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Satoshi Koike, Adam Parusiński

SOME QUESTIONS ON THE FUKUI NUMERICAL SET FOR COMPLEX FUNCTION GERMS

Abstract. The Fukui numerical set is known as a blow-analytic invariant for real analytic function germs. Taking into account the similarity between real blow-analytic properties and complex topological ones, we may ask if the Fukui numerical set is a topological invariant for complex analytic function germs. In this note we discuss the problem and give some related questions.

1. Introduction

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Take any analytic arc $\gamma : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$. Then $f(\gamma(t))$ is a convergent power series in t. We denote by $\operatorname{ord}(f(\gamma(t)))$ its order in t. Set

$$A(f) := \{\operatorname{ord}(f(\gamma(t))) \in \mathbb{N} \cup \{\infty\}; \gamma : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)\ C^\omega\}.$$

In [11] T. Fukui proved that $A(f)$ is a blow-analytic invariant in the real case. Namely, if analytic functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent, then $A(f) = A(g)$. We called $A(f)$ *the Fukui invariant* in [16], [20], but we call it *the Fukui numerical set* here. We shall give the definition of blow-analytic equivalence in the next section. As mentioned in [12], it is well-known that there is a similarity between real blow-analytic properties and complex topological ones. Therefore it is natural to ask the following question:

QUESTION 1. Suppose that analytic function germs f, $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are topologically equivalent. Then $A(f) = A(g)$?

Note that the smallest number in $A(f)$ is the multiplicity of f. Therefore Question 1 is a kind of generalisation of the Zariski conjecture [35].

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We discuss the above question in this note. We have a positive answer in case $n = 2$, which shall be shown in §3. On the other hand, it is well-known that there are families of 3 variable polynomial functions with isolated singularities which are μ-constant but not μ^*-constant. It may be natural to ask whether there is a negative example to Question 1 in such families. In §4 we analyse those polynomial functions, and see that they are not negative examples. In §5 we discuss the above question in case $n \geq 4$, relating it to some properties of isomorphic Seifert forms. Then we have some questions on Fukui numerical sets and Seifert forms. Concerning the aforementioned similarity, we pose some questions in §6 on the relation between real analytic functions and their complexifications. As a partial result to one of our questions, we show that if the complexification of a family of two variable real analytic function germs is topologically trivial, then the original real family is blow-analytically trivial.

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2. Formulae to compute the Fukui numerical set

We say that a homeomorphism germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a *blow-analytic homeomorphism* if there exist real modifications $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^n, 0)$, $\tilde{\mu} : (\tilde{M}, \tilde{\mu}^{-1}(0)) \to (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (\tilde{M}, \tilde{\mu}^{-1}(0))$ so that the following diagram is commutative:

$$
\begin{array}{ccc}
(M, \mu^{-1}(0)) & \xrightarrow{\mu} & (\mathbb{R}^n, 0) \\
\Phi \big\downarrow & & h \big\downarrow \\
(\tilde{M}, \tilde{\mu}^{-1}(0)) & \xrightarrow{\tilde{\mu}} & (\mathbb{R}^n, 0)
\end{array}
\tag{2.1}
$$

We say that two real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exists a blow-analytic homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ h$. For properties on blow-analyticity, see the surveys [12], [14].

We next recall the formulae to compute the Fukui numerical set, given in [16]. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For an analytic function germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, let $\sigma : M \to \mathbb{K}^n$ be a simplification of $f^{-1}(0)$, namely, σ is a composition of a finite number of blowings-up, M is smooth and $f \circ \sigma$ is normal crossing. We denote by E_i, $i \in J$, the irreducible components of $(f \circ \sigma)^{-1}(0)$ in $\sigma^{-1}(B_\varepsilon)$, where B_ε is a small ball in $\mathbb{K}^n$ centered at the origin. For each $i \in J$, let $N_i = \text{mult}_{E_i} f \circ \sigma$. Denote for $I \subset J$, $E_I = \bigcap_{i \in I} E_i$ and $\mathring{E}_I = E_I \setminus \bigcup_{j \in J \setminus I} E_j$.

We put

$$\mathcal{C} := \{I; \mathring{E}_I \cap \sigma^{-1}(0) \neq \emptyset\}.$$

Remark 2.1. Taking a suitable σ, we can assume that $\sigma^{-1}(0)$ is the union of some of E_i. Then $\mathcal{C} = \{I; E_I \subset \sigma^{-1}(0)\}$.

For A, $B \subset \mathbb{N} \cup \{\infty\}$, define $A + B = \{a + b \in \mathbb{N} \cup \{\infty\}; a \in A, b \in B\}$, where we set $a + b = \infty$ if $a = \infty$ or $b = \infty$. Let us put

$$\Omega_I(f) := (N_{i_1}\mathbb{N} + \cdots + N_{i_p}\mathbb{N}) \cup \{\infty\},$$

for $I = (i_1, \cdots, i_p) \in \mathcal{C}$. Then we have

Theorem 2.2. ([16]) *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be an analytic function germ and let σ be a simplification of $f^{-1}(0)$. Then we have*

$$A(f) = \bigcup_{I \in \mathcal{C}} \Omega_I(f).$$

In the real case, taking into consideration the signs, we can introduce finer invariants. Let us put

$$\mathcal{C}^+ := \{I \in \mathcal{C}; \mathring{E}_I \cap \sigma^{-1}(0) \cap \overline{P(f)} \neq \emptyset\}, \quad P(f) := \{x \in M; f \circ \sigma(x) > 0\},$$
$$\mathcal{C}^- := \{I \in \mathcal{C}; \mathring{E}_I \cap \sigma^{-1}(0) \cap \overline{N(f)} \neq \emptyset\}, \quad N(f) := \{x \in M; f \circ \sigma(x) < 0\},$$

where the overlines denote the closures in M.

