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A COMPLETE VARIETY WITH INFINITELY MANY MAXIMAL QUASI-PROJECTIVE OPEN SUBSETS

Abstract. Let K be an algebraically closed field. For every $n \geq 2$ we find an n-dimensional complete variety X_n over K, which has infinitely many maximal quasi-projective open subsets.

1. Main result

Let K be an algebraically closed field and let X be a complete algebraic variety over K. It is well known that if dim $X > 1$, then X need not be a projective variety. We have obvious necessary condition for X to be projective:

CHEVALLEY CONDITION. *For every finite set $S \subset X$, there is an affine open subset $U \subset X$ such that $S \subset U$.*

Kleiman [3] proved that if X is a smooth complete variety, then Chevalley Condition implies projectivity of X. We can change Chevalley Condition in an equivalent way, assuming that every finite set of points of X is contained in some open quasi-projective subset of X. Indeed we have more or less obvious:

PROPOSITION 1.1. *Let Y be a quasi-projective variety. Every finite set of points of Y is contained in some affine open subset of Y.*

Proof. We can assume that $Y \subset \mathbb{P}^N$. Taking a general hyperplane section we can easily reduce the general case to the case when Y is an open subset of an affine variety X. Let $S = \{y_1, \ldots, y_m\}$ be a finite subset of Y. Take

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$Z = X \backslash Y$. There is a polynomial function f such that $f_i|_Z = 0$ and $f_i(y_i) = 1$ for all $i = 1, \dots, m$. Now $S \subset X_f \subset Y$. ∎

If X is not projective then we can look for open quasi-projective subsets in X. We have:

DEFINITION 1.2. ([5]) An open subset $U \subset X$ is called a maximal quasi-projective open subset (MQOS) if U is quasi-projective and it is a maximal open subset of X with this property.

Since algebraic varieties are noetherian spaces it is easy to see that every point $x \in X$ is contained in some maximal quasi-projective open subset of X. More generally we have:

PROPOSITION 1.3. *Every open affine subset $U \subset X$ is contained in some MQOS of X.*

Włodarczyk [5] generalized Kleimans theorem and proved that any smooth complete variety contains only a finite number of MQOS (in fact his result is a little-bit more general). In fact we have:

PROPOSITION 1.4. ([5]) *Let X be a variety. If X contains only a finite number of MQOS and if it satisfies the Chevalley Condition, then it is quasi-projective.*

Proof. We use Włodarczyk's trick. Suppose X is not quasi-projective. Let $\{U_1, \dots, U_m\}$ be the family of all maximal quasi-projective subsets of X. For every $i = 1, \dots, m$ take $x_i \notin U_i$. By Chevalley Condition all points $x_1, \dots, x_m$ are in some open affine subset U. By Proposition 1.3 there is a MQOS U_i such that $U \subset U_i$. This leads to the contradiction. ∎

It is interesting, whether results of Kleiman and Włodarczyk can be extended to a non-smooth case. Let us recall the following result of Zariski:

THEOREM 1.5. ([6]) *Let X be a complete normal surface. Assume that all singular points of X are contained in some open quasi-projective subset of X. Then X is projective.*

In fact Theorem of Zariski can be easily generalized to a non-complete version:

COROLLARY 1.6. *Let X be a normal surface. Assume that all singular points of X are contained in some open quasi-projective subset of X. Then X is quasi-projective.*

Proof. By Nagata we can embed X in a complete surface $\overline{X}$. We can assume that this surface is also normal. Resolve all singularities of $\overline{X}$ which are in $\overline{X} \setminus X$. Then we obtain a new surface $\overline{X}'$ which contains X. All singular points of this new surface are contained in a quasi-projective open subset.

