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EFFECTIVE REGULARITY CRITERIA FOR ANALYTIC MAPPINGS

Abstract. We survey recent developments in the study of singularities of complex-analytic mappings from a local algebraic viewpoint. We present several effective criteria for various modes of regularity of complex-analytic mappings in terms of vertical components in fibred powers of the mappings.

Contents

1. Introduction

The purpose of this paper is to provide a systematic exposition of a certain new method of analysis of the local geometry of complex-analytic mappings. We study the relationship between degeneracies in the family of fibres of an analytic mapping and the existence of the so-called *vertical components* in fibred powers of the mapping. There are, in fact, two natural notions of a vertical component (*algebraic* and *geometric*) and we are interested also in the relationship between them.

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We gathered here a collection of criteria for various modes of regularity of analytic mappings, like Gabrielov regularity, openness, or flatness (see below). The main advantage of these results is that they reduce the continuous complexity of the family of fibres of a mapping to the discrete data that governs the presence (or lack thereof) of vertical components in a finite number of fibred powers of the mapping. In fact, in the algebraic setup, our criteria allow for a computer algorithm verification of openness and flatness, via Gröbner bases (see Section 8).

The study of vertical components began only in the late 90's with the works of M. Kwieciński [17] and P. Tworzewski [18], although the origins of this work are in M. Auslander's criterion for freeness (later generalized by Lichtenbaum [19] to the following statement).

Theorem 1.1. (cf. [5, Thm. 3.2]) *Let R be a regular local ring of dimension $n > 0$, and let F be a finite R-module. Then F is R-free if and only if the n-fold tensor power $F^{\otimes^n_R}$ is a torsion-free R-module.*

They were followed by the first effective flatness criterion of A. Galligo and M. Kwieciński [9] (a special case of our Theorem 1.8 below), and numerous papers by the author (see, e.g., [1] and [3]). The next conceptual advance was only recently obtained by the author together with E. Bierstone and P. D. Milman [4], with the discovery of a certain phenomenon in local algebra, which we discuss in detail in Section 7. Most of the results presented in this survey are not original. The main regularity criteria (below) appeared in [1], [3] and [4].

1.1. Vertical components in fibred powers

Definition 1.2. Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. An irreducible (isolated or embedded) component W_ξ of X_ξ is called *algebraic vertical* if there exists a nonzero element $a \in \mathcal{O}_{Y,\eta}$ such that the pull-back $\varphi_\xi^* a$ belongs to the associated prime $\mathfrak{p}$ in $\mathcal{O}_{X,\xi}$ corresponding to W_ξ. Equivalently, W_ξ is algebraic vertical if any sufficiently small representative of W_ξ is mapped into a proper analytic subset of a neighbourhood of η in Y. We say that W_ξ is *geometric vertical* if any sufficiently small representative of W_ξ is mapped into a nowhere dense subset of a neighbourhood of η in Y. (The notion of vertical component was introduced by Kwieciński [17], to mean what we now call a geometric vertical component.)

The concept of vertical component comes up naturally as an equivalent of torsion in algebraic geometry: Let $\varphi : X \to Y$ be a polynomial map of algebraic varieties with Y irreducible. Then the coordinate ring of the source, $A(X)$ has nonzero torsion as a module over the coordinate ring $A(Y)$ of the target if and only if there exists a nonzero element $a \in A(Y)$ such that its pull-back $\varphi^* a$ is a zerodivisor in $A(X)$. Since the set of zerodivisors equals the union of the associated primes, it follows from "prime avoidance" (see, e.g., [7, Lemma 3.3]) that $A(X)$ has nonzero torsion over $A(Y)$ if and only if there exists an irreducible (isolated or embedded) component of X whose image under φ is contained in a proper algebraic subset of Y (or, equivalently, is nowhere dense in Y). There are two natural ways of generalizing this property of irreducible components to the analytic case. For a morphism $\varphi_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces (with Y_η irreducible), one can either consider the components of the source that are mapped into nowhere dense subgerms of the target (the *geometric vertical* components), or the components that are mapped into proper analytic subgerms of the target (the *algebraic vertical* components).

The geometric approach has proved to be a very powerful tool (see Theorems 1.8 and 1.10 below). Note that in principle the existence of the algebraic vertical components is a stronger condition than the presence of the geometric vertical ones. Indeed, any algebraic vertical component (over an irreducible target) is geometric vertical, since a proper analytic subset of an irreducible analytic set has empty interior. The converse is not true though, as can be seen in the classical example of Osgood (cf. [10, Ch. II, § 5]):

$$\varphi : \mathbb{C}^2 \ni (x, y) \mapsto (x, xy, xye^y) \in \mathbb{C}^3.$$

Here the image of an arbitrarily small neighbourhood of the origin is nowhere dense in $\mathbb{C}^3$, but its Zariski closure has dimension 3 and therefore the image is not contained in a proper locally analytic subset of the target. (In general, the algebraic vertical components are precisely those geometric vertical ones along which the mapping is Gabrielov regular; see below.)

Remark 1.3. On the other hand, the algebraic approach has an advantage that all the statements about algebraic vertical components (as opposed to geometric vertical) can be restated in terms of torsion freeness of the local rings. Namely, *$\varphi_\xi : X_\xi \to Y_\eta$ has no (isolated or embedded) algebraic vertical components if and only if the local ring $\mathcal{O}_{X,\xi}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.*

In view of Remark 1.3, it is interesting to know under what conditions are the two approaches equivalent. In other words, is there a class of analytic morphisms for which the existence of geometric vertical components is equivalent to the presence of the algebraic vertical ones. Theorem 1.10 below gives a partial answer to this question. On the other hand, as we show in Example 4.2, there are examples of bad behaviour of analytic mappings that can be detected by means of geometric vertical components but not by the algebraic vertical ones. The fundamental reason for this is that, in general, the algebraic vertical components do not detect a hidden Gabrielov irregularity. Section 5 is devoted to the analysis of this problem.