Let $\lambda : U \to \mathbb{R}^n$ be an analytic arc with $\lambda(0) = 0$, where U denotes a neighbourhood of $0 \in \mathbb{R}$. We call λ *nonnegative* (resp. *nonpositive*) *for f* if $(f \circ \lambda)(t) \geq 0$ (resp. ≤ 0) in a positive half neighbourhood $[0, \delta) \subset U$. Then we define *the Fukui numerical sets with sign* by

$$A_+(f) := \{\mathrm{ord}\,(f \circ \lambda); \lambda \text{ is a nonnegative arc through } 0 \text{ for } f\},$$
$$A_-(f) := \{\mathrm{ord}\,(f \circ \lambda); \lambda \text{ is a nonpositive arc through } 0 \text{ for } f\},$$

respectively. It is easy to see that these $A_+(f)$ and $A_-(f)$ are also blow-analytic invariants. Note that $A(f) = A_+(f) \cup A_-(f)$. Then we have the following formulae to compute the Fukui numerical sets with sign:

Theorem 2.3. ([16]) *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ. Then we have*

$$A_+(f) = \bigcup_{I \in \mathcal{C}^+} \Omega_I(f), \; A_-(f) = \bigcup_{I \in \mathcal{C}^-} \Omega_I(f).$$

In [20] we proved the Thom-Sebastiani formulae for the Fukui numerical sets. On the other hand, we introduced in [21] more refined blow-analytic invariants than the above Fukui numerical sets with sign, called *the refined Fukui invariants with sign.*

3. The Fukui numerical set is a topological invariant for 2 variable functions

We first recall the notion of the tree model introduced by T.-C. Kuo and Y.-C. Lu [23]. Let $f(x,y)$ be a complex analytic function germ of multplicity m and *mini-regular in* x, that is

$$f(x,y) = u(x,y)(x^m + \sum_{i=1}^{m} a_i(y)x^{m-i}),$$

where $m = \operatorname{mult}_0 f$, u, a_i are analytic and $u(0,0) \neq 0$. Let $x = \lambda_i(y)$, $i = 1,\dots,m$, be the complex Newton-Puiseux roots of f. We define *the contact order of* λ_i and λ_j as

$$O(\lambda_i,\lambda_j) := \operatorname{ord}_0 (\lambda_i - \lambda_j)(y).$$

Let $h \in \mathbb{Q}$. We say that λ_i, λ_j are *congruent modulo* h^+ if $O(\lambda_i,\lambda_j) > h$.

The tree model $T(f)$ of f is defined as follows. First, draw a vertical line segment as the *main trunk* of the tree. Mark $m = \operatorname{mult}_0 f(x,y)$ alongside the trunk to indicate that m roots are bundled together.

Let $h_0 := \min\{O(\lambda_i,\lambda_j) | 1 \le i,j \le m\}$. Then draw a bar, B_0, on top of the main trunk. Call $h(B_0) := h_0$ the *height* of B_0.

The roots are divided into equivalence classes modulo h_0^+. We then represent each equivalence class by a vertical line segment drawn on top of B_0, and call it a *trunk*. If a trunk consists of s roots we say it has *multiplicity* s, and mark s alongside.

The same construction is repeated recursively on each trunk, getting possible more bars and trunks, etc.. The height of each bar and the multiplicity of each trunk, are defined likewise. Each trunk has a unique bar on top of it. The construction terminates at the stage where the bars have infinite heights, that is on top of a trunk that contains a single, maybe multiple, root of f.

Using the tree model, we next positively answer Question 1 in the case of analytic functions of two variables.

THEOREM 3.1. *Let f, $g : (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be analytic function germs. If f and g are topologically equivalent, then $A(f) = A(g)$.*

Proof. Let us factor f and g into irreducible components:

$$f(x,y) = f_1(x,y)^{a_1} \cdots f_m(x,y)^{a_m}, \quad g(x,y) = g_1^{b_1}(x,y) \cdots g_n^{b_n}(x,y).$$

By hypothesis, there exists a homeomorphism $h : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ such that $f = g \circ h$, consequently $m = n$.

Let $h(f_j^{-1}(0)) = g_j^{-1}(0)$, $1 \le j \le m$. Since f_j's and g_j's are irreducible, there is a neighbourhood U of 0 in $\mathbb{C}^2$ such that each f_j (resp. g_j) has an isolated singularity at $0 \in \mathbb{C}^2$ in U (resp. $h(U)$). Note that this isolated singularity means any point $P \in U \setminus \{0\}$ is a regular point of f_j. Pick a point $P \in f_j^{-1}(0) \cap U \setminus \{0\}$. Then there is a neighbourhood W of P with $W \subset U$ such that f_j (resp. g_j) is regular and f_i's (resp. g_i's), $i \neq j$, are units in W (resp. $h(W)$). Therefore there are local analytic diffeomorphisms $\sigma_1 : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, P)$ and $\tau_1 : (\mathbb{C}, f(P)) \to (\mathbb{C}, 0)$ such that $\tau_1 \circ f \circ \sigma_1(x, y) = x^{a_j}$. Similarly, there are local analytic diffeomorphisms $\sigma_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, h(P))$ and $\tau_2 : (\mathbb{C}, g(h(P))) \to (\mathbb{C}, 0)$ such that $\tau_2 \circ g \circ \sigma_2(x, y) = x^{b_j}$. Since f and g are topologically equivalent, x^{a_j} and x^{b_j} are topologically right-left equivalent as function germs: $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Therefore we have $a_j = b_j$, $1 \leq j \leq m$.

Taking a linear coordinate change if necessary, we may assume that f and g are mini-regular in x. Namely, if $f_{[k]}$ and $g_{[s]}$ are the initial forms of f and g respectively, then $f_{[k]}(x, 0) \not\equiv 0$ and $g_{[s]}(x, 0) \not\equiv 0$. Let

$$F(x, y) = f_1(x, y) \cdots f_m(x, y), \quad G(x, y) = g_1(x, y) \cdots g_m(x, y).$$

By construction, F and G are reduced. Since f and g are topologically equivalent, $(\mathbb{C}^2, F^{-1}(0))$ is topologically equivalent to $(\mathbb{C}^2, G^{-1}(0))$. By the Zariski theorem ([34]), the Puiseux characteristics of branches and their intersection numbers coincide with those of G. Then it follows from the above observation, i.e. $a_j = b_j$, $1 \leq j \leq m$, that the Puiseux characteristics of branches of f and their intersection numbers with counting multiplicities coincide also with those of g with counting multiplicities.

