DEMONSTRATIO MATHEMATICA
Vol. XLIII No 2 2010

Tadashi Tomaru

COMPLEX SURFACE SINGULARITIES AND DEGENERATIONS OF COMPACT COMPLEX CURVES

Abstract. Since 15 years ago, I have been studying some relations between complex normal surfaces ([Tt1]–[Tt5]). In this paper, we give a survey of them. After some preparations, we describe main results. Especially, we explain a method to embed resolution spaces of normal surface singularities into total spaces of degenerations of compact Riemann surfaces.

0. Introduction

After M. Artin's work [Art], normal surface singularity theory has been researching by many mathematicians (for example, P. Wagreich, H. Laufer, O. Riemenschneider, K. Saito, J. Wahl, S. S.T. Yau). On the other hand, K. Kodaira [Ko] defined the local one-parameter degeneration family of compact complex curves (=pencil of curves) correctly. He classified the configurations of singular fibers of such objects in the case of genus one (namely, elliptic pencils) and computed the homological monodromy groups and the functional invariants associated to the pencils of curves. In [NU], Y. Namikawa and K. Ueno studied similar problem for the case of genus two. The field of the degenerations of curves has been exploited via the methods of algebraic geometric, topology and complex analysis (T. Arakawa, T. Ashikaga, Y. Imayoshi, M. Ishizaka, Y. Matsumoto, J. M. Montesinos-Amilibia and S. Takamura etc.).

On the relation between singularities and pencils of curves, there have been several important works by several mathematicians (for example, V. Kulikov, U. Karras, M. Reid and J. Stevens). However it does not seem that their results are well-known. In this decade, the author has been researching the relation between singularities and pencils of curves under the

2000 *Mathematics Subject Classification*: Primary 32S10, 32S25; Secondary 14D06.
Key words and phrases: surface singularities, pencils of curves, Kodaira singularities.

influence from the activity by above Japanese mathematicians. In this paper, we give a survey of our results.

In § 1, we explain the fundamental results on normal surface singularities due to M. Artin [Art], P. Wagreich [Wag], H. Laufer [La], S. S.T. Yau [Y]. In § 2, for the relation between surface singularities and pencils of curves, we survey the results by V. I. Arnold [Arn], V. Kulikov [Ku], U. Karras [Ka1,2], M. Reid [R] and J. Stevens [St1,2]. In § 3, 4 and 5, we explain our results [Tt1-5].

1. Fundamental facts on normal surface singularities

In the following, we review some facts on normal surface singularities which relate to pencils of compact complex curves. Let (X, o) be a normal complex surface singularity. Let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution of singularity, where $E = \bigcup_{i=1}^{r} E_i$ is the irreducible decomposition of the exceptional set E. Let $\mathcal{O}_{X,o}$ be the local ring associated to (X, o) and $\mathfrak{m}_{X,o}$ the maximal ideal. Let h be an element of $\mathfrak{m}_{X,o}$, and let $E(h \circ \pi)$ be the cycle on E determined by $h \circ \pi$.

Definition 1.1. (1) For a $\mathbb{Z}$-cycle $D = \sum_{i=1}^{r} d_i E_i$ on E ($d_i \in \mathbb{Z}$), let $p_a(D) := 1 + \dfrac{D^2 + K_{\tilde{X}} D}{2}$ and call it the arithmetic genus of D. If D is an effective divisor (i. e., $d_i \geqq 0$ for any i), we put $\mathrm{red}(D) = \sum_{d_i > 0} E_i$.

(2) For the (Artin's) *fundamental cycle* $Z_E := \min\{D = \sum_{i=1}^{r} a_i E_i \mid a_i > 0$ and $DE_i \leqq 0$ for any $i\}$ (see [Art]), the values of $p_a(Z_E)$ and $(Z_E)^2$ are independent of the choice of a resolution, and so we put them $p_f(X, o)$ and $(\mathbb{Z}_X)^2$ respectively in this paper.

(3) The positive cycle $M_E := \min\{E(h \circ \pi) \mid h \in \mathfrak{m}_{X,o}\}$ on E is called the *maximal ideal cycle* on E and we have $Z_E \leqq M_E$ (see [Y]). In this paper, $\mathbb{M}_X$ (resp. $\mathbb{Z}_X$) represents the maximal ideal cycle (resp. fundamental cycle) on the minimal resolution.

Definition 1.2. (1) $p_g(X, o) := dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ (geometric genus).

(2) $p_a(X, o) := \max\{p_a(D) | D$ is a $\mathbb{Z}_{\geqq 0}$-cycle on $E\}$ (arithmetic genus of (X, o)).

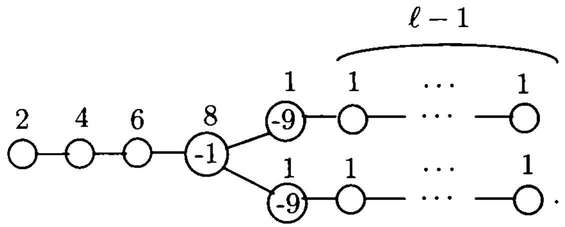
(3) If there exists a holomorphic 2-form on $X \backslash 0$, then we call (X, o) a Gorenstein singularity.

In this paper, we often use the *weighted dual graph (=w.d.graph)* Γ_E and so explain it. Let E be the exceptional set of a resolution of a singularity or the singular fiber of a pencil of curves. The w.d.graph Γ_E of E is a graph such that each vertex of Γ_E represents an irreducible component E_j weighted by E_j^2 and $g(E_j)$ (=genus), while each edge connecting to E_i and E_j, $i \neq j$, corresponds to point $E_i \cap E_j$. For example, if $E_i^2 = -b_i$ and $g(E_i) = g_i > 0$ (resp. $g_i = 0$), then E_i corresponds to a vertex that is configured as follows:

$(\text{-}b_i)$ $[g_i]$ (resp. $(\text{-}b_i)$) and ○ means $(\text{-}2)$ ·

Though $p_a(X,o)$ and $p_f(X,o)$ are topological invariants, $p_g(X,o)$ is not so. The geometric genus $p_g(X,o)$ is an analytic invariant. Since $p_g(X,o)$ is sensitive, it is useful on many situations of surface singularity theory. On the other hand, $p_f(X,o)$ is insensitive. However, $p_f(X,o)$ is useful when we consider the rough classification of singularities. It seems like that these three invariants have different roles in the classification of normal surface singularities. We show this by the following example.

Example 1.3. Let (X,o) be a hypersurface singularity defined by $x^2 + y^8 + z^{8\ell+9}$. The minimal resolution is given as follows:



Then we have $p_f(X,o) = 3$, $p_a(X,o) = 2\ell + 3$ and $p_g(X,o) = 6\ell + 6$, where $p_a(X,o)$ is computed by Tomari's formula [Tm2].

In [Art], M. Artin proved that $p_f(X,o) = 0 \Leftrightarrow p_a(X,o) = 0 \Leftrightarrow p_g(X,o) = 0$. If (X,o) satisfies such conditions, he call it a rational singularity. He proved that if (X,o) is a rational singularity, then the embedding dimension $emb(X,o)$ is equal to $\max\{3, -\mathbb{Z}_X^2 + 1\}$ and the multiplicity of (X,o) is equal to $-\mathbb{Z}_X^2$. He also classified rational singularities of multiplicity two (i.e., rational double points) as follows: $A_n : x^2 + y^2 + z^{n+1} (n \geqq 1), D_n : x^2 + y^2 z + z^{n-1} (n \geqq 4), E_6 : x^2 + y^3 + z^4, E_7 : x^2 + y^3 + yz^3, E_8 : x^2 + y^3 + z^5$.

In [Wag], P. Wagreich proved that $0 \leqq p_f(X,o) \leqq p_a(X,o) \leqq p_g(X,o)$ and also proved that $p_f(X,o) = 1 \Leftrightarrow p_a(X,o) = 1$. If (X,o) satisfies

$p_f(X,o) = 1$, (X,o) is called an elliptic singularity or a weak elliptic singularity. In this paper we use the terminology of "elliptic singularity".