Next, we want to extend the concept of vertical component to finitely generated $\mathcal{O}_{X,\xi}$-modules. Observe that, by the prime avoidance lemma, $\varphi_\xi : X_\xi \to Y_\eta$ has an algebraic (respectively, geometric) vertical component if and only if there exists a nonzero element $m \in \mathcal{O}_{X,\xi}$ such that the zero-set germ $\mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X,\xi}}(m))$ of the annihilator of m in $\mathcal{O}_{X,\xi}$ is mapped into a proper analytic (respectively, nowhere-dense) subgerm of Y_η. This can be generalized to $\mathcal{O}_{X,\xi}$-modules as follows.

Definition 1.4. Let F be a finite $\mathcal{O}_{X,\xi}$-module, and let Z_ξ be the zero-set germ of the support of F; i.e., $Z_\xi = \mathcal{V}\left(\bigcap\{\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}_{\mathcal{O}_{X,\xi}}(F)\}\right)$. We say that F has an *algebraic* (respectively, *geometric*) *vertical component* over Y_η (or over $\mathcal{O}_{Y,\eta}$) if Z_ξ has an algebraic (respectively, geometric) vertical component over Y_η in the sense of Definition 1.2; equivalently (by prime avoidance again), there exists a nonzero $m \in F$ such that the $\mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X,\xi}}(m))$ is mapped into a proper analytic (respectively, nowhere-dense) subgerm of Y_η. In the geometric case, we will call such m a *geometric vertical element* (or simply a *vertical element*) of F over Y_η (or over $\mathcal{O}_{Y,\eta}$).

Note that an analogous "algebraic vertical element" of F over Y_η is simply a (nonzero) zero-divisor of F over $\mathcal{O}_{Y,\eta}$, so there is no need to define algebraic vertical elements. A *vertical element* will always mean geometric vertical.

Remark 1.5. In the special case that $F = \mathcal{O}_{X,\xi}$, X_ξ has no geometric (respectively, algebraic) vertical components over Y_η if and only if $\mathcal{O}_{X,\xi}$ (as an $\mathcal{O}_{X,\xi}$-module) has no vertical elements (respectively, no zero-divisors) over $\mathcal{O}_{Y,\eta}$.

Now let R denote a regular local analytic $\mathbb{C}$-algebra of dimension n (i.e., $R \cong \mathbb{C}\{y_1, \dots, y_n\}$ the ring of convergent power series in n variables). By a *local analytic R-algebra* we mean a ring of the form $R\{x\}/I = \mathbb{C}\{y, x\}/I$, where I is an ideal in $\mathbb{C}\{y, x\} = \mathbb{C}\{y_1, \dots, y_n, x_1, \dots, x_m\}$, with the canonical homomorphism $R \to R\{x\}/I$.

Definition 1.6. Let F be an R-module. We say (following [9]) that F is an *almost finitely generated R-module*, when F is a finitely generated A-module, for some local analytic R-algebra A. In this case, there is a morphism of germs of analytic spaces $\varphi_\xi : X_\xi \to Y_\eta$ such that $R \cong \mathcal{O}_{Y,\eta}$, $A \cong \mathcal{O}_{X,\xi}$, $R \to A$ is the induced homomorphism $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$, and F is a finitely generated $\mathcal{O}_{X,\xi}$-module. We say that a nonzero element $m \in F$ is *vertical* over R if m is vertical over $\mathcal{O}_{Y,\eta}$ in the sense of Definition 1.4.

Remark 1.7. It is easy to see that the notion of vertical element is well-defined; i.e., independent of a choice of local R-algebra A such that F is a finitely generated A-module (see [9] for details). In particular, given an almost finitely generated R-module F, we can assume without loss of generality that F is finitely generated over the regular ring $A = R\{x\} \cong \mathbb{C}\{y, x\}$, where $x = (x_1, \ldots, x_m)$, for some $m \geq 0$.

1.2. Main regularity criteria

Recall that a morphism $\varphi_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces is called *flat* when the pull-back homomorphism $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ makes the local ring $\mathcal{O}_{X,\xi}$ into a flat $\mathcal{O}_{Y,\eta}$-module. We say that a holomorphic mapping $\varphi : X \to Y$ of analytic spaces is flat when φ_ξ is flat for every $\xi \in X$. A mapping $\varphi : X \to Y$ is *open* when it sends open subsets of X onto open subsets of Y. A morphism $\varphi_\xi : X_\xi \to Y_\eta$ is called open when φ_ξ is a germ at ξ of an open mapping. Finally, we say that $\varphi_\xi : X_\xi \to Y_\eta$ is *Gabrielov regular* when, for every isolated irreducible component W_ξ of X_ξ, $\dim_\eta \varphi(W) = \dim_\eta \overline{\varphi(W)}$ for every sufficiently small representative W of W_ξ, where $\overline{\varphi(W)}$ denotes the Zariski closure of $\varphi(W)$ in a representative of Y_η.

The failure of flatness, openness, or Gabrielov regularity of a morphism correspond to various levels of degeneracies in the family of fibres of the morphism. Douady [6] proved that every flat mapping is open. (This is rather nontrivial, but see Section 3 for a simple alternative proof). Section 3 contains also a simple example of an open and non-flat mapping. Open mappings are Gabrielov regular by definition. An example of a Gabrielov regular non-open map is given in Example 5.3.

The following criteria assert that flatness and openness of a complex-analytic mapping can be controlled by means of vertical components in *finitely many* fibred powers of the mapping (see Section 2 for definitions). In fact, it always suffices to check in the n-fold fibred power, where n is the dimension of the target.

Theorem 1.8. ([4, Thm. 1.1]) *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of complex-analytic spaces, where Y_η is smooth, of dimension n. Then φ_ξ is flat if and only if the n-fold fibred power $\varphi^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ has no (isolated or embedded) geometric vertical components.*

A special case of Theorem 1.8 when X_ξ is pure-dimensional is a result of Galligo and Kwieciński, [9, Thm. 6.1]. To obtain Theorem 1.8 in full generality, one needs to actually prove even more, namely a variant with a finite $\mathcal{O}_{X,\xi}$-module. This is Theorem 7.1, which we state and prove in Section 7.