Hence by Theorem 1.5 the surface $\overline{X}'$ is projective. Consequently X as an open subset of $\overline{X}'$ is quasi-projective. ∎

COROLLARY 1.7. *A normal surface X is quasi-projective if and only if it satisfies the Chevalley Condition.*

Let us note that also the following generalization of Corollary 1.7 is true:

THEOREM 1.8. *Let X be a normal surface. Then X has only finitely many MQOS. In fact the number of MQOS is bounded by 2^r, where $r = \#Sing\ X$.*

Proof. Let $U \subset X$ be a MQOS. Let $Sing(X) = \{x_1, \dots, x_k, y_1, \dots, y_m\}$ where all x_i belong to U and all y_i are not in U. Resolve all singularities $\{y_1, \dots, y_m\}$. Then we obtain a new surface X' and a morphism $\pi : X' \to X$. All singular points of this new surface are in a quasi-projective open subset. Hence by Corollary 1.6 the surface X' is projective. Consequently the set $X \setminus \{y_1, \dots, y_m\} = X' \setminus \pi^{-1}(\{y_1, \dots, y_m\})$ is quasi-projective. But $U \subset X \setminus \{y_1, \dots, y_m\}$ and by the maximality of U we have $U = X \setminus \{y_1, \dots, y_m\}$. Consequently we see that the number of MQOS in X is bounded by the number of subsets of $Sing(X)$. ∎

Hence the results of Kleiman and Włodarczyk hold for normal surfaces. However, we show that these results cannot be extended to the case of arbitrary surface.

We start with the non-projective surface X. If char $K \neq 2$ let C denote the nodal curve in $\mathbb{P}^2$ given by the equation $y^2z - x^3 - x^2z = 0$. If char $K = 2$ let C be given by equation $y^2z + x^3 + x^2z + xyz = 0$. If $P_0 = (0:0:1)$ is the singular point, then $C \setminus P_0$ is isomorphic to the multiplicative group $G_m = (K^*, 1, \cdot)$. For each $a \in K^*$ consider the translation of G_m given by $t \to ta$. This induces an automorphism of C which we denote by ϕ_a.

Now consider $C \times \mathbb{P}^1 \setminus \{0\}$ and $C \times \mathbb{P}^1 \setminus \{\infty\}$. We glue their open subsets $C \times \mathbb{P}^1 \setminus \{0, \infty\}$ by the isomorphism $\phi : (P, u) \to (\phi_u(P), u)$ for $P \in C$ and $u \in G_m = \mathbb{P}^1 \setminus \{0, \infty\}$. Thus we obtain a non-projective complete surface X (see [2], Ex.7.13). This surface is smooth away from the curve $Z \cong \mathbb{P}^1$ given locally as $P_0 \times K$.

THEOREM 1.9. *The surface X contains infinitely many MQOS.*

Proof. Since the surface X is non-projective it is enough to prove that it satisfies the Chevalley Condition (see Proposition 1.4). Let $f : X' \to X$ be the normalization of X. It is easy to see that X' can be covered by two smooth subsets isomorphic to $\mathbb{P}^1 \times K$, hence it is a smooth surface. By Theorem 1.5 this implies that X' is a projective surface. In fact it is a ruled surface with projection $\pi : X' \to \mathbb{P}^1$. In particular X' is a projective vector bundle $\mathbb{P}(\mathbf{E})$ associated to some vector bundle $\mathbf{E}$ on $\mathbb{P}^1$.

Now let $S \subset X$ be a finite set. Take $S' = f^{-1}(S)$. Since f is a finite mapping we have that the set S' is finite. Take a point $a \in \mathbb{P}^1$ which does not belong to $\pi(S')$. Take $U = \mathbb{P}^1 \setminus \{a\} \cong K$. Hence $S' \subset \pi^{-1}(\mathbb{P}^1 \setminus \{a\}) = \pi^{-1}(U)$. Since every vector bundle over U is trivial, we have $\pi^{-1}(U) = U \times \mathbb{P}^1$. Now let $Z \subset X$ denote the singular curve of X. Then $Z \cong \mathbb{P}^1$ and we have two disjoint sections $\rho_i : Z \to X' = \mathbb{P}(\mathbf{E})$, $i = 1, 2$ induced by the normalization f. We can assume that over U these sections are simply $\rho_1 : U \ni u \to (u, (0:1))$ and $\rho_2 : U \ni u \to (u, (1:0))$. Now the general section $\rho_x : U \ni u \to (u, (x:1))$ is disjoint from the set S' (and from sections $\rho_1(U), \rho_2(U)$). Take $\Gamma = \pi^{-1}(a) \cup \overline{\rho_x(U)}$. We have $V' = X' \setminus \Gamma = \pi^{-1}(U) \setminus \rho_x(U) \cong K^2$, hence V' is an affine open subset. Note that $S' \subset V'$.