Let (X,o) be an elliptic singularity, and let $(\tilde{X},E)$ be a resolution of (X,o). In [La], H. Laufer defined a positive cycle $Z_{\min}$ on E by $Z_{\min}$: $=\min\{F|p_a(D) \leqq 0$ for any cycle D with $0 < D < F$ and $p_a(F) = 1\}$. He proved that the following three conditions are equivalent: (i) $Z_E = Z_{\min}$, (ii) $-KE_i = Z_E E_i$ for any component E_i of E, (iii) (X,o) is a Gorenstein singularity of $p_g(X,o) = 1$. If an elliptic singularity satisfies these conditions, he called it a *minimally elliptic singularity.* He proved that if (X,o) is a minimally elliptic singularity, then the embedding dimension $emb(X,o)$ is equal to $\max\{3, -\mathbb{Z}_X^2\}$, and the multiplicity of (X,o) is equal to $\min\{2, -\mathbb{Z}_X^2\}$. He also classified minimally elliptic hypersurface singularities (i.e., the multiplicity $\leqq 3$).

In [R], M. Reid also independently studied Gorenstein singularities of $p_g(X,o) = 1$. He also classified such hypersurface singularities of multiplicity of 2 or 3 and pointed out relations between minimally elliptic singularities and elliptic pencils.

On the other hand, V. I. Arnold [Arn] studied real or complex function germs. He introduced the invariant *modality* $\mu(X,o)$ for hypersurface singularity $(X,o) = \{f = 0\}$. He show that $\mu(X,o) = 0$ if and only if (X,o) is a rational double point. He also classified hypersurface singularities of $\mu = 1$ or 2, and call singularities of $\mu = 1$ (resp. $\mu = 2$) *uni-modal* (resp.*bi-modal) singularities.* Uni-modal singularities are exhausted as follows: (i) simple elliptic singularities $x^a + y^b + z^c + txyz : (a,b,c) = (3,3,3), (2,4,4)$ or $(2,3,6)$, (ii) cusp singularities $T_{p,q,r}$ $(=x^p + y^q + z^r + txyz : \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1)$, (iii) 14 exceptional singularities (for example, $x^2+y^3+z^7+ayz^5, x^2+y^3+z^8+ayz^6$ etc.). For bi-modal case, there are 22 types singularities up to one parameter deformation.

Further, V. Kulikov [Ku] showed that uni-modal and bi-modal singularities are obtained through some procedure (see Definition 2.2 of this paper) from Kodaira's list [Ko] of pencils of elliptic curves.

Inspiring by Kulikov's observation, U. Karras [Ka1,2] introduced the notion of *Kodaira singularities* in terms of pencils of curves. He also applied it to the deformation theory of surface singularities. Further, J. Stevens [St1, 2] studied a subclass of Kodaira singularities (called *Kulikov singularities*) and proved some relations between them and deformations of curve singularities.

Moreover, K. Saito [Sa] studied some classes of quasi-homogeneous hypersurface elliptic singularities and considered pencils of curves associated to those singularities.

2. Kodaira singularities and Kulikov singularities

Here we describe the precise definition of pencils of curves and the definition of Kodaira (and Kulikov) singularities.

DEFINITION 2.1. Let S be a non-singular complex surface and $\Delta \subset \mathbb{C}$ a small open disc around the origin. If $\Phi\colon S \to \Delta$ is a proper surjective holomorphic map and the generic fiber $S_t\colon = \Phi^{-1}(t)$ $(t \neq 0)$ is a smooth curve, it is called a *quasi-pencil of curves.* Further, if the generic fiber S_t is a connected smooth curve of genus g, then we call it *a pencil of curves* or *a pencil of curves of genus g.* In this situation, we call $S_o = \Phi^{-1}(o)$ *the singular fiber* or *the degenerate fiber.*

DEFINITION 2.2. ([Kal], [St1,2]) Let $\Phi\colon S \to \Delta$ be a pencil of curves of genus g which has reduced components. Let $P_1, \cdots, P_r \in \mathrm{supp}(S_o)$ be non-singular points of S_o (i.e., they are contained in components whose coefficients of S_o equal to one and also smooth points of $\mathrm{red}(S_o)$). Let $S' \xrightarrow{\sigma} S$ be a finite succession of blowing-ups with centers $P_1, \cdots, P_r$. Let $\tilde{X}$ be an open neighborhood of the proper transform $E \subset S'$ of $\mathrm{supp}(S_o)$ by σ. By contracting E in $\tilde{X}$, we obtain a normal surface singularity (X, o). Then, the contraction map $\varphi\colon (\tilde{X}, E) \longrightarrow (X, o)$ is a resolution of (X, o) and so we have the following:

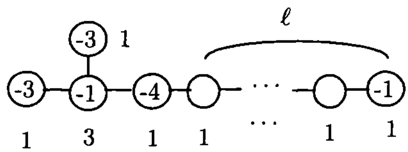
$$\begin{array}{ccccc} & S & \xleftarrow{\ \sigma\ } & S' \supset & (\tilde{X}, E) \\ {}^{\Phi}\swarrow & & & & \downarrow{\scriptstyle \varphi} \\ \Delta & & & & (X, o). \end{array}$$

If a normal surface singularity is isomorphic to a singularity obtained in this way, then it is called a *Kodaira singularity* of genus g (or *Kodaira singularity* associated to Φ). Also, if σ is just one blowing-up at every center P_i $(i = 1, \cdots, r)$ in the above construction, then (X, o) is called a *Kulikov singularity* of genus g (or *Kulikov singularity* associated to Φ). We can easily see that if (X, o) is a Kodaira singularity, the fundamental cycle coincides with the maximal ideal cycle on the minimal resolution. (i.e., $\mathbb{Z}_X = \mathbb{M}_X$).

Let Γ be the w.d.graph of the exceptional set of the minimal good resolution of a normal surface singularity. If there exists a Kodaira singularity whose w.d.graph for the minimal good resolution coincides with Γ, then Γ is called *a Kodaira graph.*

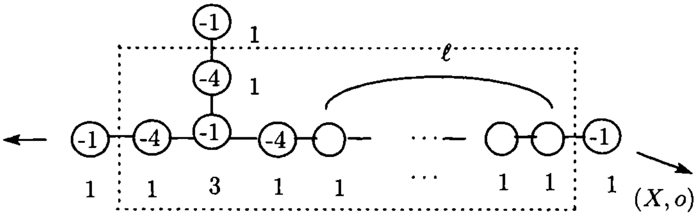
PROPOSITION 2.3. ([Kal], 2.7) *Let $(\tilde{X}, E)$ be a resolution of a normal surface singularity (X, o). Let Γ be a w.d.graph of E. Then Γ is a Kodaira graph if and only if each irreducible component E_i with $E_i Z_E < 0$ appears with multiplicity one in the fundamental cycle Z_E.*

EXAMPLE 2.4. The following figure gives an example of Kulikov singularity of genus 1.



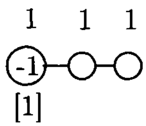
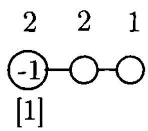
$(\ell \geqq 1)$

(elliptic pencil)



$(X, o) = \{x^3 + y^3 + z^{3\ell+1} = 0\}.$

In general, we can not always determine from the w.d.graph whether a singularity is a Kodaira singularity or not. For example, if we put $(X_1, o) = \{z^2 + y^3 + x^{18} = 0\}$ and $(X_2, o) = \{z^2 + y(y^4 + x^6) = 0\}$, then they are elliptic singularities whose w.d.graphs for minimal resolution coincides with each other. We can easily check that (X_1, o) is a Kulikov singularity from 4.13 since $p_e(X_1, o, x) = 1$ and x is a reduced element. However, (X_2, o) is not a Kodaira singularity, because $\mathbb{Z}_{X_2} \neq \mathbb{M}_{X_2}$. Those cycles are given as follows:

$\mathbb{Z}_{X_2} =$  and $\mathbb{M}_{X_2} =$  .

THEOREM 2.5. ([Ka1], 2.9) *Let (X, o) be a rational or minimally elliptic singularity with w.d.graph Γ. Then (X, o) is a Kodaira singularity if and only if Γ is a Kodaira graph.*

From the definition, we can easily see that the w.d.graph of Kodaira singularity is equal to the one of a Kulikov singularity. Therefore, if we find a Kodaira singularity which is not Kulikov, we need to discuss the analytic type of singularity and it is not so easy. The author considered this problem under a special situation.