It remains an open problem, in general, whether the geometric vertical components can be replaced with the algebraic vertical ones in the statement of Theorem 1.8 (compare with Theorem 1.10 below). The few cases known so far, when such a strengthening is possible are listed in the following theorem.

Theorem 1.9. (cf. [3, Thm. 1.1]) *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of complex-analytic spaces, where Y_η is smooth, of dimension n. Suppose that one of the following conditions is satisified:*

1. *$n < 3$;*
2. *$\varphi_\xi : X_\xi \to Y_\eta$ is a Nash morphism of Nash germs;*
3. *X_ξ is pure-dimensional and its singular locus is mapped into a proper analytic subgerm of Y_η;*
4. *the local ring of the source $\mathcal{O}_{X,\xi}$ is Cohen-Macaulay.*

Then, φ_ξ is flat if and only if the n-fold fibred power $X^{\{n\}}_{\xi^{\{n\}}}$ contains no algebraic vertical components.

In short, in all of the above cases, some of the geometric vertical components of Theorem 1.8 must, in fact, be algebraic vertical (see Section 7 for details).

Theorem 1.10. ([1, Thm. 2.2]) *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Let X_ξ be of pure dimension, and let Y_η be irreducible, of dimension n. Then the following conditions are equivalent:*

1. *φ_ξ is a germ of an open mapping;*
2. *$X^{\{n\}}_{\xi^{\{n\}}}$ has no isolated geometric vertical components;*
3. *$X^{\{n\}}_{\xi^{\{n\}}}$ has no isolated algebraic vertical components.*

Remark 1.11. In light of Remark 1.3, the equivalence (1) $\Leftrightarrow$ (3) in the above theorem can be restated as follows:
The morphism $\varphi_\xi : X_\xi \to Y_\eta$ is open if and only if the reduced *local ring $(\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}})_{\mathrm{red}}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.*

On the other hand, the equivalence (2) $\Leftrightarrow$ (3) means that, under the pure-dimensionality assumption on the domain, at least one of the restrictions of $\varphi^{\{n\}}_{\xi^{\{n\}}}$ to an isolated geometric vertical component of $X^{\{n\}}_{\xi^{\{n\}}}$ must be Gabrielov regular.

Note that since openness is a local property, Theorem 1.10 can easily be "globalized" to the case of an analytic mapping $\varphi : X \to Y$ of analytic spaces, where X is of pure dimension and Y is locally irreducible of dimension n. Thus, our result is a stronger version of a theorem by Kwieciński and Tworzewski [18, Thm. 3.2], asserting that openness is equivalent to the lack of geometric vertical components in the n'th fibred power. We prove Theorem 1.10 in Section 4.

As we show in Example 4.2, the pure-dimensionality constraint on X in the above theorem is unavoidable. Roughly speaking, this is a consequence of the fact that fibred products do not behave well with respect to Gabrielov regularity. It is thus interesting to know when does the Gabrielov regularity of a morphism imply that of its fibred powers. This question is answered in Theorem 1.12 below.

For a morphism $\varphi_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces, let k be its maximal fibre dimension and let $(A_j)_\xi$ denote the germ of a locus of points of fibre dimension greater than or equal to j, for $j = 0, 1, \dots, k$ (see Section 4 for details). Then we have:

THEOREM 1.12. ([3, Prop. 4.2]) *Suppose Y_η is irreducible. The following conditions are equivalent:*

(1) $\varphi^{\{d\}}_{\xi\{d\}} : X^{\{d\}}_{\xi\{d\}} \to Y_\eta$ *is Gabrielov regular, for all* $d \geq 1$;
(2) *all the restrictions* $\varphi_\xi|_{(A_j)_\xi}$ *are Gabrielov regular* ($j = 0, \dots, k$).

Notice the difference between the assumptions on the target space in Theorems 1.8 and 1.9 vs. Theorems 1.10 and 1.12. In fact, the assumptions of the latter theorems are the weakest possible that one needs to impose on the target to make sense of the notion of vertical component. (Otherwise, for instance, each of the irreducible components of the set $X = \{xy = 0\}$ would be an algebraic vertical component for the identity mapping $\mathrm{id}_X : X \to X$, which is absurd.) This is in contrast with Theorem 1.8, where any weakening of the regularity assumption on the target space leads to an open problem. As of today, it is not even known whether, for a singular irreducible Y_η, there exists a constant N (depending only on Y_η) such that non-flatness of a morphism $\varphi_\xi : X_\xi \to Y_\eta$ could be detected by vertical components in the N-fold fibred power of φ_ξ. So far, the best result in this direction is the following theorem of Kwieciński.

THEOREM 1.13. ([17, Thm. 1.1]) *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Suppose that Y_η irreducible. Then φ_ξ is non-flat if and only if there exists an (isolated or embedded) algebraic vertical component in the d-fold fibred power $X^{\{d\}}_{\xi\{d\}}$, for* some $d \geq 1$.

It is also symptomatic that, contrary to openness, to characterize flatness, one needs to control both the isolated and embedded components in the fibred powers. (The embedded components are the zero-set germs of the embedded associated primes of the local ring). This reflects the fact that flatness often depends on infinitesimal data, which is invisible from the topological point of view. In Section 3 we give a simple example of an open mapping which is not flat (see also Example 2.1).

The structure of the paper is as follows. In the next section we recall the notions of fibred product and analytic tensor product. Section 3 is concerned with the comparison between openness and flatness: a simple proof of Douady's theorem on openness of a flat morphism is derived from Theorem 1.10. Section 4 contains the proof of Theorem 1.10 and some related results. In Section 5 we give a proof of Theorem 1.12 and an example of a Gabrielov regular mapping with irregular fibred powers. Sections 6 and 7 are devoted to the proof of Theorems 1.8 and 1.9. The last section contains computable criteria for openness and flatness of polynomial mappings, derived from Theorems 1.10 and 7.1.