Moreover, $\Gamma = f^{-1}(f(\Gamma))$. Take $V = X \setminus f(\Gamma)$. Observe that $V' = f^{-1}(V)$, this implies that the mapping $f : V' \to V$ is finite. Since V' is affine, we have that the set V is affine, too (see [1], 1.5, p. 63). However $S \subset V$, hence X satisfies the Chevalley Condition. ∎

Corollary 1.10. *For every $n \geq 2$ there exists an n-dimensional complete variety X_n, which has infinitely many maximal quasi-projective open subsets.*

Proof. Take $X_n = X \times \mathbb{P}^{n-2}$, where X is the non-projective surface defined above. Since X_n contains closed non-projective subvarieties of type $X \times \{a\}$, it is non-projective as well. Hence, it is enough to show that X_n satisfies the Chevalley Condition. Let $S \subset X_n$ be a finite set of points. Consider the projection $\pi : X_n \to X$ and put $S' = \pi(S)$. By the above consideration there exists an affine open set U in X such that $S' \subset U$. Now $S \subset U \times \mathbb{P}^{n-2}$ and we conclude by Proposition 1.1. ∎

Remark 1.11. Let us note that we also give an example of a smooth projective variety X' and a complete variety X, such that there is a finite surjective mapping $\pi : X' \to X$ and the variety X is not projective. On the other hand it is well-known that if we assume additionally that X is smooth, then the variety X has to be projective (see [1], 4.7).

We finish this note stating:

Conjecture. (Białynicki-Birula) *Every normal variety contains only finitely many MQOS.*

2. Appendix

For the convenience of the reader we show that the surface X is non-projective (we will follow hints from [2]). Recall that C denotes the nodal curve given in $\mathbb{P}^2$ by the equation $y^2z - x^3 - x^2z = 0$ (for char $K = 2$ by $y^2z + x^3 + x^2z + xyz = 0$). Consider the parametrization $\pi : \mathbb{P}^1 \ni (t:s) \to$

$(4ts(t-s) : 4ts(t+s) : (t-s)^3) \in \mathbb{P}^2$ (in case of char $K = 2$ consider $(t:s) \to (ts(t+s) : ts(at+bs) : (at+bs)^3)$, where a and b are the roots of $x^2+x+1=0$). For simplicity we will consider only the case char $K \neq 2$, we leave the (similar) case char $K=2$ to the reader as an exercise.

Except for the points $A = (0:1)$ and $B = (1:0)$ the morphism π is an isomorphism $C \setminus P_0 \cong K^*$. Let ϕ denote the automorphism $\mathbb{P}^1 \times K^* \ni ((t : s), u) \to ((ut : s), u) \in \mathbb{P}^1 \times K^*$. Since ϕ coincide on $A \times K^*$ and $B \times K^*$, it can be lifted to a $C \times K^*$ automorphism $\phi : (\pi((t:s)), u) \to (\pi((ut:s)), u)$.

Notice that X is obtained by gluing $C \times (\mathbb{P}^1 \setminus \{\infty\})$ with $C \times (\mathbb{P}^1 \setminus \{0\})$ along their open subsets $C \times K^*$ via the isomorphism ϕ. Both the projections $C \times (\mathbb{P}^1 \setminus \{\infty\}) \to \mathbb{P}^1 \setminus \{\infty\}$ and $C \times (\mathbb{P}^1 \setminus \{0\}) \to \mathbb{P}^1 \setminus \{0\}$ are proper morphisms, hence the projection $X \to \mathbb{P}^1$ is a proper morphism, thus X is a complete algebraic variety.