THEOREM 2.6. ([Tt4], 3.8) *Let (X, o) be a normal surface singularity obtained by the contraction of the zero-section of a negative line bundle L on a non-singular complex projective curve E.*

(i) (X,o) *is a Kodaira singularity if and only if* $L \sim -\sum_{i=1}^{r} n_i P_i$ *(linearly equivalent), where* $n_i > 0$ *for any* i.

(ii) (X,o) *is a Kulikov singularity if and only if* $L \sim -\sum_{i=1}^{r} P_i$*, where* $P_1, \cdots, P_d$ *are* r *mutually distinct points.*

(iii) *In the case of* (i) *(resp.* (ii)*),* (X,o) *is a Kodaira (resp. Kulikov) singularity associated to the trivial pencil; it is obtained by taking* n_i *blowing-up at* $Q_i := (P_i, 0) \in E \times \mathbb{C}$ *for* $i = 1, \cdots, r$*. Moreover we have* $n_1 = \cdots = n_r = 1$ *in the case of* (ii).

Let E be a non-hyperelliptic curve and let P be any point of E. If (X,o) is a normal surface singularity obtained by the contraction of the zero-section of the negative line bundle associated to $-2P$, then (X,o) is a Kodaira singularity, but not a Kulikov singularity from 2.6.

3. Embeddings of resolution spaces into pencils of curves

If $\Phi\colon S \to \Delta$ is a pencil of curves, the intersection matrix of any connected one-dimensional analytic proper subset E in $\mathrm{supp}(S_o)$ is negative definite from Zariski's lemma ([BPV], p. 90). Hence E is contracted to a normal surface singularity by Grauert's result ([G], p. 367]). From now on we consider the converse problem. We prepare some definitions.

Definition 3.1. Let $\pi\colon (\tilde{X}, E) \to (X,o)$ be a resolution of a normal surface singularity. Let $\Phi\colon S \to \Delta$ be a pencil of curves such that $(S, \mathrm{supp}(S_o)) \supset (\tilde{X}, E)$ (i.e., $S \supset \tilde{X}$ and $\mathrm{supp}(S_o) \supset E$).
(i) If $h \in \mathfrak{m}_{X,o}$ satisfies $h \circ \pi = \Phi$, then Φ is called *a pencil of curves extending* $h \circ \pi$ *or an extension of* $h \circ \pi$. Namely it implies the following diagram:

$$(3.1)\qquad \begin{array}{ccc} (X,o) & \xleftarrow{\ \pi\ } & (\tilde{X}, E) \subset (S, \mathrm{supp}(S_o)). \\ & {}_{h}\searrow & \swarrow{}_{\Phi} \\ & \Delta & \end{array}$$

(ii) Under the situation of (i), if there is no (-1) curve in $\mathrm{supp}(S_o)\backslash E$ which does not intersect E, then we call Φ a *pencil of curves minimally extending* $h \circ \pi$ or a *minimal extension of* $h \circ \pi$.

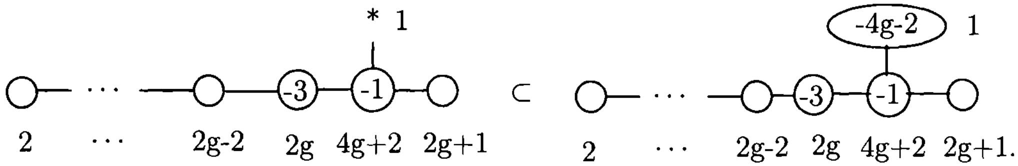
Definition 3.2. Let R be a ring and h a non-zero element of R. Then h is called a *perfect power element* if there is an element $g \in R$ satisfying $h = g^k$ for some positive integer $k \geqq 2$.

The author proved the following.

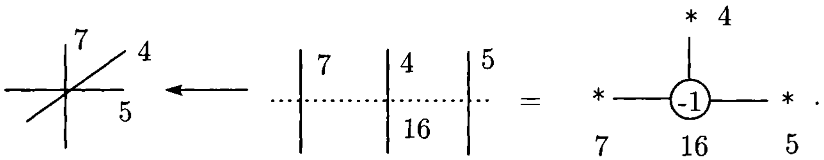
THEOREM 3.3. ([Tt4], 2.4) *Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$. Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be a good resolution such that $red((h \circ \pi)_{\tilde{X}})$ is a simple normal crossing divisor on $\tilde{X}$. Then there exists a quasi-pencil of curves $\Phi\colon S \to \Delta$ such that Φ is a minimal extension of $h \circ \pi$. Further, if h is not a perfect power element, then Φ is a pencil of curves.*

We prove Theorem 3.3 by gluing $\tilde{X}$ and resolution spaces of some cyclic quotient singularities. Also, we consider $h \circ \pi$ and some holomorphic functions on cyclic quotient singularities. By gluing them, we obtain a pencil of curves $\Phi\colon S \to \Delta$. By computing examples, we explain this.

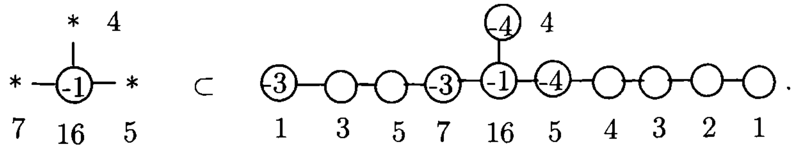
EXAMPLE 3.4. (1) Let $(X, o) = (\mathbb{C}^2, o)$ and $h_1 = x^2 + y^{2g+1}$ $(g \geqq 1)$. Then the minimal embedding resolution of the curve singularity defined by $h_1 = 0$ is given by the left configuration in the following. Also, the pencil of curves of genus g constructed as in 2.2 is given by the right configuration:



(2) Let (C, o) be a non-reduced curve singularity defined by $x^7y^5(x-y)^4 = 0$. After one blowing up at the origin of $\mathbb{C}^2$, we have the following configuration:



From $\dfrac{16}{7} = [[3, 2, 2, 3]] \left(= 3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3}}}\right)$ and $\dfrac{16}{5} = [[4, 2, 2, 2, 2]]$, we have the following:



Then we can see that its genus is 6 by the adjunction formula.

REMARK 3.5. Here we remark that any pencils of curves is birational to a pencil of curves which is constructed as in 3.3 (see [Tt4], Theorem 2.7).

REMARK 3.6. In any dimension, E. Looijenga [Lo, p. 301] proved similar results in a different way. Namely, let (X, o) be a normal isolated singularity. If $h \in \mathfrak{m}_{X,o}$ is a reduced element, then it defines a one parameter smoothing of (X, o). Using formal completion argument presented by M. Artin, Looijenga proved that if $h \in \mathfrak{m}_{X,o}$ is an element that gives a one parameter smoothing $h\colon (X, o) \to (\mathbb{C}^1, o)$, then there is a flat projective morphism $\psi\colon Z \to \mathbb{C}^1$ and an embedding $\phi\colon X \to Z$ satisfying $h = \psi \circ \phi$.

4. Pencil genus for normal surface singularities

In [Tt4], the author introduced an invariant for normal surface singularities by using pencils of curves.

DEFINITION 4.1. Let (X, o) be a normal surface singularity, and let $h \in \mathfrak{m}_{X,o}$ be not a perfect power element.

(i) We define a holomorphic invariant for (X, o) as follows:
$p_e(X, o) :=$ min{the genus of a pencil of curves including a resolution of (X, o)}.
(ii) We also define a holomorphic invariant for a pair of (X, o) and h as follows:
$p_e(X, o, h) :=$ min{the genus of a pencil of curves extending $h \circ \pi$ for a resolution π of (X, o)}.
Then, $p_e(X, o)$ (resp. $p_e(X, o, h)$) is called the *pencil genus* of (X, o) (resp. a pair of (X, o) and h).