2. Analytic tensor product and fibred product

Most of the arguments in this article are of local algebraic nature. To establish the notation and keep the article self-contained, we recall here the concepts of analytic tensor product and fibred product of analytic spaces, that are used throughout the paper.

Our main objects of study are morphisms of germs of complex-analytic spaces, that is, germs of holomorphic mappings between complex-analytic spaces. By a complex-analytic space we mean a $\mathbb{C}$-ringed space, whose underlying topological space is Hausdorff, and the local models are defined by coherent (not necessarily radical!) ideal sheaves on open sets in $\mathbb{C}^n$ (readers not familiar with these notions may find a quick treatment in [8, Ch. 0]). In particular, unlike some classical expositions (e.g., [20]), we do not assume that our spaces are reduced. On the contrary, we are very much interested in their embedded nilpotent structure, especially in the study of flatness. Also, when studying fibred products of analytic spaces, the requirement that the spaces be reduced is not natural, because a fibred product of reduced spaces may as well be non-reduced, even when the product is over a smooth target (see Example 2.1 below).

The analytic tensor product is defined in the category of finitely generated modules over local analytic $\mathbb{C}$-algebras (i.e., rings of the form $\mathbb{C}\{z_1, \dots, z_n\}/I$ for some ideal I) by the usual universal mapping property for tensor product (cf. [10]): Let $\varphi_i : R \to A_i$ $(i = 1, 2)$ be homomorphisms of local analytic $\mathbb{C}$-algebras. Then there is a unique (up to isomorphism) local analytic $\mathbb{C}$-algebra $A_1 \tilde{\otimes}_R A_2$, together with homomorphisms $\theta_i : A_i \to A_1 \tilde{\otimes}_R A_2$ $(i = 1, 2)$, such that $\theta_1 \circ \varphi_1 = \theta_2 \circ \varphi_2$, and for every pair of homomorphisms of local analytic $\mathbb{C}$-algebras $\psi_1 : A_1 \to B$, $\psi_2 : A_2 \to B$ satisfying $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$, there is a unique homomorphism of local analytic $\mathbb{C}$-algebras $\psi : A_1 \tilde{\otimes}_R A_2 \to B$ making the associated diagram commute. The algebra $A_1 \tilde{\otimes}_R A_2$ is called the *analytic tensor product* of A_1 and A_2 *over* R.

For finite modules M_1 and M_2 over local analytic R-algebras A_1 and A_2, respectively, there is a unique (up to isomorphism) finite $A_1\tilde{\otimes}_R A_2$-module $M_1\tilde{\otimes}_R M_2$, together with an R-bilinear mapping $\rho : M_1 \times M_2 \to M_1\tilde{\otimes}_R M_2$, such that for every R-bilinear $\kappa : M_1 \times M_2 \to N$, where N is a finite $A_1\tilde{\otimes}_R A_2$-module, there is a unique homomorphism of $A_1\tilde{\otimes}_R A_2$-modules $\lambda : M_1\tilde{\otimes}_R M_2 \to N$ satisfying $\kappa = \lambda \circ \rho$. The module $M_1\tilde{\otimes}_R M_2$ is called the *analytic tensor product* of M_1 and M_2 *over* R.

In our considerations, it will be convenient to express the analytic tensor product of modules over a local analytic $\mathbb{C}$-algebra in terms of ordinary tensor product of certain naturally associated modules: Given homomorphisms of local analytic $\mathbb{C}$-algebras $\varphi_i : R \to A_i$, and finitely generated A_i-modules M_i ($i = 1, 2$), the modules $M_1\tilde{\otimes}_R A_2$ and $A_1\tilde{\otimes}_R M_2$ are finitely generated over $A_1\tilde{\otimes}_R A_2$, and there is a canonical isomorphism

$$M_1\tilde{\otimes}_R M_2 \;\cong\; (M_1\tilde{\otimes}_R A_2) \otimes_{A_1\tilde{\otimes}_R A_2} (A_1\tilde{\otimes}_R M_2).$$

In particular, if $A_1 = R\{x\}/I_1$ and $A_2 = R\{t\}/I_2$, where $x = (x_1, \dots, x_l)$, $t = (t_1, \dots, t_m)$ are systems of variables and $I_1 \subset R\{x\}$, $I_2 \subset R\{t\}$ are ideals, then

$$\begin{aligned} A_1\tilde{\otimes}_R A_2 &\cong (A_1\tilde{\otimes}_R R\{t\}) \otimes_{R\{x\}\tilde{\otimes}_R R\{t\}} (R\{x\}\tilde{\otimes}_R A_2) \\ &\cong (R\{x,t\}/I_1 R\{x,t\}) \otimes_{R\{x,t\}} (R\{x,t\}/I_2 R\{x,t\}) \\ &\cong R\{x,t\}/(I_1 R\{x,t\} + I_2 R\{x,t\}). \end{aligned}$$

The fibred product of analytic spaces is defined by a dual universal mapping property (see [8]): Let $\varphi_i : X_i \to Y$ ($i = 1, 2$) denote holomorphic mappings of complex-analytic spaces. Then there exists a unique (up to isomorphism) complex-analytic space $X_1 \times_Y X_2$, together with holomorphic maps $\pi_i : X_1 \times_Y X_2 \to X_i$ ($i = 1, 2$), such that $\varphi_1 \circ \pi_1 = \varphi_2 \circ \pi_2$, and for every pair of holomorphic maps $\psi_1 : X \to X_1$, $\psi_2 : X \to X_2$ satisfying $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$, there is a unique holomorphic map $\psi : X \to X_1 \times_Y X_2$ making the associated diagram commute. The space $X_1 \times_Y X_2$ is called the *fibred product* of X_1 and X_2 *over* Y (more precisely, *over* φ_1 and φ_2). There is a canonical holomorphic mapping $\varphi_1 \times_Y \varphi_2 : X_1 \times_Y X_2 \to Y$, given by $\varphi_1 \times_Y \varphi_2 = \varphi_i \circ \pi_i$ (where $i = 1$ or 2).