On the other hand, by Theorem VIII in [16], the Fukui numerical set $A(f)$ can be computed using the tree model $T(f)$. Since the tree model $T(f)$ is completely determined by the Puiseux characteristics of branches of f and their intersection numbers with counting multiplicities, it follows that $A(f) = A(g)$. ∎

From the proof of Theorem 3.1, we see that the topological equivalence of two variable complex function germs implies the coincidence of their tree models. The converse is also valid. Namely, we have

THEOREM 3.2. ([29]) *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be analytic function germs. Then f and g are topologically equivalent if and only if the tree models of f and g coincide.*

On the other hand, we can introduce the notion of the real tree model for two variable real analytic function germs. See [21, 22] for the definition of it. Firstly K. Kurdyka and L. Paunescu consider the notion of the real part of the tree model in [26]. Our real tree model is the extension of theirs, and using it we can characterise the blow-analytic equivalence for two variable real analytic functions as follows:

Theorem 3.3. ([21]) *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs. Then f and g are blow-analytically equivalent if and only if the real tree models of f and g are isomorphic.*

As seen as above, the Fukui numerical set is a topological invariant for two variable complex analytic functions. Therefore we may ask the following question:

Question 2. If a family of analytic function-germs $\{f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)\}$ with isolated singularities has a constant Fukui numerical set, then is the family $\{f_t\}$ topologically trivial?

We can construct a negative example as follows:

Example 3.4. Let $f_t : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $t \in \mathbb{C}$, be a polynomial function defined by

$$f_t(x, y) = (1 - t)x(y^2 - x^2) + tx(y^2 - x^3).$$

Then for any $t \in \mathbb{C}$, f_t has an isolated singularity and we have $A(f_t) = \{3, 4, 5, \cdots\} \cup \{\infty\}$. But we can see that $f_0(x, y) = x(x^2 - y^2)$ and $f_1(x, y) = x(y^2 - x^3)$ are not topologically equivalent as function germs.

4. Fukui numerical sets of some special 3 variable functions

For a positive integer $a \in \mathbb{N}$, set $\mathbb{N}_{\geq a} = \{m \in \mathbb{N}; m \geq a\}$.

4.1. List of the Fukui numerical sets for the Brieskorn polynomials

We first make a convention. By the Brieskorn polynomials in 2 variables we mean $f(x, y) = ax^p + by^q$ $(p \leq q)$ where $a \neq 0$ and $b \neq 0$. Since their analytic types depend only on the signs of a and b in the real case (resp. on whether a and b are non-zero in the complex case), in order to simplify the notation, in this note we consider only Brieskorn polynomials of the form $f(x, y) = \pm x^p \pm y^q$ (resp. $f(x, y) = x^p + y^q$).

Let $(p, q) = d$ where (p, q) denotes $\gcd(p, q)$. Then there are $p_1, q_1 \in \mathbb{N}$ such that $p = p_1 d$, $q = q_1 d$ and $(p_1, q_1) = 1$. Set $[p, q] = LCM(p, q) = p_1 q_1 d = p q_1 = p_1 q$.

Here we recall the list of the Fukui numerical sets for real Brieskorn polynomials $f(x, y) = \pm x^p \pm y^q$, $(x, y) \in \mathbb{R}^2$, $p \leq q$, given in [20]. See the next page for the list.

We can easily compute the Fukui numerical sets for complex Brieskorn polynomials $f(x, y) = x^p + y^q$, $(x, y) \in \mathbb{C}^2$, $p \leq q$ as follows:

$$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}. \tag{4.1}$$

We give also the Fukui numerical sets for the product function.

$f(x,y)$	Fukui invariants
$\pm x^p \pm y^q$, p, q odd	$A(f) = A_+(f) = A_-(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
p odd, q even	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$\pm x^p + y^q$	$A_+(f) = A(f)$, $A_-(f) = p\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$\pm x^p - y^q$	$A_-(f) = A(f)$, $A_+(f) = p\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
p even, q odd	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$x^p \pm y^q$	$A_+(f) = A(f)$, $A_-(f) = q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$-x^p \pm y^q$	$A_-(f) = A(f)$, $A_+(f) = q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$\pm(x^p - y^q)$, p,q even	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$x^p - y^q$	$A_+(f) = p\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$, $A_-(f) = q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$-x^p + y^q$	$A_+(f) = q\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$, $A_-(f) = p\mathbb{N} \cup \mathbb{N}_{\geq [p,q]} \cup \{\infty\}$
$\pm(x^p + y^q)$, p,q even	$A(f) = p\mathbb{N} \cup q\mathbb{N} \cup \{\infty\}$
$x^p + y^q$	$A_+(f) = A(f)$, $A_-(f) = \{\infty\}$
$-x^p - y^q$	$A_-(f) = A(f)$, $A_+(f) = \{\infty\}$

EXAMPLE 4.1. Let $f(x,y) = cx^py^q$, $c \neq 0$, be a polynomial function defined over $\mathbb{R}^2$ or $\mathbb{C}^2$. Then we have

$$A(f) = \{ap + bq : a, b \in \mathbb{N}\} \cup \{\infty\}.$$

In addition, in the real case we have the following:

(i) If p or q is odd, then $A_+(f) = A_-(f) = A(f)$.
(ii) If p and q are even and $c > 0$, then $A_+(f) = A(f)$ and $A_-(f) = \{\infty\}$.
(iii) If p and q are even and $c < 0$, then $A_-(f) = A(f)$ and $A_+(f) = \{\infty\}$.

4.2. Other formulae for the Fukui numerical sets

Throughout this section, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$. Let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, $n \geq 2$, be an analytic function germ of the form:

$$g(x_1, \cdots, x_n) = f(x_1, x_2) + \phi(x_1, x_2) + \sum_{j=3}^{n} x_j \psi_j(x_1, \cdots, x_n), \tag{4.2}$$

where $f(x_1, x_2) = ax_1^p + bx_2^q$, $ab \neq 0$, $p \leq q$, and $j^{[p,q]}\phi(0) = 0$ as a weighted $[p,q]$-jet with respect to the system of weights (q_1, p_1), p_1 and q_1 as in Subsection 4.1. Here we recall the results for semi-quasi homogeneous functions.