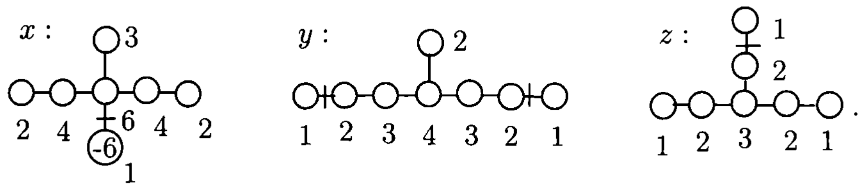
EXAMPLE 4.2. For rational double points, we can easily see that $p_e(A_n) = 0$ and $p_e(D_n) = 0$. Also, we have $p_e(E_\ell) = 1$ ($\ell = 6, 7, 8$) (see [Tt4], 3.12).

THEOREM 4.3. *Let (X, o) be a normal surface singularity and let $h \in \mathfrak{m}_{X,o}$ be not a perfect power element. Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be a good resolution such that red$((h \circ \pi)_{\tilde{X}})$ is simple normal crossing on $\tilde{X}$. Suppose that $\Phi\colon S \to \Delta$ is a pencil of curves of genus g extending $h \circ \pi$. If $g = p_e(X, o, h)$ and Φ is a minimal extension of $h \circ \pi$, then any connected component of* $\operatorname{supp}(S_o)\backslash E$ *is a $\mathbb{P}^1$-chain. Conversely, if any connected component of* $\operatorname{supp}(S_o)\backslash E$ *is a $\mathbb{P}^1$-chain, then $g = p_e(X, o, h)$. Therefore, the genus of any pencil of curves constructed from $h \circ \pi$ as in Theorem 3.3 is equal to $p_e(X, o, h)$.*

From 4.3, we have $p_e(\mathbb{C}^2, o, x^2 + y^{2g+1}) = g$ and $p_e(\mathbb{C}^2, o, x^7y^5(x-y)^4) = 6$.

EXAMPLE 4.4. Let $(X, o) = (\{x^2 + y^3 + z^4 = 0\}, o)$ (a rational double point of type E_6). (X, o) is a double covering over $\mathbb{C}^2$ branched along a plane curve $C := \{y^3 + z^4 = 0\}$. Let $V \xrightarrow{\sigma} \mathbb{C}^2$ be the minimal embedded resolution of C. Taking the double covering over V branched along $\sigma^*(C)$

(total transform of C) and contracting some (-1)-curves, we can obtain the minimal resolution $(\tilde{X}, E) \xrightarrow{\pi} (X, o)$ and the divisor $(x \circ \pi)_{\tilde{X}}$ (cf. Lemmas 3.1 in [Tt2] or Lemmas 4.3 in [Tt4]). Similarly, we can obtain the divisors $(y \circ \pi)_{\tilde{X}}$ and $(z \circ \pi)_{\tilde{X}}$. Applying Theorem 3.3 for them, the singular fibers of the pencils of curves constructed from x, y and z are given as follows:



From the adjunction formula and Theorem 4.3, we can see that $p_e(X, o, x) = 3$, $p_e(X, o, y) = 1$ and $p_e(X, o, z) = 1$.

Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ be not a perfect power element. Let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution such that $\mathrm{red}((h \circ \pi)_{\tilde{X}})$ is a simple normal crossing divisor. Let $E(h \circ \pi)$ be the cycle on E determined by $h \circ \pi$ and $v_{E_i}(h \circ \pi)$ be the vanishing order of $h \circ \pi$ on E_i. Then $E(h \circ \pi) = \sum_{i=1}^{r} v_{E_i}(h \circ \pi) E_i$. Let $\Lambda(h \circ \pi) = \sum_{j=1}^{r(h)} \gamma_j C_j$ ($\gamma_j \in \mathbb{N}$) be the non-exceptional part of the divisor $(h \circ \pi)_{\tilde{X}}$ (i.e., $\Lambda(h \circ \pi) = (h \circ \pi)_{\tilde{X}} - E(h \circ \pi)$), and let put $C = \sum_{j=1}^{r(h)} C_j$, where C_j is an irreducible component for any j. Let $n_1, \cdots, n_{r(h)}$ be positive integers denoted by $n_j = v_{E_{i_j}}(h \circ \pi)$ if E_{i_j} intersects C_j.

THEOREM 4.5. ([Tt4], 2.11) *Under the situation above, we have the following:*

$$p_e(X, o, h) = p_a(E(h \circ \pi)) - E(h \circ \pi)^2 - \frac{1}{2}\Big\{(E(h \circ \pi) + E)(\Lambda(h \circ \pi) - C) + r(h) + \sum_{j=1}^{r(h)} \gcd(n_j, \gamma_j)\Big\}.$$

Further, if h is a reduced element, then

$$p_e(X, o, h) = p_a(E(h \circ \pi)) - E(h \circ \pi)^2 - r(h).$$

For the normalization $\nu\colon \tilde{C} \longrightarrow C$ of a curve singularity (C, o), the conductor number $\delta(C, o)$ is defined by $\dim_{\mathbb{C}}(\mathcal{O}_{\tilde{C}}/\nu^* \mathcal{O}_{C,o})$ (cf. [Na]).

COROLLARY 4.6. (i) *Let (X, o) be a normal surface singularity and $h \in \mathfrak{m}_{X,o}$ a reduced element. Let $\delta(h)$ be the conductor number of a curve singu-*

larity $(X \cap \{h=0\}, o)$. *Then*

$$p_e(X,o,h) = \delta(h) - r(h) + 1.$$

(ii) *Let* $(X,o) = \{z^n = h(x,y)\}$ *be a normal hypersurface singularity. Then*

$$p_e(X,o,z) = p_e(\mathbb{C}^2, o, h) = \delta(h) - r(h) + 1 = \frac{\mu(h) - r(h) + 1}{2},$$

where $\mu(h)$ *is the Milnor number of a plane curve singularity* $(\{h=0\}, o) \subset (\mathbb{C}^2, o)$.

Since the Milnor number of a curve singularity $(\{x^a + y^b = 0\}, o)$ is $(a-1)(b-1)$, we can easily check the computations of the genus of 4.4 by using 4.6. Further we have the following estimate on $p_e(X,o)$.

THEOREM 4.7. ([Tt4], 3.5) *Let* (X,o) *be a normal surface singularity. Then*

$$p_f(X,o) \leqq p_e(X,o) \leqq p_a(X,o) + \mathrm{mult}(X,o) - r(h).$$

Especially, if (X,o) *is a rational singularity, then* $0 \leqq p_e(X,o) \leqq \mathrm{mult}(X,o) - 1$. *Also, if* (X,o) *is an elliptic singularity* (i.e., $p_f(X,o) = 1$), *then* $1 \leqq p_e(X,o) \leqq \mathrm{mult}(X,o)$.

From the definition, we can easily see that if (X,o) is a Kodaira singularity, then $p_e(X,o) = p_f(X,o)$. However, the converse is not true. For example, any D_n-singularity is not a Kodaira singularity, but it satisfies $p_e(X,o) = p_f(X,o)$.

DEFINITION 4.8. If (X,o) satisfies $p_e(X,o) = p_f(X,o)$, then (X,o) is called *a weak Kodaira singularity. Further, if* $p_e(X,o) = p_f(X,o) = g$, *then* (X,o) *is called a weak Kodaira singularity of genus* g.

From the definition, any Kodaira singularity is a weak Kodaira singularity. If (X,o) is a rational double point of type E_6, E_7 or E_8, then $p_f(X,o) = 0 < p_e(X,o) = 1$. Hence they are not weak Kodaira singularity. We can find many weak Kodaira singularities which are not Kodaira. In [Tt3], we studied normal surface singularities obtained through some procedures. But the description is complicated. Hence, in the following, we explain singularities which are obtained through a more simpler procedure than [Tt3]. For any pencil of curves $\Phi\colon S \to \Delta$, if an irreducible component A_i in $\mathrm{supp}(S_o)$ satisfies $A_i \cdot (\mathrm{supp}(S_o) \backslash A_i) = 1$, then it is called *an edge curve.* Also, if an edge curve is a (-1)-curve, then it is called a (-1)-edge curve.