Given a holomorphic map $\varphi : X \to Y$ of complex-analytic spaces, the *d-fold fibred power* of X over Y, denoted $X^{\{d\}}$, is defined recursively as $X^{\{2\}} = X \times_Y X$, $X^{\{d\}} = X^{\{d-1\}} \times_Y X$. Let further $\xi \in X$, $\eta = \varphi(\xi)$, and let $\varphi_\xi : X_\xi \to Y_\eta$ denote the germ of φ at ξ. We denote by $\varphi^{\{d\}} : X^{\{d\}} \to Y$ the canonical map from the d-fold fibred power of X over Y to Y, and by $\varphi^{\{d\}}_{\xi^{\{d\}}} : X^{\{d\}}_{\xi^{\{d\}}} \to Y_\eta$ its germ at the diagonal point $\xi^{\{d\}} := (\xi, \dots, \xi) \in X^d$.

Example 2.1. ([13, Ch. III, Exercise 9.3]) The fibred product of reduced spaces need not be reduced itself: Let $X \subset \mathbb{C}^4$ be a union of two copies of $\mathbb{C}^2$ that intersect precisely at the origin, say

$$X_1 = \{(y_1, y_2, t_1, t_2) \in \mathbb{C}^4 : t_1 = t_2 = 0\} \qquad \text{and}$$
$$X_2 = \{(y_1, y_2, t_1, t_2) \in \mathbb{C}^4 : t_1 - y_1 = t_2 - y_2 = 0\},$$

and let $\varphi : X \to \mathbb{C}^2$ be the projection onto the y variables. Then the fibred power $X^{\{2\}} = X \times_{\mathbb{C}^2} X$ has an embedded component, namely the origin in $\mathbb{C}^4$.

Suppose that $\varphi_1 : X_1 \to Y$ and $\varphi_2 : X_2 \to Y$ are holomorphic mappings of analytic spaces, with $\varphi_1(\xi_1) = \varphi_2(\xi_2) = \eta$. Then the local rings $\mathcal{O}_{X_i,\xi_i}$ $(i = 1, 2)$ are $\mathcal{O}_{Y,\eta}$-modules and, by the uniqueness of fibred product and of analytic tensor product, the local ring $\mathcal{O}_{Z,(\xi_1,\xi_2)}$ of the fibred product $Z = X_1 \times_Y X_2$ at (ξ_1, ξ_2) is canonically isomorphic to $\mathcal{O}_{X_1,\xi_1} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X_2,\xi_2}$. Therefore, given a holomorphic germ $\varphi_\xi : X_\xi \to Y_\eta$, we will identify the d-fold analytic tensor power $\mathcal{O}_{X,\xi}^{\tilde{\otimes}^d_{\mathcal{O}_{Y,\eta}}} = \mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \dots \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}$ with the local ring of the d-fold fibred power $\mathcal{O}_{X^{\{d\}},\xi^{\{d\}}}$, for $d \geq 1$.

3. Openness vs. flatness

The relationship between Theorems 1.8 and 1.10 (the criteria for flatness and openness in terms of vertical components in fibred powers) shows that openness can be viewed as a geometric analogue of flatness. Flatness intuitively means that fibres of a morphism change in a continuous way. If one disregards the embedded structure of the fibres, that is, if one only considers their geometry, then continuity in the family of fibres means openness of the morphism. This roughly explains why for the study of openness it is enough to consider the isolated irreducible components in fibred powers, whilst both the isolated and embedded components must be considered in the study of flatness.

In fact, every flat morphism is open. This result was first obtained by A. Douady [6]; another proof was given by R. Kiehl [15]. Our openness criterion, Theorem 1.10, implies a quick alternative proof (see Proposition 3.2 below).

The proof of Proposition 3.2 below is based on the observation that if $\varphi_\xi : X_\xi \to Y_\eta$ is flat then every restriction of φ_ξ to an irreducible component of X_ξ is flat as well (Lemma 3.1 below). The converse is not true though: flatness of all the restrictions to irreducible components does not imply flatness of the morphism, as can be observed in Example 2.1.

Observe that for openness the opposite is true: Clearly, openness of all the restrictions to isolated irreducible components implies openness of the morphism, but openness of a morphism does not imply openness of all its restrictions to isolated irreducible components. Consider, for instance, the mapping given by

$$\varphi : X \ni (x, y, s, t) \mapsto (x + s, xy + t) \in Y = \mathbb{C}^2,$$

where $X = X_1 \cup X_2$, $X_1 = \{(x, y, s, t) \in \mathbb{C}^4 : s = t = 0\}$, and $X_2 = \{(x, y, s, t) \in \mathbb{C}^4 : x = 0\}$. Then φ is open, but $\varphi|_{X_1}$ is not. (In particular, the above is a simple example of an open and non-flat mapping, by Proposition 3.2 and Lemma 3.1 below.)

The following lemma is probably known, but we prove it here for completeness.

Lemma 3.1. *If $\varphi_\xi : X_\xi \to Y_\eta$ is a flat morphism of germs of analytic spaces with Y_η irreducible, then every restriction $\varphi_\xi|_{W_\xi} : W_\xi \to Y_\eta$ to an (isolated or embeddded) irreducible component W_ξ of X_ξ is also flat.*

Proof. Let $\varphi_\xi : X_\xi \to Y_\eta$ be a flat morphism of germs of analytic spaces with Y_η irreducible, and let W_ξ be an (isolated or embedded) irreducible component of X_ξ. Without loss of generality, we can assume that X_ξ is a subgerm of $\mathbb{C}^m_0$ for some $m \geq 1$. We can then embed X_ξ into $Y_\eta \times \mathbb{C}^m_0$ via the graph of φ_ξ. Therefore the local ring $\mathcal{O}_{X,\xi}$ of the germ X_ξ can be thought of as a quotient of the local ring of $Y_\eta \times \mathbb{C}^m_0$; i.e., $\mathcal{O}_{X,\xi} = \mathcal{O}_{Y,\eta}\{x\}/I$ for some ideal I in $\mathcal{O}_{Y,\eta}\{x\}$, where $x = (x_1, \dots, x_m)$ is a system of m complex variables.