Let $Y = \mathbb{P}^1 \times (\mathbb{P}^1 \setminus \{0,\infty\})$, $Y_1 = \mathbb{P}^1 \times (\mathbb{P}^1 \setminus \{\infty\})$, $Z = C \times (\mathbb{P}^1 \setminus \{0,\infty\})$ and $Z_1 = C \times (\mathbb{P}^1 \setminus \{\infty\})$. The parametrization π naturally extends to $Y \to Z$ and $Y_1 \to Z_1$ which also will be denoted by π. Consider the exact sequence:

$$0 \longrightarrow \pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^* \longrightarrow \mathcal{K}^*/\mathcal{O}_Z^* \longrightarrow \mathcal{K}^*/\pi_*\mathcal{O}_Y^* \longrightarrow 0.$$

Using the global sections functor we obtain a commutative diagram.

$$\begin{array}{ccccccc}
 & & K[u,1/u]^* & & K[u,1/u]^* & & \\
 & & \| & & \| & & \\
0 & \longrightarrow & \Gamma(Z,\mathcal{O}_Z^*) & \longrightarrow & \Gamma(Z,\pi_*\mathcal{O}_Y^*) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(Z,\mathcal{K}^*) & \longrightarrow & \Gamma(Z,\mathcal{K}^*) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \Gamma(Z,\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*) & \longrightarrow & \Gamma(Z,\mathcal{K}^*/\mathcal{O}_Z^*) & \longrightarrow & \Gamma(Z,\mathcal{K}^*/\pi_*\mathcal{O}_Y^*) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma(Z,\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*) & \longrightarrow & \Gamma(Z,\mathcal{K}^*/\mathcal{O}_Z^*)/\Gamma(Z,\mathcal{K}^*) & \longrightarrow & \Gamma(Z,\mathcal{K}^*/\pi_*\mathcal{O}_Y^*)/\Gamma(Z,\mathcal{K}^*) & & \\
 & & \| & & \| & & \\
 & & \mathrm{Pic}(Z) & & \mathrm{Pic}(Y) & &
\end{array}$$

Using the "snake lemma" we obtain the following exact sequence:

$$0 \to K[u,1/u]^* \to K[u,1/u]^* \to \Gamma(Z,\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*) \to \mathrm{Pic}(Z) \to \mathrm{Pic}(Y).$$

Moreover, the morphism $K[u,1/u]^* \to K[u,1/u]^*$ is the identity, consequently the morphism $K[u,1/u]^* \to \Gamma(Z,\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*)$ is zero. Finally we have exact sequence

$$(2.1)\qquad 0 \longrightarrow \Gamma(Z, \pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*) \longrightarrow \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(Y).$$

Consider the variety Y. Let $D = Y_1 \setminus Y = \mathbb{P}^1 \times \{0\}$. We have an exact sequence:

$$\mathbb{Z}D \to \mathrm{Pic}(Y_1) \to \mathrm{Pic}(Y) \to 0.$$

Since D is a principal divisor we have $\mathrm{Pic}(Y) = \mathrm{Pic}(Y_1) = \mathrm{Pic}(\mathbb{P}^1 \times K) = \mathrm{Pic}(\mathbb{P}^1)$. In fact if $\pi : Y \to \mathbb{P}^1$ is the projection, then every element of $\mathrm{Pic}(Y)$ is of the form $\pi^*(\alpha)$, where $\alpha \in \mathrm{Pic}(\mathbb{P}^1)$ and the mapping $\mathrm{Deg} : \mathrm{Pic}(Y) \ni \pi^*\alpha \to \deg \alpha \in \mathbb{Z}$ is an isomorphism. Now we will study the group $\Gamma(Z, \pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*)$ in order to describe $\mathrm{Pic}(Z)$ and, eventually, the divisors on X.