DEFINITION 4.9. (i) Let $\bar{\Phi}\colon \bar{S} \longrightarrow \Delta$ be a non-multiple pencil of curves without any (-1)-edge curve. Let $S^{(0)} = \bar{S} \xleftarrow{\sigma_1} S^{(1)}$ be blow-ups at non-singular points $P_1^{(1)}, \cdots, P_{t_1}^{(1)}$ of $\mathrm{red}(S_o^{(0)})$. As the next step, let $P_1^{(2)}, \cdots, P_{t_2}^{(2)} \in \bigcup_{j=1}^{t_1} \sigma_1^{-1}(P_j^{(1)})$ be non-singular points of $\mathrm{red}(S_o^{(1)})$ and let $S^{(1)} \overset{\sigma_2}{\longleftarrow} S^{(2)}$ be blow-ups at these points. After continuing this process m times, we get $S^{(0)} = \bar{S} \overset{\sigma_1}{\longleftarrow} S^{(1)} \overset{\sigma_2}{\longleftarrow} \cdots \overset{\sigma_m}{\longleftarrow} S^{(m)} = S$ and put $\sigma = \sigma_1 \circ \cdots \circ \sigma_m$. Hence we get a new pencil $\Phi = \bar{\Phi} \circ \sigma \colon S \longrightarrow \Delta$ and call this procedure *a Kulikov process of type I started from* $P_1^{(1)}, \cdots, P_{t_1}^{(1)}$ (*or I-process started from* $P_1^{(1)}, \cdots, P_{t_1}^{(1)}$).

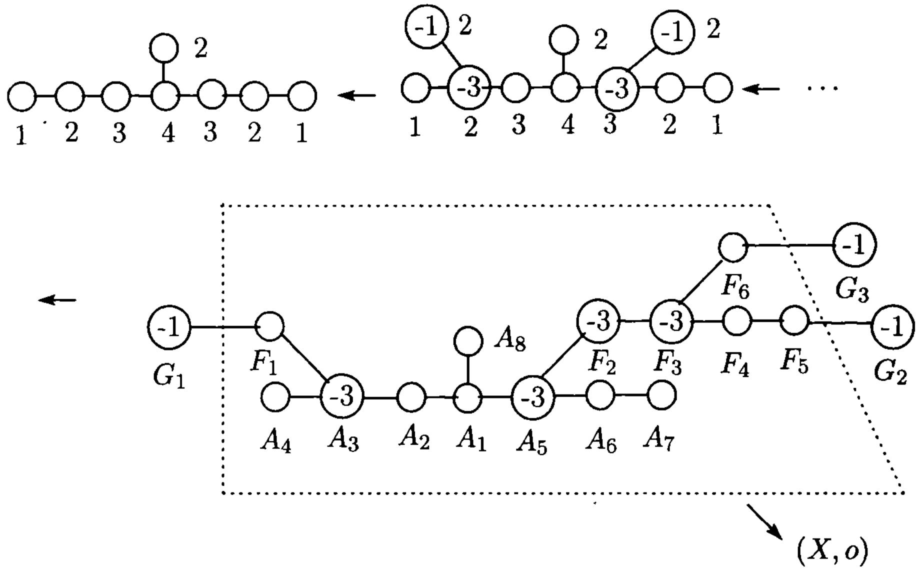
(ii) In I-process of (i), if a component $\bar{A}_{k_j}$ of $\mathrm{supp}(\bar{S}_o)$ contains $P_j^{(1)}$ $(j = 1, \cdots, t_1)$ and $A_{k_j} = \sigma_*^{-1}\bar{A}_{k_j}$ (i.e., this means the strict transform of $\bar{A}_{k_j}$ by σ), then we call A_{k_j} *a root component* of this I-process. Let $B_1, \cdots, B_{t_1}$ be connected components of $B := \mathrm{supp}(S_o) \setminus \mathrm{supp}(\sigma_*^{-1}\bar{S}_o)$. Each B_j $(j = 1, \cdots, t_1)$ is constructed from all components which are produced by blowing-ups at infinitesimally near points of $P_j^{(1)}$. We call such B_j *a branch* of $\mathrm{supp}(S_o)$ by this I-process.

(iii) For any component $H_j^{(i)}$ of a branch B_j, let $\ell(H_j^{(i)})$ be the number of blow-ups to produce $H_j^{(i)}$ from the root component A_j, and we call it *the length of* $H_j^{(i)}$. Also we define $\ell(A_k) = 0$ for any component A_k of the strict transform of $\mathrm{supp}(S_o)$ through σ. Further, let $c_R(H_j^{(i)}) = \mathrm{Coeff}_{A_{k_j}} S_o$ (i.e., coefficient of the root of $H_j^{(i)}$) if A_{k_j} is the root of $H_j^{(i)}$.

Definition 4.10. Let $\bar{\Phi} \colon \bar{S} \longrightarrow \Delta$ be a non-multiple pencil of curves without any (-1)-edge curve. Let $\bar{S} \overset{\sigma}{\longleftarrow} S$ be a birational map given by the I-process started from $P_1, \cdots, P_k$. Let $A = \mathrm{supp}(\sigma_*^{-1}\bar{S}_o)$ and let F be the union of all components in branches by the I-process except for (-1)-edge curves. Let $\tilde{X}$ be a small neighborhood of $A \cup F$ and let (X, o) be a normal surface singularity obtained by contracting $A \cup F$ in $\tilde{X}$. We call such (X, o) *a singularity obtained from this I-process.* If we put $G := \mathrm{supp}(S_o) \setminus (A \cup F)$, then any connected component G_i of G is (-1)-curve and we call G_i *an edge curve.*

Theorem 4.11. ([Tt3], 2.5) *Let $\bar{\Phi} \colon \bar{S} \longrightarrow \Delta$ be a non-multiple minimal pencil of curves of genus $g \geqq 1$ (i.e., $\bar{S}$ does not contain (-1)). Let (X, o) be a normal surface singularity obtained from I-process $\bar{S} \overset{\sigma}{\longleftarrow} S$ and $(\tilde{X}, E) \subset (S, \mathrm{supp}(S_o))$ the associated good resolution, where $E = A \cup F$. Then, (X, o) is a weak Kodaira singularity of genus g if and only if $\ell(G_i) \geqq c_R(G_i)$ for any edge curve G_i.*

EXAMPLE 4.12. Let consider a I-process started from two points and consider a normal surface singularity (X, o) as follows:



Then $C_R(G_1) = \ell(G_1) = 2$, $C_R(G_2) = 3 < \ell(G_2) = 5$ and $C_R(G_3) = 3 < \ell(G_3) = 4$. Hence (X, o) is a weak Kodaira elliptic singularity from 4.11.

The following result generalizes the results by Karras [Ka2] and Stevens [St2].

PROPOSITION 4.13. ([Tt3], 2.7) *Let $\bar{\Phi}\colon \bar{S} \longrightarrow \Delta$ be a non-multiple minimal pencil of genus 1. Let (X, o) be a normal surface singularity obtained by a I-process $\bar{S} \leftarrow S$. Then we have the following.*

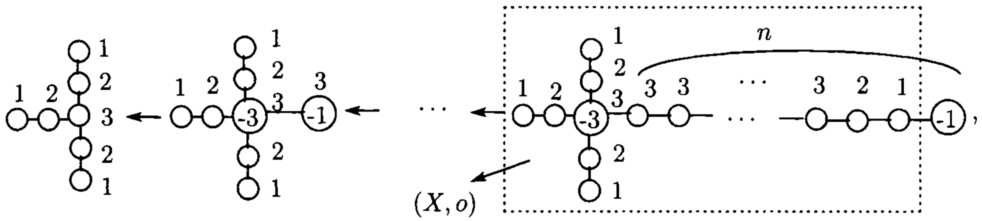
(i) $p_g(X, o) = \min\left\{ \left[\dfrac{\ell(G_j)}{c_R(G_j)} \right] \mid G_j \text{ is any } (-1) \text{ edge curve} \right\}$, *where* $[a] = \max\{n \in \mathbb{Z} \mid n \leqq a\}$ *for any* $a \in \mathbb{R}$. *Further, if (X, o) is an elliptic singularity, then $p_g(X, o)$ coincides with the length of the elliptic sequence in the sense of Yau [Y].*

(ii) *Suppose that $\ell(G_j) \geqq c_R(G_j)$ for any (-1) edge curve G_j. Then, (X, o) is a Gorenstein singularity if and only if there is a constant integer k such that $\ell(G_j) = k \cdot c_R(G_i)$ for any (-1) edge curve G_j.*

(iii) *(X, o) is a minimally elliptic singularity* (i. e., *$p_g(X, o) = 1$ and (X, o) is a Gorenstein singularity*) *if and only if $\ell(G_j) = c_R(G_j)$ for any (-1) edge curve G_j.*

Therefore, if (X, o) is the singularity of 4.12, then it is a non-Gorenstein elliptic singularity of $p_g(X, o) = 1$.