THEOREM 4.2. *Given a system of weights $\alpha = (\alpha_1, \cdots, \alpha_n)$. Let f_t : $(\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, $t \in I = [0,1]$, be an analytic family of analytic function germs. Suppose that for each $t \in I$, the weighted initial form of f_t with respect to α is of the same weighted degree and has an isolated singularity at $0 \in \mathbb{K}^n$. Then we have*

(1) *In the real case,* $\{f_t\}_{t\in I}$ *is blow-analytically trivial over* I. *(T. Fukui - L. Paunescu* [13], *T. Fukui - E. Yoshinaga* [10])
(2) *In the complex case,* $\{f_t\}_{t\in I}$ *is topologically trivial over* I. *(V.I. Arnol'd* [1], *H. King* [19], *J. Damon - T. Gaffney* [8], *A. Parusiński* [28])

It follows from the theorem above that f and $f+\phi$ are blow-analytically equivalent (resp. topologically equivalent) in the real case (resp. in the complex case) as two variable analytic function germs. Since the Fukui numerical sets are blow-analytic invariants for real analytic functions by T. Fukui [11] (resp. a topological invariant for two variable complex analytic functions by Theorem 3.1), we have $A(f)=A(f+\phi)$, $A_+(f)=A_+(f+\phi)$ and $A_-(f)=A_-(f+\phi)$ (resp. $A(f)=A(f+\phi)$). Thus we have

ASSERTION 4.3. *Under the above assumptions on* g, *we have*

$$A(g)\supset A(f),\ A_+(g)\supset A_+(f),\ A_-(g)\supset A_-(f)\quad (\textit{resp. } A(g)\supset A(f))$$

in the real case (resp. in the complex case). Here $A(f)$, $A_+(f)$ *and* $A_-(f)$ *(resp.* $A(f)$*) are given in the list of Subsection* 4.1 *(resp. in* (4.1)*).*

Next let $g:(\mathbb{K}^n,0)\to(\mathbb{K},0)$, $n\geq 2$, be an analytic function germ of form (4.2), where $f(x_1,x_2)=cx_1^px_2^q$, $c\neq 0$ and $j^m\phi(0)=0$ for some $m\geq p+q$. Then we have

ASSERTION 4.4. *Under the above assumptions on* g, *we have*

$$A(g)\supset A(f)\cap\mathbb{N}_{\leq m},\ A_+(g)\supset A_+(f)\cap\mathbb{N}_{\leq m},\ A_-(g)\supset A_-(f)\cap\mathbb{N}_{\leq m}$$
$$(\textit{resp. } A(g)\supset A(f))$$

in the real case (resp. in the complex case). Here $A(f)$, $A_+(f)$ *and* $A_-(f)$ *(resp.* $A(f)$*) are given in Example* 4.1.

Let $g:(\mathbb{K}^n,0)\to(\mathbb{K},0)$, $n\geq 2$, be an analytic function germ of form (4.2), where $f(x_1,x_2)=ax_1^p+bx_2^q$, $ab\neq 0$, $p\leq q$, $j^{[p,q]}\phi(0)=0$ as a weighted $[p,q]$-jet with respect to the system of weights (q_1,p_1) and $j^{[p,q]-2}\psi_j(0)=0$ as a normal jet, $3\leq j\leq d$.

PROPOSITION 4.5. *Suppose that* $ab<0$, *or* p *or* q *is odd in the real case (resp. suppose that* $ab\neq 0$ *in the complex case). Then we have*

$$A(g)=A(f),\ A_+(g)=A_+(f),\ A_-(g)=A_-(f)\quad (\textit{resp. } A(g)=A(f)).$$

Proof. We show only the real case, since the complex case follows similarly. By the argument of Assertion 4.3,

$$A(f+\phi)=A(f),\ A_+(f+\phi)=A_+(f),\ A_-(f+\phi)=A_-(f).$$

Since $A(f),A_+(f),A_-(f)\supset\{[p,q],[p,q]+1,\cdots\}\cup\{\infty\}$ by the table in Subsection 4.1

$$A(g)\subset A(f),\ A_+(g)\subset A_+(f),\ A_-(g)\subset A_-(f).$$

On the other hand, it follows from the fact

$$(f+\phi)(\lambda_1(t),\lambda_2(t)) = g(\lambda_1(t),\lambda_2(t),0,\cdots,0)$$

that $A(f+\phi)\subset A(g)$, $A_+(f+\phi)\subset A_+(g)$ and $A_-(f+\phi)\subset A_-(g)$. Thus

$$A(g)=A(f),\ A_+(g)=A_+(f),\ A_-(g)=A_-(f). \ \blacksquare$$

Corollary 4.6. *Let $f:(\mathbb{K}^n,0)\to(\mathbb{K},0)$, $n\geq 2$, be an analytic function germ of the form:*

$$f(x_1,\cdots,x_n)=ax_1^m+bx_2^m+\sum_{j=3}^{n}x_jh_j(x_1,\cdots,x_n)$$

with $j^{m-2}h_j(0)=0$, $j=3,\cdots,n$. Suppose that m is odd and $ab\neq 0$, or m is even and $ab<0$ in the real case (resp. suppose that $ab\neq 0$ in the complex case). Then we have

$$A(f)=\{m,m+1,m+2,\cdots\}\cup\{\infty\}.$$

4.3. Applications to some special examples

The Briançon-Speder family ([7]) and the Oka family ([27]) are well-known as families of 3 variable complex polynomial functions which are μ-constant but not μ^*-constant. In this subsection we analyse their Fukui numerical sets of those families.

Example 4.7. Let $f_t:(\mathbb{K}^3,0)\to(\mathbb{K},0)$, $t\in\mathbb{K}$, be Briançon-Speder's family defined by

$$f_t(x,y,z)=z^5+tzy^6+y^7x+x^{15}.$$

We first compute the Fukui numerical sets in the real case. Regarding z^5+x^{15} as $f(z,x)$ and tzy^5+y^6x as $\psi(z,x,y)$, it follows from Assertion 4.3 that

$$A(f_t),\ A_+(f_t),\ A_-(f_t)\supset\{5,15,16,17,\cdots\}\cup\{\infty\}\ (t\in\mathbb{R}).$$

We next regard y^7x as $f(y,x)$, x^{15} as $\phi(y,x)$, z^4+ty^6 as $\psi(y,x,z)$ and $m=14$. Then it follows from Assertion 4.4 that

$$A(f_t),\ A_+(f_t),\ A_-(f_t)\supset\{8,9,\cdots 14\}\ (t\in\mathbb{R}).$$

Thus $A(f_t),\ A_+(f_t),\ A_-(f_t)\supset\{5,8,9,10,\cdots\}\cup\{\infty\}\ (t\in\mathbb{R})$.

