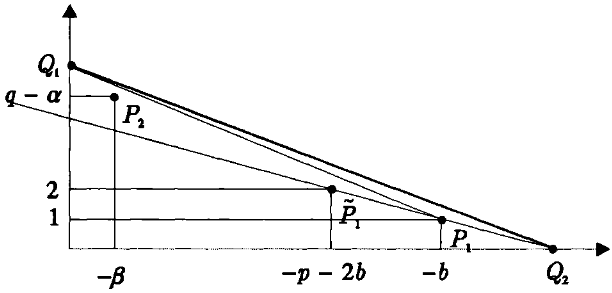
$$\frac{\alpha d - 1}{q-1} = \alpha b - \beta.$$

Because $\alpha d - 1 \geq 0$ and $q - 1 > 0$, then $\alpha b - \beta \geq 0$, as desired.

(ii) $b = -1$. Then $a = 1$ (because $p > q$), so we get the case (i).

(iii) $a = q-1$, $q \geq 3$. Let $Q_1 = (0,q)$ and $Q_2 = (p,0)$. From Theorem 3, $\lambda'(f_0) = d$ and the jump is realized by the point $P_1 = (-b,1)$. By 1^0 GCD$(q-1,-b) = 1$ hence the point $(-\beta, q-\alpha-1)$ with non-zero coordinates realizes the jump equal to 1 for the function $f(x,y) = x^{-b} - y^{q-1}$. We claim that the points $P_1 = (-b,1)$ and $P_2 = (-\beta, q-\alpha)$ i.e. the deformation $f_s(x,y) = f_0(x,y) + sx^{-b}y + sx^{-\beta}y^{q-\alpha}$ realize the jump equal to $d+1 = \lambda'(f_0)+1$. In fact, it suffices to show that the broken line $\overline{Q_1P_2P_1Q_2}$ is convex i.e. P_2 lies over the line $L_{Q_2P_1}$. Since $q \geq 3$ then the equality $(q-1)p + bq = d$

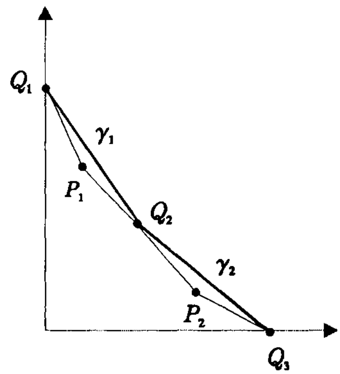
implies that $-p-2b>0$. Then the point $\tilde{P}_1=(-p-2b,2)$ lies on $L_{Q_2P_1}$ and the area of the triangle $Q_1P_1\tilde{P}_1$ is equal to $\frac{d}{2}$.



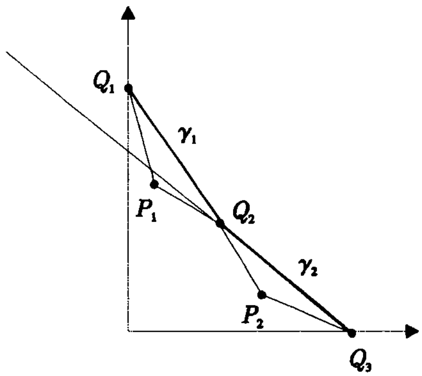
Since P_2 realizes the jump for $x^{-b}-y^{q-1}$ then $\rho(P_2,L_{Q_1P_1})\leq\rho(\tilde{P}_1,L_{Q_1P_1})$. Suppose to the contrary that P_2 lies beneath the line $L_{Q_2P_1}$. Then P_2 would lie on the right of $\tilde{P}_1$ i.e. $-\beta\geq -p-2b$. Moreover, its second coordinate $q-1>1$. The only point which satisfies these conditions is $\tilde{P}_1$, which contradicts the supposition.

(iv) $b=-(p-1)$. The case is imposible (because $p>q$).

(v) $1<a<q-1$, $1<-b<p-1$. Then $p+b>1$ and $q-a>1$. Hence, from 1^0 and 2^0 there exist points P_1, P_2 realizing the jumps equal to 1 for the functions $f_1(x,y)=x^{-b}-y^a$ and $f_2(x,y)=x^{p+b}-y^{q-a}$, respectively. Denote $Q_1=(0,q)$, $Q_2=(-b,q-a)$, $Q_3=(p,0)$ and $\gamma_1=\overline{Q_1Q_2}$, $\gamma_2=\overline{Q_2Q_3}$.



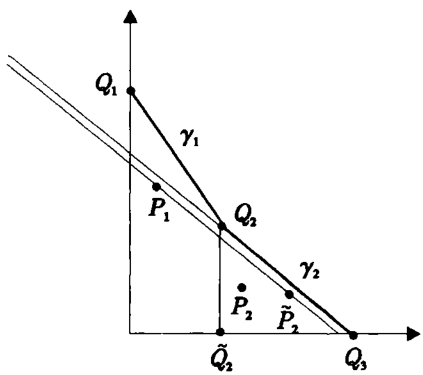
We claim that the broken line $\overline{Q_1P_1Q_2Q_3}$ or $\overline{Q_1Q_2P_2Q_3}$ is convex (in other words, one of the points P_1, P_2 changes the Newton polygon of $f_s^{(-b,q-a)}$ only on segment γ_1 or γ_2). In fact, let $h_i=\rho(P_i,\gamma_i)$ be the distance of P_i to the segment γ_i, $i=1,2$. We may assume that $h_1\leq h_2$ (the case $h_2\leq h_1$ is analogous). If our claim would be false, then the point P_1 would lie beneath the line L_{γ_2} containing the segment γ_2.



Hence and from the convexity of $\overline{Q_1Q_2Q_3}$

$$\tilde{h} := \rho(P_1, L_{\gamma_2}) < \rho(P_1, L_{\gamma_1}) = h_1.$$

In consequence $\tilde{h} < h_2$. If we translate the point P_1 of a multiple of the length of segment γ_2 along the direction of the line L_{γ_2}, then we obtain a point $\tilde{P}_2$, which will have integers coordinates and lie in a rectangle with one side γ_2 and second of lenght $\tilde{h}$. Since always $\tilde{h} < \frac{\sqrt{2}}{2}$, then it is easy to check that $\tilde{P}_2$ will lie in the triangle $Q_2\tilde{Q}_2Q_3$, where $\tilde{Q}_2 = (-b, 0)$.



But $\rho(\tilde{P}_2, \gamma_2) = \rho(P_1, L_{\gamma_2}) = \tilde{h} < h_2 = \rho(P_2, \gamma_2)$ which contradicts the choice of the point P_2.

We have proved that $\overline{Q_1P_1Q_2Q_3}$ or $\overline{Q_1Q_2P_2Q_3}$ is convex. In consequence, the points P_1, Q_2 or Q_2, P_2 realize the jump $d + 1 = \lambda'(f_0) + 1$. ∎

Summing up, we may formulate the known facts on Milnor numbers associated to a singularity.

Corollary 3. *Let f_0 be a non-degenerate and convenient singularity, which Newton polygon is reduced to one segment. If $\Lambda'(f_0) = (\mu_0, \mu_1, \ldots, \mu_k)$ is the sequence of Milnor numbers associated to f_0, then*

1. $\mu_0 = \mu_0(f_0)$,
2. μ_1 *is given in Corollary* 2,
3. $\mu_2 = \mu_1 - 1$.

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