Let $\mathfrak{p}$ be the associated prime of I in $\mathcal{O}_{Y,\eta}\{x\}$ corresponding to W_ξ. Now, $\varphi_\xi|_{W_\xi} : W_\xi \to Y_\eta$ is flat if and only if $\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}$ is a flat $\mathcal{O}_{Y,\eta}$-module. By a well known characterization of flatness [14, Prop. 6.2], the latter is equivalent to

$$\widetilde{\operatorname{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) = 0,$$

where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}_{Y,\eta}$ and $\widetilde{\operatorname{Tor}}$ denotes the derived functor for analytic tensor product (see Section 2).

The short exact sequence of $\mathcal{O}_{Y,\eta}$-modules

$$0 \to \mathfrak{p}/I \to \mathcal{O}_{Y,\eta}\{x\}/I \to \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p} \to 0$$

induces a long exact sequence of the $\widetilde{\operatorname{Tor}}$ modules, of which a part is

$$\begin{aligned} \widetilde{\operatorname{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{X,\xi}) \to \widetilde{\operatorname{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) \xrightarrow{\psi} \\ (\mathcal{O}_{Y,\eta}/\mathfrak{m})\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}(\mathfrak{p}/I) \xrightarrow{\chi} (\mathcal{O}_{Y,\eta}/\mathfrak{m})\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}\mathcal{O}_{X,\xi} \to \\ (\mathcal{O}_{Y,\eta}/\mathfrak{m})\tilde{\otimes}_{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) \to 0. \end{aligned}$$

Since $\mathcal{O}_{X,\xi}$ is $\mathcal{O}_{Y,\eta}$-flat, then $\widetilde{\operatorname{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{X,\xi}) = 0$, and hence ψ is injective. Therefore, $\widetilde{\operatorname{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) = 0$ if and only if χ is injective.

Observe that χ is *not* injective if and only if there exists a nonzero series $f \in \mathfrak{p} \setminus I$ that factors (in $\mathcal{O}_{Y,\eta}\{x\}$, but not in $\mathfrak{p}$!) as $f = gh$ for some $g \in \mathfrak{m}$ and $h \in \mathcal{O}_{Y,\eta}\{x\} \setminus \mathfrak{p}$. Then $g \in \mathfrak{p}$, since $\mathfrak{p}$ is prime and $h \notin \mathfrak{p}$. Thus, the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{Y,\eta}$ contains a nonzero element of an associated prime of I, and hence $\mathcal{O}_{Y,\eta}$ contains a zero-divisor of $\mathcal{O}_{X,\xi}$. It follows from the characterization of flatness in terms of relations (see, e.g., [7, Cor. 6.5]) that $\mathcal{O}_{Y,\eta}$ has a zero-divisor itself, which contradicts the assumption that Y_η be irreducible. We thus showed that χ must be injective, which completes the proof. ∎

Proposition 3.2. *Let $\varphi : X \to Y$ be a holomorphic map of complex-analytic spaces. If φ is flat, then it is open.*

Proof. We will proceed in three steps:

Step 1. Suppose first that Y is locally irreducible and X is of pure dimension. Let $\xi \in X$ and $\eta = \varphi(\xi)$. Then flatness of $\varphi_\xi : X_\xi \to Y_\eta$ implies flatness of all its fibred powers (as flatness is preserved by any base change [14, Prop. 6.8], and compositions of flat mappings are flat). In particular, $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ is $\mathcal{O}_{Y,\eta}$-flat, hence torsion-free. By Remark 1.3, $X^{\{n\}}_{\xi^{\{n\}}}$ has no algebraic vertical components over Y_η, and so $\varphi_\xi : X_\xi \to Y_\eta$ is open, by Theorem 1.10. Thus φ is open, because ξ was arbitrary.

Step 2. Next, assume only that Y is locally irreducible. Since openness of a germ φ_ξ follows from openness of every restriction $\varphi_\xi|_{W_\xi}$ to an isolated irreducible component W_ξ of X_ξ, by Lemma 3.1 the problem reduces to Step 1.

Step 3. In the general case we may assume that Y is reduced. Let $\nu \colon \widetilde{Y} \to Y$ denote the normalization map. Then $\widetilde{Y}$ is locally irreducible and there is a commutative square

$$\begin{array}{ccc} X \times_Y \widetilde{Y} & \longrightarrow & X \\ {\scriptstyle\tilde{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \widetilde{Y} & \xrightarrow{\ \nu\ } & Y. \end{array}$$

Now, $\tilde{\varphi}$ is flat, because flatness is preserved by any base change (see [14, Prop. 6.8]), and hence open, by Step 2. Since Y has the quotient topology with respect to ν, this implies that φ is open. ∎

4. Vertical components and variation of fibre dimension

In this section, we describe a relationship between the filtration of the target of an analytic mapping $\varphi : X \to Y$ by fibre dimension and the isolated irreducible components of the n-fold fibred power $X^{\{n\}}$, where $n = \dim Y$.

We will also discuss the relationship between the occurence of the two kinds of vertical components in fibred powers.

Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, where Y_η is irreducible and of dimension n. Let Y be an irreducible representative of Y_η, let X be a representative of X_ξ, such that the isolated irreducible components of X are precisely the representatives in X of the isolated irreducible components of X_ξ, and $\varphi(X) \subset Y$, where φ represents the germ φ_ξ. Let $\mathrm{fbd}_x\varphi$ denote the *fibre dimension* $\dim_x \varphi^{-1}(\varphi(x))$ of φ at a point $x \in X$.