EXAMPLE 4.14. Let (X, o) be a normal surface singularity obtained as follows:



where the multiplicity of each component in the figure of the right hand side is the coefficient of the fundamental cycle $\mathbb{Z}_X$. If $n \geqq 3$, then (X, o) is a weak Kodaira elliptic singularity of $p_g(X, o) = \left[\frac{n}{3}\right]$. However, it is not a Kodaira singularity, because the w.d.graph is not a Kodaira graph from 2.3.

We have the following characterization of Kodaira or Kulikov singularities.

THEOREM 4.15. *Let (X, o) be a normal surface singularity.*

(i) *(X, o) is a Kodaira singularity if and only if there exists $h \in \mathfrak{m}_{X,o}$ which is not a perfect power element satisfying $p_e(X, o, h) = p_f(X, o)$ and $E(h \circ \pi) = Z_E$, where $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ is a resolution such that* red$(h \circ \pi)_{\tilde{X}}$ *is simple normal crossing.*

(ii) *(X, o) is a Kulikov singularity if and only if there exists a reduced element $h \in \mathfrak{m}_{X,o}$ with $p_e(X, o, h) = p_f(X, o)$.*

5. Cyclic covers of surface singularities and pencils of curves

In [Tt2], the author studied hypersurface Kodaira singularities defined by $z^n = h$, where $h \in \mathbb{C}\{x, y\}$ and $n > 1$. Let $(X, o) = \{z^n = h(x, y)\}$ be a normal hypersurface singularity, and so h is a reduced element. Let $\mu(h)$ (resp. $r(h)$) be the Milnor number of a curve singularity $(\{h = 0\}, o)$ (resp. the number of irreducible factors of h). Let $\pi\colon (V, F) \longrightarrow (\mathbb{C}^2, o)$ be the minimal embedded resolution of $(\{h = 0\}, o)$ and $F = \bigcup_{i=1}^{s} F_i$ be the irreducible decomposition. Let $N_h(F_i)$ be the vanishing order of $h \circ \sigma$ on F_i and put $N_h = \max\{N_h(F_i) | 1 \leqq i \leqq s\}$. Then we have the following.

THEOREM 5.1. ([Tt2], 4.5) (i) *If n divides $ord(h)$ (=the order of h), then (X, o) is a Kodaira singularity of genus $\dfrac{(n-1)(ord(h)-2)}{2}$ and $\mathbb{Z}_X^2 = -n$.*

(ii) *If $n \geqq N_h$, then (X, o) is a Kulikov singularity of genus $\dfrac{\mu(h) - r(h) + 1}{2}$ and $\mathbb{Z}_X^2 = -r(h)$.*

In [Tt4], the author generalized (ii) as the results of cyclic covers of singularities.

DEFINITION 5.2. Let $(Y, o) \subset (\mathbb{C}^N, o)$ be a normal singularity and I its defining ideal in $\mathbb{C}\{y_1, \cdots, y_N\}$. Further, let $h \in \mathfrak{m}_{Y,o}$ be an element and $\tilde{h} \in \mathbb{C}\{y_1, \cdots, y_N\}$ be an element corresponding to h. Let (X, o) $(\subset (\mathbb{C}^{N+1}, o))$ be a singularity defined by the ideal generated by I and $z^n - \tilde{h}(y_1, \cdots, y_N)$ in $\mathbb{C}\{y_1, \cdots, y_N, z\}$. Then (X, o) is called the *n-fold cyclic covering of (Y, o) defined by $z^n = h$.*

In this section, we assume that h is not a perfect power element. Then (X, o) is a normal singularity if and only if h is a reduced element in $\mathcal{O}_{Y,o}$ ([TW], Theorem 3.2). For example, hypersurface singularities defined by $z^n = h(x, y)$ is a normal n-fold cyclic covering of $(\mathbb{C}^2, o)$ defined by $z^n = h$ when h is a reduced element.

Let (X, o) be a normal surface singularity and let $\pi\colon (\tilde{X}, E) \longrightarrow (X, o)$ be a resolution such that $\mathrm{red}((h\circ\pi)_{\tilde{X}})$ is simple normal crossing. Let $E = \bigcup_{i=1}^{r} E_i$ and $\mathrm{supp}(\Lambda(h\circ\pi)_{\tilde{X}}) = \bigcup_{i=1}^{s} C_j$ be irreducible decompositions, where $\Lambda(h\circ\pi)_{\tilde{X}}$ is the proper transform of a divisor $\{h = 0\}$ through π.

DEFINITION 5.3. Under the situation above, put $a_i = v_{E_i}(h \circ \pi)$ for any i, $b_j = v_{C_j}(h \circ \pi)$ for any j and $N_h(\pi) = \max\{\mathrm{lcm}(a_i, b_j) | E_i C_j \neq 0\}$. Define a positive integer $N_h(X, o)$ as follows:

(i) $N_h(X, o) = \min\{N_h(\pi) \mid \pi$ is a resolution such that $\mathrm{red}(h\circ\pi)_{\tilde{X}}$ is simple normal crossing$\}$.

(ii) If $\gcd(a_1, \cdots, a_r, b_1, \cdots, b_s) = 1$, then h is called a *semi-reduced element.*

It is obvious that h is semi-reduced if h is reduced.

THEOREM 5.4. *Let $(Y, o) \subset (\mathbb{C}^N, o)$ be a normal surface singularity and $h \in \mathfrak{m}_{Y,o}$ a semi-reduced element. Let (X, o) be the normalization of the n-fold cyclic covering of (Y, o) defined by $z^n = h$. If $n \geqq N_h(Y, o)$, then (X, o) is a weak Kodaira singularity of genus $p_e(Y, o, h)$.*

THEOREM 5.5. *Let (Y, o) be a normal surface singularity and $h \in \mathfrak{m}_{Y,o}$ a reduced element. Let (X, o) be the n-fold cyclic covering of (Y, o) defined by $z^n = h$. If $n \geqq N_h(Y, o)$, then (X, o) is a Kulikov singularity of genus $\delta(h) - r(h) + 1$ and $\mathbb{Z}_X^2 = -r(h)$.*

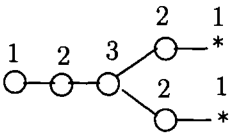
COROLLARY 5.6. *Let $(Y, o) = \{h(x, y, z) = 0\} \subset \mathbb{C}^3$ be a normal hypersurface singularity. If x is a reduced element of $\mathcal{O}_{Y,o}$, then a hypersur-*

face singularity $(X,o)=\{h(x^n,y,z)=0\}$ *is a Kulikov singularity of genus* $\dfrac{\mu(f)-r(f)+1}{2}$ *and* $\mathbb{Z}_X^2=-r(f)$ *if* $n \geqq N_x(Y,o)$, *where* $f:=h(0,y,z)$.

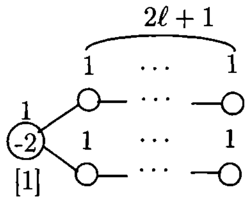
Also, we obtain 5.1 (ii) as a corollary of 5.5.