Let $\gamma=(\gamma_1,\gamma_2,\gamma_3):(\mathbb{R},0)\to(\mathbb{R}^3,0)$ be an analytic arc:

$$\gamma_j(t)=a_j^{(1)}t+a_j^{(2)}t^2+\cdots,\ 1\leq j\leq 3.$$

If $a_3^{(1)}\neq 0$, then $\operatorname{ord}(f_t\circ\gamma)=5$. If $a_3^{(1)}=0$, then $\operatorname{ord}(f_t\circ\gamma)\geq 8$. Therefore,

$$A(f_t)=\ A_\pm(f_t)=\{5,8,9,10,\cdots\}\cup\{\infty\}\ (t\in\mathbb{R}).$$

Using a similar argument, we can compute the Fukui numerical set in the complex case as follows:

$$A(f_t)=\{5,8,9,10,\cdots\}\cup\{\infty\}\ (t\in\mathbb{C}).$$

Modifying the Briançon-Speder family, T. Fukui and L. Paunescu constructed a family of 3 variable semi-quasi homogeneous functions in [13] which is μ-constant but not μ^*-constant as a family of complex functions. Let us recall the family $g_t : (\mathbb{K}^3, 0) \to (\mathbb{K}, 0)$, $t \in \mathbb{K}$, defined by

$$g_t(x, y, z) = z^5 + tzy^7 + y^8x + x^{15}.$$

By the same argument as above, we can easily see that

$$A(g_t) = A_{\pm}(g_t) = \{5, 9, 10, 11, \cdots\} \cup \{\infty\} \ (t \in \mathbb{R})$$

in the real case, and

$$A(g_t) = \{5, 9, 10, 11, \cdots\} \cup \{\infty\} \ (t \in \mathbb{C})$$

in the complex case.

Example 4.8. Let $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $t \in \mathbb{C}$, be Oka's family defined by

$$f_t(x, y, z) = x^8 + y^\ell + z^\ell + tx^5z^2 + x^3yz^3 \quad (\ell \geq 16).$$

Then we have

$$A(f_t) = \{7, 8, 9, \cdots\} \cup \{\infty\} \ (t \in \mathbb{C}).$$

Proof. We show only the case $\ell = 16$. The other cases follow similarly.

It is obvious that $A(f_t) \subset \{7, 8, 9, \cdots\} \cup \{\infty\}$.

Regarding $y^{16} + z^{16}$ as $f(y, z)$ and $x^7 + tx^4z^2 + x^2yx^3$ as $\psi(y, z, x)$, it follows from Assertion 4.3 that

$$A(f_t) \supset \{16, 17, 18, \cdots\} \cup \{\infty\}.$$

On the other hand, we can easily see $7, 8, 9, \cdots, 15 \in A(f_t)$ as follows:

- 7 is attained by the arc $\gamma(s) = (s, as, s)$ for $a \neq -t$.
- 8 is attained by the arc $\gamma(s) = (s, s, s^2)$.
- 9 is attained by the arc $\gamma(s) = (s, -ts - s^2 + s^3, s)$.
- 10 is attained by the arc $\gamma(s) = (s^2, s, s)$.
- 11 is attained by the arc $\gamma(s) = (s^2, s^2, s)$.
- 12 is attained by the arc $\gamma(s) = (s^2, as^3, s)$ for $a \neq -t$.
- 13 is attained by the arc $\gamma(s) = (s^3, s, s)$.
- 14 is attained by the arc $\gamma(s) = (s^3, s^2, s)$.
- 15 is attained by the arc $\gamma(s) = (s^3, s^3, s)$.

Thus we have $A(f_t) = \{7, 8, 9, \cdots\} \cup \{\infty\} \ (t \in \mathbb{C})$. ∎

5. Seifert forms and topological types of complex functions

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity. Then we denote by $\Gamma(f)$ the Seifert form corresponding to $f^{-1}(0)$. For the definition of the Seifert form, see V. I. Arnol'd, S. M. Guzein-Zard, A. N. Varchenko [2] or A. H. Durfee [9].

Let us recall an interesting family of plane curves on $\mathbb{C}^2$ constructed by P. Du Bois and F. Michel.

THEOREM 5.1. ([6]) *Let*

$$f_{a,b}(x,y) = ((y^2 - x^3)^2 - x^{b+6} - 4yx^{\frac{b+9}{2}})((x^2 - y^5)^2 - y^{a+10} - 4xy^{\frac{a+15}{2}}),$$

where $b \geq 11$ *and* $b \neq a+8$. *Then the Seifert forms* $\Gamma(f_{a,b})$ *and* $\Gamma(f_{b-8,a+8})$ *are isomorphic, but* $f_{a,b}$ *and* $f_{b-8,a+8}$ *are not topologically equivalent as function germs.*

The following result in higher dimensions is in contrast to the above one in the plane curve case.

THEOREM 5.2. (A. H. Durfee [9], M. Kato [17], H. C. King [18]) *Let* $f,\ g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $n \geq 4$, *be analytic function germs with isolated singularities. Suppose that the Seifert forms* $\Gamma(f)$ *and* $\Gamma(g)$ *are isomorphic. Then* f *and* g *are topologically equivalent.*

REMARK 5.3. In case $n = 3$ the same result as Theorem 5.2 does not always hold. Considering the suspension of the above Du Bois - Michel functions, E. Artal Bartolo constructed 3 variable polynomial functions in [3] so that they are not topologically equivalent but the Seifert forms corresponding to their zero-sets are isomorphic.

On Seifert forms, known is a kind of Thom-Sebastiali type's result.

THEOREM 5.4. (A. G. Gabrielov [15], K. Sakamoto [31]) *Let* $g : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *and* $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be analytic function germs with isolated singularities. Define* $f : (\mathbb{C}^{m+n}, 0) \to (\mathbb{C}, 0)$ *by* $f(x,y) = g(x) + h(y)$. *Then* $\Gamma(f)$ *is isomorphic to* $(-1)^{mn}\Gamma(g) \otimes \Gamma(h)$.