Put $l := \min\{\mathrm{fbd}_x\varphi : x \in X\}$, $k := \max\{\mathrm{fbd}_x\varphi : x \in X\}$, and $A_j := \{x \in X : \mathrm{fbd}_x\varphi \geq j\}$, $l \leq j \leq k$. Then $X = A_l \supset A_{l+1} \supset \cdots \supset A_k$ and, by upper-semicontinuity of fibre dimension (see Cartan–Remmert Theorem [20, §V.3.3, Thm. 5]), the A_j are analytic in X. Define $B_j := \varphi(A_j) = \{y \in Y : \dim \varphi^{-1}(y) \geq j\}$, $l \leq j \leq k$. Upper-semicontinuity of $\mathrm{fbd}_x\varphi$ (as a function of x) implies that the germs $(A_j)_\xi$ and $(B_j)_\eta$ are independent of the choices of representatives made.

Note that, except for B_k (cf. proof of Proposition 4.1 below), the B_j may not even be semianalytic in general. This fact is responsible for a complicated relationship between the algebraic vertical and geometric vertical components in the fibred powers of X over Y (see Section 5), but will not affect our considerations here, which rely only on the properties of B_k.

Proposition 4.1. ([1, Prop. 2.1]) *Under the assumptions above, let $\bigcup_{i\in I} W_i$ be the decomposition of $(X^{\{n\}})_{\mathrm{red}}$ into finitely many isolated irreducible components through $\xi^{\{n\}}$. Then:*

(1) *For each $j = l, \dots, k$, there is an index subset $I_j \subset I$ such that*
$$B_j = \bigcup_{i \in I_j} \varphi^{\{n\}}(W_i).$$

(2) *Let $y \in B_j$ and let $s = \dim \varphi^{-1}(y)$ $(s \geq j)$. If Z is an isolated irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(y)$, of dimension ns, and W is an irreducible component of $X^{\{n\}}$ containing Z, then $\varphi^{\{n\}}(W) \subset B_j$.*

Proof. For (2), fix $j \geq l+1$. (The statement is trivial for $j = l$, since $B_l = \varphi(X)$.) Suppose that there exists $x = (x_1, \dots, x_n) \in W$ such that $\varphi(x_1) \in Y \setminus B_j$ (and hence $\varphi(x_i) \in Y \setminus B_j$, $i \leq n$). Then $\mathrm{fbd}_{x_i}\varphi \leq j-1$, $i = 1, \dots, n$; hence $\mathrm{fbd}_x\varphi^{\{n\}} \leq n(j-1) = nj - n$. In particular, the generic fibre dimension of $\varphi^{\{n\}}|_W$ is at most $nj-n$. Since $\mathrm{rank}(\varphi^{\{n\}}|_W) \leq \dim Y = n$, then $\dim W \leq (nj-n)+n = nj$ (see, e.g., [20, §V.3]).

Now we have $W \supset Z$, $\dim W \leq nj$, $\dim Z = ns \geq nj$, and both W and Z are irreducible analytic sets in $X^{\{n\}}$. This is possible only if $W = Z$; hence $\varphi^{\{n\}}(W) = \varphi^{\{n\}}(Z) = \{y\} \subset B_j$; a contradiction. Therefore $\varphi^{\{n\}}(W) \subset B_j$, completing the proof of (2).

Part (1) follows immediately, since if $y \in B_j$ and Z is an irreducible component of $(\varphi^{\{n\}})^{-1}(y)$ of the maximal dimension, then there exists an isolated irreducible component W of $X^{\{n\}}$ that contains Z. ∎

We are now ready to prove our openness criterion.

Proof of Theorem 1.10. First, notice that if $\varphi_\xi : X_\xi \to Y_\eta$ is open, then $\varphi^{\{d\}}_{\xi\{d\}} : X^{\{d\}}_{\xi\{d\}} \to Y_\eta$ is open for every $d \geq 1$, and hence every isolated irreducible component of $X^{\{d\}}_{\xi\{d\}}$ is dominating Y_η. This proves (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial, as every algebraic vertical component is geometric vertical over an irreducible target (see Section 1).

For the proof of (3) $\Rightarrow$ (1), let us suppose that φ_ξ is not open. Let $\varphi : X \to Y$ be its non-open representative, where Y is an irreducible representative of Y_η, of dimension n, X is a representative of X_ξ, such that the isolated irreducible components $X_1, \dots, X_s$ of X are precisely the representatives in X of the isolated irreducible components of X_ξ, and $\varphi(X) \subset Y$. Put $m := \dim X$. As above, let $k = \max\{\mathrm{fbd}_x\varphi : x \in X\}$, $A_k = \{x \in X : \mathrm{fbd}_x\varphi = k\}$, and $B_k = \varphi(A_k) = \{y \in Y : \dim \varphi^{-1}(y) = k\}$. Then the fibre dimension of φ is constant on the analytic set A_k. By the Remmert Rank Theorem (see [20, §V.6, Thm. 1]), B_k is locally analytic in Y, of dimension $\dim A - k$ (which is at most $\dim X - k = m - k$). Shrinking Y if necessary, we can assume that B_k is an analytic subset of Y. Therefore, by Proposition 4.1 and irreducibility of the germ Y_η, it is enough to show that the analytic germ $(B_k)_\eta$ is a proper subgerm of Y_η.

Suppose that k is the generic fibre dimension for all the components $X_1, \dots, X_s$ simultanously. Then, by the Remmert Rank Theorem again, either $\dim X - k < n$ and then $(B_k)_\eta = (\varphi(X))_\eta$ is a proper analytic subgerm of Y_η, or else $\dim X - k = n$ and φ is open. Since an isolated algebraic vertical component in φ_ξ induces such a component in every fibred power $\varphi^{\{d\}}_{\xi\{d\}}$, we may assume that, for some isolated irreducible component X_j of X, the generic fibre dimension of $\varphi|_{X_j}$ is at most $k-1$. Hence

$$\dim Y \;\geq\; \dim X_j - \text{ generic fbd}\varphi|_{X_j} \;\geq\; m - k + 1 \;>\; \dim B_k.$$

Consequently, $\dim(B_k)_\eta < \dim Y = \dim Y_\eta$, so that $(B_k)_\eta \subsetneq Y_\eta$. ∎

The pure-dimensionality constraint on X in our theorem is unavoidable. If X is not of pure dimension, it may happen that the exceptional fibres of one component are generic for a component of higher dimension, and therefore they do not give rise to an isolated algebraic vertical component in any fibred power. This phenomenon is illustrated in the following example, where not only is the morphism φ not open, but also it is not even regular in the sense of Gabrielov.