Let us start by identifying the stalks of $\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*$ at an arbitrary point $\mathbf{p} \in Z$. For $\mathbf{p} \in Z \setminus \{(0:0:1)\} \times K^*$ the morphism π is an isomorphism, hence the quotient $(\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*)_\mathbf{p}$ is in fact $\{1\}$. Let $\mathbf{p} = ((0:0:1), p)$, we can describe the stalk $(\pi_*\mathcal{O}_Y^*)_\mathbf{p}$ as $(\mathcal{O}_Y^*)_{((0:1),p))} \bigcap (\mathcal{O}_Y^*)_{((1:0),p))} = \{\gamma = \frac{f}{g} : f, g \in K(u)[t,s], \deg_{t,s} f = \deg_{t,s} g$, the coefficients of f and g are regular at p and the leading terms in t and s do not vanish at $p\}$ and $(\mathcal{O}_Z^*)_\mathbf{p} = \{\gamma \in (\pi_*\mathcal{O}_Y^*)_\mathbf{p} : \gamma((0:1),u) = \gamma((1:0),u)\}$. Take $\gamma \in (\pi_*\mathcal{O}_Y^*)_\mathbf{p}$. We have $\gamma = \frac{\alpha_k t^k + \alpha_{k-1}t^{k-1}s + \ldots + \alpha_0 s^k}{\beta_k t^k + \beta_{k-1}t^{k-1}s + \ldots + \beta_0 s^k}$, where $\alpha_i, \beta_i \in K(u), \alpha_i(p), \beta_i(p) \in K, \alpha_k(p), \beta_k(p) \neq 0, \alpha_0(p), \beta_0(p) \neq 0$. Consider $\delta = \frac{c_1 t + c_0 s}{t-s}$, where $c_0 = -\alpha_0/\beta_0$ and $c_1 = \alpha_k/\beta_k$. Since $\gamma((0:1),u) = \delta((0:1),u)$ and $\gamma((1:0),u) = \delta((1:0),u)$ we have that every element of $(\pi_*\mathcal{O}_Y^*)_\mathbf{p}/(\mathcal{O}_Z^*)_\mathbf{p}$ is equivalent to $\frac{t-\alpha s}{t-s}$ for some $\alpha \in K(u)$ regular and non zero at p. Moreover, since $\frac{t-\alpha s}{t-s} \cdot \frac{t-\beta s}{t-s}$ and $\frac{t-\alpha\beta s}{t-s}$ coincide at points $((0:1),u)$ and $((0:1),u)$ we have $\frac{t-\alpha s}{t-s} \cdot \frac{t-\beta s}{t-s} = \frac{t-\alpha\beta s}{t-s}$ in $(\pi_*\mathcal{O}_Y^*)_\mathbf{p}/(\mathcal{O}_Z^*)_\mathbf{p}$.

Note that $\frac{t-\alpha s}{t-\beta s} \neq 1$ in $(\pi_*\mathcal{O}_Y^*)_\mathbf{p}/(\mathcal{O}_Z^*)_\mathbf{p}$ for $\alpha \neq \beta$. Indeed, if $\frac{t-\alpha s}{t-\beta s} \in (\mathcal{O}_Z^*)_\mathbf{p}$ then $\frac{t-\alpha s}{t-\beta s}((0:1),u) = \frac{t-\alpha s}{t-\beta s}((1:0),u)$ and consequently $\alpha(u)/\beta(u) = 1$. Hence $\alpha = \beta$.

Since we know the stalks of $\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*$, we are in a position to identify the global sections of this sheaf. Recall that a global section can be identified with a set $\{(U_i, f_i)\}$, where $\bigcup U_i = Z$, $f_i \in \Gamma(U_i, \pi_*\mathcal{O}_Y^*)$ and $\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_Z^*)$. Outside the line $\{(0:0:1)\} \times K^*$ all sections are equivalent to 1. Let (U_1, f_1) and (U_2, f_2) be such that $\mathbf{p} \in U_1 \cap U_2 \cap \{(0:0:1)\} \times K^*$. Let $f_1 = \frac{t-\alpha s}{t-s}$ and $f_2 = \frac{t-\beta s}{t-s}$. Then $\frac{f_1}{f_2} = \frac{t-\alpha s}{t-\beta s} \in \Gamma(U \cap V, \mathcal{O}_Z^*) \subset (\mathcal{O}_Z^*)_\mathbf{p}$ and thus $\alpha = \beta$ and $f_1 = f_2$. Therefore a global section of $\pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*$ can be expressed as $\{(U,1),(V,f)\}$, where $U = Z \setminus \{(0:0:1)\} \times K^*$, V is an open set containing $\{\{(0:0:1)\} \times K^*\}$ and $f = \frac{t-\alpha s}{t-s}$ with $\alpha \in \Gamma(K^*, \mathcal{O}_{K^*}^*) =$

$\{au^k : a \in K^*, k \in \mathbb{Z}\}$. We have $\frac{t-\alpha_1 u^{k_1}s}{t-s} \cdot \frac{t-\alpha_2 u^{k_2}s}{t-s} = \frac{t-\alpha_1\alpha_2 u^{k_1+k_2}s}{t-s}$ thus we have an isomorphism of groups $\Gamma(Z, \pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*) \cong K^* \times \mathbb{Z}$.