EXAMPLE 5.7. Let (Y,o) be a hypersurface singularity defined by $z^2 = y(x^2+y^3)$ (i.e., D_5). Then we have $p_e(Y,o,x)=1$ for a reduced element x. Let consider a cyclic cover (X,o) defined by $u^n=x$ (i.e., $(\{z^2=y(u^{2n}+y^3)\},o)$). Assume $n \geqq 3$. Then (X,o) is a Kulikov singularity of genus 1 by 5.6. In the following figures, (i) shows the divisor of (x) on the minimal resolution of (Y,o) and (ii)-(iv) show the fundamental cycles on resolutions of (X,o):

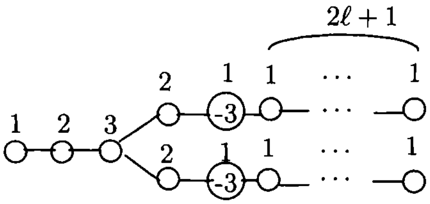
(i) divisor (x) on $\tilde{Y}$:



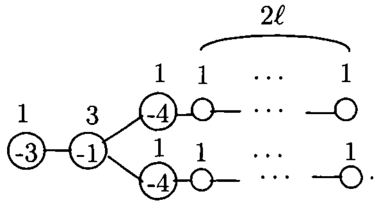
(ii) $n=3\ell$:



(iii) $n=3\ell+1$:



(iv) $n=3\ell+2$:



6. Normal surface singularities with $\mathbb{C}^*$-action and $\mathbb{C}^*$-pencils of curves

DEFINITION 6.1. Let $(X,o)\subset(\mathbb{C}^N,o)$ be a normal surface singularity embedded into $(\mathbb{C}^N,0)$. Let consider a $\mathbb{C}^*$-action on $\mathbb{C}^N$ as follows:

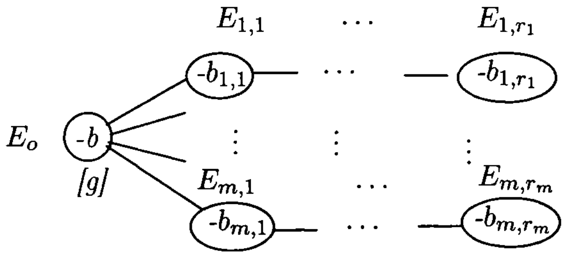
$$t\cdot(x_1,\cdots,x_N)=(t^{p_1}x_1,\cdots,t^{p_N}x_N),$$

where $t\in\mathbb{C}^*$ and $p_1,\cdots,p_N$ are relatively prime positive integers. If (X,o) is invariant under the action, we say that (X,o) has a good $\mathbb{C}^*$-action.

If (X,o) is a normal surface singularity with good $\mathbb{C}^*$-action, then we call it a normal $\mathbb{C}^*$-surface singularity. Normal $\mathbb{C}^*$-surface singularities form a special class in normal surface singularities. However, the class contains many important singularities. For example, the class contains all quotient singularities (so all rational double points). Also, every surface singularity obtained by the contraction of the zero-section of a holomorphic negative line bundle is a normal $\mathbb{C}^*$-surface singularity. In this section, we consider relations between normal $\mathbb{C}^*$-surface singularities and pencils of curves with $\mathbb{C}^*$-action.

Let (X, o) be a normal $\mathbb{C}^*$-surface singularity. Then there exists a $\mathbb{C}^*$-equivariant resolution $\pi : (\tilde{X}, E) \longrightarrow (X, o)$. Namely, there exists a $\mathbb{C}^*$-action on $\tilde{X}$ such that π is a $\mathbb{C}^*$-equivariant map.

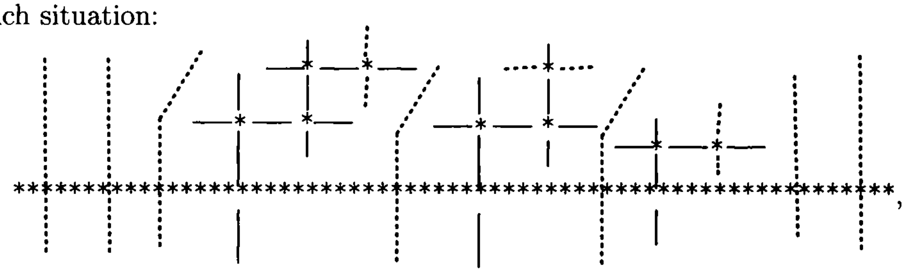
THEOREM 6.2. ([OW]) *Let (X, o) be a normal $\mathbb{C}^*$-surface singularity. Then there always exists a $\mathbb{C}^*$-equivariant resolution $(\tilde{X}, E)$ such that the w.d.graph of E is star-shaped as follows:*



where

$$\frac{d_i}{e_i} = [[b_{i,1}, \cdots, b_{i,r_i}]] := b_{i,1} - \cfrac{1}{b_{i,2} - \cfrac{1}{\ddots - \cfrac{1}{b_{i,r_i}}}} \quad (b_{i,j} \geqq 2) \text{ and } b > \sum_{i=1}^{m} e_i/d_i.$$

We call this resolution *the minimal good $\mathbb{C}^*$-resolution of* (X, o) and call E_o *the central curve.* Every connected component of $E \backslash E_o$ is contracted to a cyclic quotient singularity. We call such connected component a cyclic branch. Under the $\mathbb{C}^*$-action on $\tilde{X}$, any point of E_o is a 0-dimensional orbit (i.e., fixed point) and the intersection point of two connected components of a cyclic branch is also 0-dimensional orbit. If $E_{i,j}$ is a component of a cyclic branch, then it contains a 1-dimensional orbit. The following figure explains such situation:



where dotted lines are 1-dimensional orbits.

THEOREM 6.3. ([OW], [F2] and [P]) *Let (X, o) be a normal $\mathbb{C}^*$-surface singularity and let $(\tilde{X}, E)$ its minimal good $\mathbb{C}^*$-resolution. Then the analytic type is determined by the following three datum:*

(i) *analytic type of the central curve E_o,*
(ii) *analytic type of the normal bundle of E_o in $\tilde{X}$,*
(iii) *intersection points of E_o and all branches.*

DEFINITION 6.4. For any non-negative integer k, *Pinkham-Demazure divisor* on E_o is defined as follows:

$$D^{(k)} := kN^*_{E_o} - \sum_{i=1}^{r} \{\frac{ke_i}{d_i}\} p_i,$$

where $N^*_{E_o}$ is the restriction of the conormal bundle associated to the embedding of E_o into $\tilde{X}$ and $p_i := E_o \cap E_{i,1}$ for any i.

THEOREM 6.5. ([P]) *The affine graded ring R_X associated to (X, o) is given by*

$$R_X \simeq \bigoplus_{k=0}^{\infty} H^0(E_o, \mathcal{O}_{E_o}(D^{(k)}) t^k.$$

The above representation of R_X is called the Pinkham construction (or Pinkham-Demazure construction (see [Wat])). In the following, we explain it by computing the defining equation of a simple elliptic singularity of type $\tilde{E}_6$.

EXAMPLE 6.6. Let E_o be an elliptic curve. We choose a point P_o in E_o. Let (X, o) be a normal surface singularity obtained by the contraction of the zero-section of a negative line bundle $[-P_o]$ (i.e., $\tilde{E}_6$). Let f be a meromorphic function on E_o which has a pole of order -2 at P_o. Let g be a derivative of f. The function g has a pole of order -3. By Weierstrass's canonical form, we assume a relation $g^2 + f^3 + 1 = 0$. Then we have the following:

$$\begin{array}{ll}
H^0(E_o, \mathcal{O}_{E_o}(P_o))t : & t \\
H^0(E_o, \mathcal{O}_{E_o}(2P_o))t^2 : & t^2, ft^2 \\
H^0(E_o, \mathcal{O}_{E_o}(3P_o))t^3 : & t^3, ft^3, gt^3, \\
H^0(E_o, \mathcal{O}_{E_o}(4P_o))t^4 : & t^4, ft^4, gt^4, f^2t^4 \\
H^0(E_o, \mathcal{O}_{E_o}(6P_o))t^5 : & t^6, f^3t^6, f^2t^6, ft^6, gt^6, g^2t^6, fgt^6.
\end{array}$$

If we put $z := t$, $y := ft^2$ and $x := gt^3$, then we have the defining equation $x^2 + y^3 + z^6 = 0$ from $g^2 + f^3 + 1 = 0$ and $dim_{\mathbb{C}} H^0(E_o, \mathcal{O}_{E_o}(6P_o)) = 6$.

DEFINITION 6.7. Let $\Phi : S \longrightarrow \mathbb{C}$ be a pencils of curves. Assume that there exists an effective holomorphic $\mathbb{C}^*$-action on S. If we have $\Phi(t \cdot p) =$

$t^d\Phi(p)$ for any $t \in \mathbb{C}^*$ and any $p \in S$, then we call $\Phi : S \longrightarrow \mathbb{C}$ *a $\mathbb{C}^*$-pencil of curves.*

Then we have the following, which is a $\mathbb{C}^*$-equivariant version of Theorem 3.3.