Suppose that there are two analytic functions $f,\ g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with isolated singularities such that $A(f) \neq A(g)$ but $\Gamma(f)$ and $\Gamma(g)$ are isomorphic. Note that f and g are not topologically equivalent in this case (see Theorem 3.1). Let m be a positive integer such that "$m \in A(f)$ but $m \notin A(g)$" or "$m \in A(g)$ but $m \notin A(f)$". Define analytic functions $F,\ G : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ by

$$F(x,y,z,w) = f(x,y) + z^{m+1} + w^{m+1}, \quad G(x,y,z,w) = g(x,y) + z^{m+1} + w^{m+1}.$$

Then, by construction, $A(F) \neq A(G)$. On the other hand, it follows from Theorem 5.4 that $\Gamma(F)$ is isomorphic to $\Gamma(G)$. By Theorem 5.2, we see that F and G are topologically equivalent as function germs.

The above argument gives rise to the following question naturally:

QUESTION 3. Are there two analytic functions $f,\ g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with isolated singularities such that $A(f) \neq A(g)$ but $\Gamma(f)$ and $\Gamma(g)$ are isomorphic?

In this respect, we analyse the Du Bois - Michel functions $f_{a,b}$ and $f_{b-8,a+8}$ for $b \geq 11$ and $b \neq a+8$, mentioned in Theorem 5.1. The resolution tree of $f_{b-8,a+8}^{-1}(0)$ is obtained from that of $f_{a,b}^{-1}(0)$, by exchanging some end parts of two branches of the tree. We note that the multiplicities of the exceptional divisors which support the end parts are the same. Therefore it follows from Theorem 2.2 that $A(f_{a,b}) = A(f_{b-8,a+8})$ unfortunately. Taking this fact into consideration, we may ask also the following question:

Question 4. Let $f,\ g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be analytic functions with isolated singularities. Suppose that the Seifert forms corresponding to their zero-sets are isomorphic. Then $A(f) = A(g)$?

6. Real blow-analiticity and complex topological triviality

Let $I = [a, b]$ be a closed interval of $\mathbb{R}$, and let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a germ of a real analytic function at $\{0\} \times I$. Therefore we think of F as a real analytic function defined over a small neighbourhood of $\{0\} \times I$ in $\mathbb{R}^n \times \mathbb{R}$. Suppose that $F(0;t) = 0$. Let $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in J$, be the analytic function defined by $f_t(x) = F(x;t)$. Here J is a small open set in $\mathbb{R}$ containing I.

Let $F_{\mathbb{C}} : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ be the complexification of F. Therefore $F_{\mathbb{C}}$ is a complex analytic function defined over a small neighbourhood of $\{0\} \times I$ in $\mathbb{C}^n \times \mathbb{C}$. Let $f_{t,\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $t \in \tilde{I}$, be the complex analytic function defined by $f_{t,\mathbb{C}}(x) = F_{\mathbb{C}}(x;t)$. Here $\tilde{I}$ is a small open set in $\mathbb{C}$ containing J. Therefore $f_{t,\mathbb{C}}$ is the complexification of f_t for $t \in J$.

As mentioned in the Introduction, there is a similarity between real blow-analytic properties and complex topological ones. Therefore we first ask

Question 5. Let I be a closed interval, and let $\{f_t\}_{t \in I}$ be a family of analytic function germs with algebraically isolated singularities. If $\{f_t\}$ is blow-analytically trivial over I, then is $\{f_{t,\mathbb{C}}\}$ topologically trivial over $\tilde{I}$?

We can easily construct a negative example to this question as follows:

Example 6.1. Let $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $t \in I$, be a family of real polynomial functions with algebraically isolated singularities defined by

$$f_t(x,y) = (x^2 + y^2)^2 + tx^5 + x^7,$$

where I is a closed interval containing $0, 1 \in \mathbb{R}$. Then it follows from the main theorem in [24] that $\{f_t\}$ is blow-analytically trivial over I. On the other hand, it is easy to see that $f_{0,\mathbb{C}}$ and $f_{1,\mathbb{C}}$ are not topologically equivalent.

We next ask the following opposite question:

Question 6. If $\{f_{t,\mathbb{C}}\}$ is topologically trivial over $\tilde{I}$, then is $\{f_t\}$ blow-analytically trivial over I?

Concerning this question, we have an affirmative result in case $n = 2$ without the assumption of algebraically isolated singularities. More precisely, we have

THEOREM 6.2. *Let I be a closed interval, and let $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ with $F(0;t) = 0$ be a germ of a real analytic function at $\{0\} \times I$ in the above sense. Suppose that $\{f_{t,\mathbb{C}}\}_{t\in\tilde{I}}$ is topologically trivial over $\tilde{I}$. Then $\{f_t\}$ is blow-analytically trivial over I.*

Proof. By assumption, $f_{t,\mathbb{C}} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, $t \in \tilde{I}$, is topologically trivial over $\tilde{I}$. Let

$$F_C(x,y,t) = G_1(x,y,t)^{\ell_1} \cdots G_q(x,y,t)^{\ell_q}$$

be the decomposition of $F_\mathbb{C}$ to irreducible components, and let

$$G(x,y,t) = G_1(x,y,t) \cdots G_q(x,y,t).$$

Then $(\mathbb{C}^2 \times \tilde{I}, G^{-1}(0))$ is topologically trivial over $\tilde{I}$. For $t \in \tilde{I}$, let $g_t(x,y) = G(x,y,t)$. Thanks to the triviality above, we may assume that for each $t \in \tilde{I}$, g_t is reduced and has an isolated singularity at $0 \in \mathbb{C}^2$. Let

$$g_t(x,y) = g_{t,1}(x,y) \cdots g_{t,m}(x,y)$$

be the decomposition of each g_t to irreducible components. Note that m is independent of t.