Recall that $\Gamma(Z, \pi_*\mathcal{O}_Y^*/\mathcal{O}_Z^*)$ is a subgroup of $\mathrm{Pic}(Z)$. A divisor $\{(U,1), (V, \frac{t-\alpha s}{t-s})\}$ is equivalent to $\{(U, \frac{t-s}{2(t-\alpha s)}), (V,1)\} = \{(U, \frac{4ts(t-s)}{8ts(t-\alpha s)}), (V,1)\} = \{(U, \frac{x}{(1+\alpha)x+(1-\alpha)y}), (V,1)\}$.

Consider the divisor $D = \{(U, \frac{x}{x-y}), (V,1)\}$. Its image in $\mathrm{Pic}(Y)$ is $D' = \{(K^* \times K^*, \frac{t-s}{-2s}), ((\mathbb{P}^1 \setminus \{(1:1)\}) \times K^*, 1)\}$. Since $\mathrm{Deg}\, D' = 1$, it is a generator of $\mathrm{Pic}(Y)$. Consequently, the last mapping in the exact sequence (2.1) is onto and we have an isomorphism $\mathrm{Pic}(Z) \ni \{(U, \frac{x}{(1+a)x+(1-a)y} \cdot (\frac{x}{(1+u)x+(1-u)y})^k \cdot (\frac{x}{x-y})^n), (V,1)\} \to (a,k,n) \in K^* \times \mathbb{Z} \times \mathbb{Z}$.

Note, that we can proceed in a similar way with Z_1. We have $\Gamma(Z_1, \pi_*\mathcal{O}_{Y_1}^*/\mathcal{O}_{Z_1}^*) = \{(U,1), (V, \frac{t-\alpha s}{t-s})\}$, where $\alpha \in \Gamma(K, \mathcal{O}_K^*) = K^*$. Thus $\mathrm{Pic}(Z_1) \cong K^* \times \{0\} \times \mathbb{Z}$.

Let us now examine the action of ϕ^* on $\mathrm{Pic}(Z)$. We have $\phi^*(\frac{x}{(1+\alpha)x+(1-\alpha)y}) = \phi^*(\frac{t-s}{t-\alpha s}) = \frac{ut-s}{ut-\alpha s} = \frac{t-s}{t-\alpha s} = \frac{x}{(1+\alpha)x+(1-\alpha)y}$ and $\phi^*(\frac{x}{x-y}) = \phi^*(\frac{t-s}{-2s}) = \frac{tu-s}{-2s} = \frac{t-s}{-2s} \cdot \frac{t-s}{t-u^{-1}s} = \frac{x}{x-y} \cdot \frac{x}{(1+u^{-1})x+(1-u^{-1})y}$. Identifying $\mathrm{Pic}(Z)$ with $K^* \times \mathbb{Z} \times \mathbb{Z}$ we can say that the morphism ϕ^* has the form $(a,k,n) \to (a, k-n, n)$.

Take $D \in \mathrm{Pic}(X)$. Using the identification $\mathrm{Pic}(Z_1) \cong K^* \times \{0\} \times \mathbb{Z}$ we can say that $D|_{C\times(\mathbb{P}^1\setminus\{\infty\})}$ and $D|_{C\times(\mathbb{P}^1\setminus\{0\})}$ represent elements $(a,0,n)$ and $(b,0,k)$ respectively. Both expressions must coincide on $C \times (\mathbb{P}^1 \setminus \{0,\infty\})$ via the morphism ϕ^*, meaning $(a,0,n) = \phi^*((b,0,k)) = (b,-k,k)$, thus $a = b$ and $n = k = 0$. Therefore $D|_{C\times(\mathbb{P}^1\setminus\{\infty\})}$ has to be of the form $\{(U, \frac{x}{(1+\alpha)x+(1-\alpha)y}), (V,1)\}$. Note that the divisor $D|_{C\times\{0\}}$ is of the form $A - B$ ($A, B \in \mathit{Reg}(C \times \{0\})$), hence it has degree 0.

Now assume that X is projective. Let $D' \in \mathrm{Pic}(X)$ be a general hyperplane section. The divisor $D'|_{C\times\{0\}}$ is effective and nonzero, thus it has a positive degree. This is a contradiction.

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