THEOREM 6.8. *Let (X, o) be a normal $\mathbb{C}^*$-surface singularity and R_X the affine graded ring associated to (X, o). Let $h \in R_X$ be not a perfect power homogeneous element. Let $\pi : (\tilde{X}, E) \longrightarrow (X, o)$ be the minimal good $\mathbb{C}^*$-resolution of (X, o). Then there exists a $\mathbb{C}^*$-pencil of curves $\Phi : S \longrightarrow \mathbb{C}$ which satisfies the following $\mathbb{C}^*$-equivariant diagram :*

$$\begin{array}{ccc} (X,o) & \xleftarrow{\ \pi\ } & (\tilde{X},E) \subset (S, \operatorname{supp}(S_o)). \\ & {\searrow}^{h} \qquad \qquad {\swarrow}_{\Phi} & \\ & \Delta & \end{array} \tag{6.1}$$

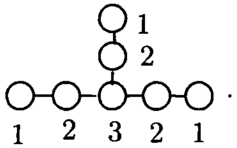
The author [Tt5] proved 6.8 in a different way from 3.3. Let us explain the outline. We consider a $\mathbb{P}^1$-bundle $\pi : \bar{S} \longrightarrow E_o$ on a curve E_o and consider a meromorphic function f on $\bar{S}$. After taking suitable blowing-ups $\sigma : \tilde{S} \longrightarrow \bar{S}$, we consider $\Phi := f \circ \sigma$ on $\tilde{S}$. By taking a suitable open subset S in $\tilde{S}$ and the restriction of Φ onto S, then we get a $\mathbb{C}^*$-pencil of curves. By using the slice theorem, we showed that every $\mathbb{C}^*$-pencil of curves is constructed in this way. From it, we can show the following.

THEOREM 6.9. (i) *The singular fiber of any $\mathbb{C}^*$-pencil of curves become star-shaped after suitable blowing-ups. In the situation, the analytic type of a $\mathbb{C}^*$-pencil of curves is determined the numerical conditions (i.e., w.d.graph) and Pinkham-Demazure data (i.e., the analytic type of the central curve E_o and N_{E_o} and intersection points of E_o and branches).*

(ii) *For a $\mathbb{C}^*$-pencil of curves $\Phi : S \longrightarrow \Delta$, we have the following:*

$$\bigoplus_{k=0}^{\infty} \mathbb{C} \cdot \Phi^k \simeq \bigoplus_{k=0}^{\infty} H^0(E_o, \mathcal{O}_{E_o}(D^{(k)})t^k.$$

EXAMPLE 6.10. Let consider an elliptic pencil whose singular fiber has the following w.d.graph:



Since $D^{(k)} = 2kP_0 - \left\{\frac{2k}{3}\right\}P_1 - \left\{\frac{2k}{3}\right\}P_2 - \left\{\frac{2k}{3}\right\}P_3$, we have $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D^{(k)}))$ $= \mathbb{C}\Phi^{\frac{k}{3}}$ if $3|k$ and zero if $3 \nmid k$.

References

[Arn] V. I. Arnold, *Normal forms of functions*, Invent. Math. 35 (1976), 87–109.

[Art] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. 88 (1963), 129–138.

[As] T. Ashikaga, *Surface singularities on cyclic coverings and an inequality for the signature*, J. Math. Soc. Japan 51 (1999), 485–521.

[BPV] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Ergebnisse der Mathematik, Band 4, Springer-Verlag, 1984.

[EW] W. Ebeling, C. T. C. Wall, *Kodaira singularities and an extension of Arnold's strange duality*, Compositio Math. 56 (1985), 3–77.

[F1] A. Fujiki, *On isolated singularities with* $\mathbb{C}^*$*-action. Master thesis*, Kyoto University (197), (in Japanese).

[F2] A. Fujiki, *On resolution of cyclic quotient singularities*, Publ. RIMS Kyoto Univ. 10 (1974), 293–328.

[G] H. Grauert, *Über Modifikationen und exceptionelle analytische Mengen*, Math. Ann. 146 (1962), 311–368.

[Ka1] U. Karras, *On pencils of curves and deformations of minimally elliptic singularities*, Math. Ann. 247 (1980), 43–65.

[Ka2] U. Karras, *Methoden zur Berechnung von Algebraischen Invarianten und zur Konstruktion von Deformationen Normaler Flachensingularitaten*, Habilitationschrift, Dortmund 1981.

[Ko] K. Kodaira, *On compact analytic surface II*, Ann. of Math. 77 (1963), 563–626.

[Ku] V. S. Kulikov, *Degenerate elliptic curves and resolution of uni- and bimodal singularities*, Funct. Anal. Appl. 9 (1975), 69–70.

[La] H. Laufer, *On minimally elliptic singularities*, Amer. J. Math. 99 (1977), 1257–1295.

[Lo] E. Looijenga, *Riemann-Roch and smoothing of singularities*, Topology 25 (3) (1986), 293–302.

[Na] M. Namba, *Geometry of Projective Algebraic Curves*, Marcel Dekker, (1984).

[NU] Y. Namikawa, K. Ueno, *The complete classification of fibres in pencils of curves of genus two*, Manuscripta Math. 9 (1973), 143–186.

[O] T. Okuma, *Plurigenera of Surface Singularities*, Nova Science Publishers, New York, 2000.

[OW] P. Orlik, P. Wagreich, *Isolated singularities of algebraic surfaces with* $\mathbb{C}^*$*-action*, Ann. Math. 93 (1971), 205–228.

[P] H. Pinkham, *Normal surface singularities with* $\mathbb{C}^*$*-action*, Math. Ann. 227 (1977), 183–193.

[R] M. Reid, *Elliptic Gorenstein singularities of surfaces*, Preprint (1978).

[Sa] K. Saito, *Algebraic surfaces for regular systems of weights, Algebraic Geometry*, Commutative Algebra II - in honor of Masayoshi NAGATA (H. Hijikata et al., eds.), Kinokuniya, Tokyo, (1987), 517–614.

[St1] J. Stevens, *Elliptic surface singularities and smoothings of curves*, Math. Ann. 267 (1984), 239–249.

[St2] J. Stevens, *Kulikov Singularities*, Thesis, Leiden, 1985.

[Tm1] M. Tomari, *A* p_g*-formula and elliptic singularities*, Publ. Res. Inst. Math. Sci. Kyoto Univ. 21 (2), (1985), 297–354.

[Tm2] M. Tomari, *Maximal-ideal-adic filtration on $\mathbb{R}^1\psi^*\mathcal{O}_{\tilde{V}}$ for normal two-dimensional singularities*, Complex analytic singularities, Adv. Stud. Pure Math., North-Holland, Amsterdam, 8 (1987), 633–647.

[TW] M. Tomari, K-i. Watanabe, *Cyclic covers of normal graded rings*, Kodai Math. J. 24 (2001), 436–457.

[Tt1] T. Tomaru, *On Gorenstein surface singularities with fundamental genus $p_f \geqq 2$ which satisfy some minimality conditions*, Pacific J. Math. 170 (1995), 271–295.

[Tt2] T. Tomaru, *On Kodaira singularities defined by $z^n = f(x, y)$*, Math. Z. 236 (2001), 133–149.

[Tt3] T. Tomaru, *On some classes of weakly Kodaira singularities*, Séminaires & Congrés, "Singularités Franco-Japonaises" 10 (2005), 323–340.

[Tt4] T. Tomaru, *Pencil genus for normal surface singularities*, J. Math. Soc. Japan 59 (1) (2007), 35–80.

[Tt5] T. Tomaru, *Normal $\mathbb{C}^*$-surface singularities and $\mathbb{C}^*$-pencils of curves*, (in preparation).

[Wag] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math. 92 (1970), 421–454.

[Wat] K-i. Watanabe, *Some remarks concerning Demazure's construction of normal graded rings*, Nagoya Math. J. 83 (1981), 203–211.

[Y] S. S.-T. Yau, *On maximally elliptic singularities*, Trans. Amer. Math. Soc. 257 (2) (1980), 269–329.

SCHOOL OF HEALTH SCIENCES
GUNMA UNIVERSITY
MAEBASHI, GUNMA 371-8514
JAPAN
E-mail: ttomaru@health.gunma-u.ac.jp

Received November 28, 2009.