In the case where $m = 1$ and $0 \in \mathbb{C}^2$ is a regular point of g_t for some $t \in \tilde{I}$, $0 \in \mathbb{C}^2$ is a regular point of g_t for any $t \in \tilde{I}$. Therefore we can regard $V = G^{-1}(0) \subset \mathbb{C}^2 \times \tilde{I}$ as a desingularised variety through the identity map. Otherwise, by B. Teissier [31], there is a simplification of $(\mathbb{C}^2 \times \tilde{I}, G^{-1}(0))$, $\Pi : M \to \mathbb{C}^2 \times \tilde{I}$, whose restriction to V' gives a strong simultaneous resolution of V, where V' is the strict transform of V by Π. Let $\lambda : \mathbb{C}^2 \times \tilde{I} \to \tilde{I}$ be the canonical projection.

Let us express Π as follows:

$$\Pi = \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_r : M = M_r \to M_{r-1} \to \cdots \to M_2 \to M_1 \to M_0 = \mathbb{C}^2 \times \tilde{I},$$

where each $\Pi_i : M_i \to M_{i-1}$ is a blow up with smooth centre $C_i \subset M_{i-1}$, $1 \le i \le r$. We denote by E_i the exceptional set of $\beta_i = \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_i$, by V_i the strict transform of V by β_i, and by ΣV_{i-1} the singular set of V_{i-1}, $1 \le i \le r$, where $V_0 = V$. Let D_i be the exceptional divisor created by a blow up Π_i with centre C_i, $1 \le i \le r$. In these notations $\Pi = \beta_r$, $V' = V_r$ and $E_i = \bigcup_{1 \le j \le i} D_j$, $1 \le i \le r$. Let $\mathcal{E} = E_r$, and let β_0 be the identity map. By the construction of Hironaka's desingularisation, $C_i \subset E_{i-1} \cup \Sigma V_{i-1}$, $1 \le i \le r$. It follows that $C_1 = \Sigma V = \{0\} \times \tilde{I}$ and $C_i \subset E_{i-1}$, $2 \le i \le r$. Let $\breve{\Pi} = \Pi|_{V'} : V' \to V$. Then $\breve{\Pi}^{-1}(\Sigma V) = \mathcal{E} \cap V'$, denoted by $\breve{\mathcal{E}}$. Since $\breve{\Pi}$ gives a strong simultaneous resolution of V, $\lambda \circ \breve{\Pi}|_{\breve{\mathcal{E}}} : \breve{\mathcal{E}} \to \tilde{I}$ is simple i.e. a locally trivial deformation. Using this property, we can show the following:

Assertion 6.3. *For $1 \leq i \leq r$, C_i is one-dimensional, and each C_{i+1} is contained in $V_i \cap E_i$, $1 \leq i \leq r-1$. In addition, C_i, $1 \leq i \leq r$, is not contained even locally in $(\lambda \circ \beta_{i-1})^{-1}(t)$ for any $t \in \tilde{I}$.*

Proof. $C_1 = \{0\} \times \tilde{I}$ is 1-dimensional, and C_i, $2 \leq i \leq r$, is zero-dimensional or one-dimensional. Assume that the centre C_i is zero-dimensional for some i with $2 \leq i \leq r$. Let C_i be locally $P_0 \in E_{i-1} \cup \Sigma V_{i-1}$. Then $E_i \cap V_i$ is one-dimensional, and it contains a one-dimensional subset mapped to P_o by Π_i. Let $S_i = (\Pi_i \circ \cdots \circ \Pi_r)^{-1}(P_0) \cap V'$. Then S_i is not empty, in fact, it is one-dimensional. This contradicts the local triviality of $\lambda \circ \breve{\Pi}|_{\breve{\mathcal{E}}}$. Therefore it follows that C_i is one-dimensional for $1 \leq i \leq r$.

Note that $C_{i+1} \cap V_i$ is not empty at each stage. Assume that C_{i+1} is not contained in $V_i \cap E_i$ for some i, $1 \leq i \leq r-1$. Since C_{i+1} is one-dimensional, $C_{i+1} \cap V_i$ is locally one point. Let P_0 be such a point. Then $V_{i+1} \cap D_{i+1}$ is one-dimensional, and it contains a one-dimensional subset in $\Pi_{i+1}^{-1}(P_0)$ since $C_{i+1} \cap V_i = \emptyset$ in a punctured neighbourhood of P_0. Similarly to the above, this contradicts the local triviality of $\lambda \circ \breve{\Pi}|_{\breve{\mathcal{E}}}$. Therefore we have $C_{i+1} \subset E_i \cap V_i$.

We next assume that there is i, $2 \leq i \leq r$, such that C_i is locally contained in $(\lambda \circ \beta_{i-1})^{-1}(t_0)$ for some $t_0 \in \tilde{I}$. As seen above, $C_i \subset E_{i-1} \cap V_{i-1}$. Therefore $E_{i-1} \cap V_{i-1} \cap (\lambda \circ \beta_{i-1})^{-1}(t_0)$ is one-dimensional. If $E_{i-1} \cap V' \cap (\lambda \circ \Pi)^{-1}(t_0)$ is one-dimensional, it contradicts the local triviality of $\lambda \circ \breve{\Pi}|_{\breve{\mathcal{E}}}$ also in this case. If $E_{i-1} \cap V' \cap (\lambda \circ \Pi)^{-1}(t_0) = \emptyset$ or zero-dimensional, then there is, by construction, E_{j-1}, $i < j \leq r$, such that $E_{j-1} \cap V' \cap (\lambda \circ \Pi)^{-1}(t_0)$ is one-dimesional. Repeating this argument, we see that $\mathcal{E} \cap V' \cap (\lambda \circ \Pi)^{-1}(t_0)$ is one-dimensional which is a contradiction. Therefore C_i, $1 \leq i \leq r$, is not locally contained in $(\lambda \circ \beta_{i-1})^{-1}(t)$ for any $t \in \tilde{I}$. ∎

The map $\lambda : C_1 = \{0\} \times \tilde{I} \to \tilde{I}$ is the canonical projection. Therefore it is submersive. In general, we have the following:

Assertion 6.4. *For $1 \leq i \leq r$, the map $\lambda \circ \beta_{i-1} : C_i \to \tilde{I}$ is submersive. It follows from the construction that $\lambda \circ \Pi|_{D_i} : D_i \to \tilde{I}$ is submersive for $1 \leq i \leq r$.*

Proof. Assume that $\lambda \circ \beta_{i-1} : C_i \to \tilde{I}$ is not submersive at $P_0 \in C_i$. Then $D_i \cap V_i$ has a one-dimensional subset which is mapped by Π_i onto a neighbourhood of P_0 in C_i. If $D_i \cap V'$ has a one-dimensional subset mapped by $\Pi_i \circ \cdots \circ \Pi_r$ onto a neighbourhood of P_0 in C_i, it will contradict the local triviality of $\lambda \circ \breve{\Pi}|_{\breve{\mathcal{E}}}$. If not, there is $j > i$ such that $D_j \cap V'$ has a similar one-dimensional subset to the above, which is also a contradiction. Therefore the submersiveness follows. ∎

ASSERTION 6.5. *If $D_i \cap D_j$, $i \neq j$, (resp, $D_i \cap V'$) is not empty, then $\lambda \circ \Pi|_{D_i \cap D_j} : D_i \cap D_j \to \tilde{I}$ (resp. $\lambda \circ \Pi|_{D_i \cap V'} : D_i \cap V' \to \tilde{I}$) is submersive. As a map germ at $\mathcal{E} \cap V'$, $\lambda \circ \Pi|_{V'} : V' \to \tilde{I}$ is submersive.*

In addition, at most two of $D_1, \cdots, D_r$ and V' intersect at any $\xi \in \mathcal{E}$.

Proof. The first statements follow from the construction of $C_{i+1} \subset E_i \cap V_i$, $0 \leq i \leq r-1$, and Assertion 6.4.

Assume that V', D_i and D_j, $i \neq j$, intersect at $\xi \in \mathcal{E}$. Then $D_i \cap V'$ and $D_j \cap V'$ intersect at $\xi \in \mathcal{E}$. This contradicts the local triviality of $\lambda \circ \breve{\Pi}|_{\breve{\mathcal{E}}}$. We next assume that D_i, D_j and D_k, $1 \leq i < j < k < r$, intersect at $\xi \in \mathcal{E}$ in M. Therefore ξ is not a point of V'. D_j is created by a blow up with centre $C_j \subset D_i$. If D_k is also created by a blow up with centre $C_k \subset D_i$, the two one-dimensional subsets of $\Sigma V_i \cap D_i$ intersect at $\xi \in C_j$. If so, we have to consider a one-point blow up at ξ not a blow up with centre C_j near ξ at the j-th stage. But our centre is one-dimensional. Therefore this case does not happen. If D_k is created by a blow up with centre $C_k \subset D_j$, ξ was a special point in C_j. In this case also we have to consider a one-point blow up at ξ near ξ at the j-th stage. Therefore the last statement follows. ∎

Let ξ be an arbitrary point of $\mathcal{E}$ such that $\lambda \circ \Pi(\xi) = t_\xi$. Then it follows from Assertions 6.4 and 6.5 that there is a local coordinate system, centred at ξ, (X, Y, T) such that

$$G \circ \Pi(X, Y, T) = U(X, Y, T) X^a Y^b$$

where $T = t - t_\xi$ (more precisely, this means that $\lambda \circ \Pi(X, Y, T) = t - t_\xi$) and $U(X, Y, T)$ is a unit near ξ.

Let $F_{\mathbb{C}}$ be the complexification of $F(x, y, t) = f_t(x, y)$. Then Π gives also a simplification of $(\mathbb{C}^2 \times \tilde{I}, F_{\mathbb{C}}^{-1}(0))$, and near ξ,

$$F_{\mathbb{C}} \circ \Pi(X, Y, T) = V(X, Y, T) X^c Y^d$$

where $V(X, Y, T)$ is a unit near ξ.

By Hironaka's construction, Π can be chosen invariant under complex conjugation. Let π be the restriction of Π to the real part of M, and let E be the exceptional set of π. Then for any $\xi \in E$, there is a local coordinate sysytem, centred at ξ, (X, Y, T) such that

$$F \circ \pi(X, Y, T) = W(X, Y, T) X^u Y^v$$

where $T = t - t_\xi$ and $W(X, Y, T)$ is a unit near ξ. Then the blow-analytic triviality of $\{f_t\}$ follows from a similar argument to the proof of Theorem 1 in [25] using Cartan Theorem B.

This completes the proof of Theorem 6.2. ∎

Taking the above result and the observations seen in subsection 4.3 into consideration, we may pose the following question:

QUESTION 7. Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $t \in I$, be a μ-constant family of complex analytic function germs with isolated singularities. Here I is an open disk in $\mathbb{C}$. Then is the Fukui numerical set $A(f_t)$ constant over I?

REMARK 6.6. We have an affirmative answer to the above question in case $n = 2$. Let $F : (\mathbb{C}^2 \times I, \{0\} \times I) \to (\mathbb{C}, 0)$ be a function germ defined by $F(x, y; t) = f_t(x, y)$. In the two variable case μ-constancy is equivalent to μ^*-constancy. By B. Teissier [32], the latter condition implies the Whitney regularity of the pair $(F^{-1}(0) \setminus \{0\} \times I, \{0\} \times I)$. In addition, by [29] or J. Briançon, Ph. Maisonobe and M. Merle [5], the Whitney regularity implies the Thom (a_F)-regularity of the stratification $\{\mathbb{C}^2 \times I \setminus F^{-1}(0), F^{-1}(0) \setminus \{0\} \times I, \{0\} \times I\}$ of $\mathbb{C}^2 \times I$. Then we can show that the family $\{f_t\}_{t \in I}$ is topologically trivial over I, using Thom's 2nd Isotopy Lemma. Therefore it follows from Theorem 3.1 that $A(f_t)$ is constant over I.

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Satoshi Koike:
DEPARTMENT OF MATHEMATICS
HYOGO UNIVERSITY OF TEACHER EDUCATION
942-1 SHIMOKUME, KATO
HYOGO 673-1494, JAPAN
E-mail: koike@hyogo-u.ac.jp
Adam Parusiński:
LABORATOIRE J. A. DIEUDONNÉ U.M.R. C.N.R.S. NO. 6621
UNIVERSITÉ DE NICE SOPHIA-ANTIPOLIS
28, Parc Valrose
06108 NICE CEDEX 02, FRANCE
Email: Adam.PARUSINSKI@unice.fr

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