EXAMPLE 4.2. Let $X = X_1 \cup X_2$, where $X_1 = \{(x,y,s,t,z) \in \mathbb{C}^5 : s = t = z = 0\}$ and $X_2 = \{(x,y,s,t,z) \in \mathbb{C}^5 : x = 0\}$. Define $\varphi : X \to Y = \mathbb{C}^3$ as

$$\varphi(x,y,s,t,z) = (x+s, xy+t, xye^y + z).$$

Observe that (the germ at the origin of) $\varphi|_{X_1}$ is an Osgood mapping (cf. Section 1) and hence it is not Gabrielov regular. Therefore φ is not open. But the exceptional fibre $\{x = s = t = z = 0\}$ of $\varphi|_{X_1}$ is in no sense exceptional for $\varphi|_{X_2}$. One can easily verify that in any fibred power of X over Y, any isolated (!) irreducible component is either *purely geometric* vertical (with the image equal to that of $\varphi|_{X_1}$) or maps onto (the germ at the origin of) Y and so is not vertical in neither sense.

Fortunately, pathologies like the one above can be avoided in the mixed-dimensional case by assuming that the maximal fibre dimension is not generic along all components of X of maximal dimension.

PROPOSITION 4.3. *Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of complex-analytic spaces. Suppose that Y_η is irreducible,* $\dim Y_\eta = n$, $\dim X_\xi = m$, *and the maximal fibre dimension of φ_ξ is not generic on some m-dimensional irreducible component of X_ξ. Then the n-fold fibred power $\varphi^{\{n\}}_{\xi\{n\}} : X^{\{n\}}_{\xi\{n\}} \to Y_\eta$ contains an isolated algebraic vertical component.*

Proof. As above, let $\varphi : X \to Y$ be a representative of φ_ξ, where Y is irreducible and of dimension n. Let $k = \max\{\mathrm{fbd}_x\varphi : x \in X\}$, $A_k = \{x \in X : \mathrm{fbd}_x\varphi = k\}$, and $B_k = \varphi(A_k) = \{y \in Y : \dim \varphi^{-1}(y) = k\}$. Then, as in the proof of Theorem 1.10 above, after shriking Y if necessary, we may assume that B_k is analytic in Y, of dimension at most $\dim X - k$. Let X_j be an isolated irreducible component of X, of dimension $m = \dim X$, for which the generic fibre dimension of $\varphi|_{X_j}$ is at most $k-1$. We have

$$\dim Y \;\geq\; \dim X_j - \text{ generic fbd}\varphi|_{X_j} \;\geq\; m - k + 1.$$

Then $\dim B_k \leq m - k < \dim Y$; hence $\dim(B_k)_\eta < \dim Y = \dim Y_\eta$, so that $(B_k)_\eta \subsetneq Y_\eta$. ∎

5. Vertical components and Gabrielov regularity

In this section we show that, in general, the fibred product does not behave well with respect to Gabrielov regularity. We then present a proof of Theorem 1.12, which gives a necessary and sufficient condition for Gabrielov regularity to be preserved by fibred powers.

Recall that a morphism $\varphi_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces is called Gabrielov regular when, for every isolated irreducible component W_ξ of X_ξ, $\dim_\eta \varphi(W) = \dim_\eta \overline{\varphi(W)}$ for every sufficiently small representative W of

W_ξ, where $\overline{\varphi(W)}$ denotes the Zariski closure of $\varphi(W)$ in a representative of Y_η. Equivalently, $\dim_\eta \overline{\varphi(W)}$ equals the generic rank of $\varphi|_W$ (see, e.g., [21]).

Under the assumptions of Theorem 1.10, let as before Y be an irreducible representative of Y_η, let X be a pure-dimensional representative of X_ξ, such that the isolated irreducible components of X are precisely the representatives in X of the isolated irreducible components of X_ξ, and $\varphi(X) \subset Y$, where φ represents the germ φ_ξ. The following result is a simple corollary of Proposition 4.1. Define $S = \{y \in Y : \dim \varphi^{-1}(y) > l\}$, where as before $l = \min\{\mathrm{fbd}_x \varphi : x \in X\}$. Then $\dim S < \dim Y$, since X is pure-dimensional.

PROPOSITION 5.1. ([1, Prop. 3.1]) *Suppose that* $\dim_\eta \overline{S} = n$, *where* $\overline{S}$ *denotes the Zariski closure of* S *in* Y. *Then* $X^{\{n\}}_{\xi\{n\}}$ *contains an isolated* purely geometric *vertical component; i.e., an isolated geometric vertical component which is not algebraic vertical.*

As a consequence, there exist Gabrielov regular morphisms $\varphi_\xi\colon X_\xi \to Y_\eta$ such that, for some $d \geq 1$, $\varphi^{\{d\}}_{\xi\{d\}}$ is not Gabrielov regular (see Example 5.3 below).

Proof of Proposition 5.1. By Proposition 4.1(1), there exist irreducible components $W_1, \dots, W_p$ of $X^{\{n\}}$ such that $S = \bigcup_{i=1}^p \varphi^{\{n\}}(W_i)$. (Recall that $S = B_{l+1}$ according to the notation from Section 4). We claim that $\dim_\eta \overline{\varphi^{\{n\}}(W_j)} = n$ for some $j \in \{1, \dots, p\}$. Indeed, if $\dim_\eta \overline{\varphi^{\{n\}}(W_i)} < n$ for all i, then we would have

$$n = \dim_\eta \overline{S} = \dim_\eta \overline{\bigcup_{i=1}^p \varphi^{\{n\}}(W_i)} = \max_{i=1,\dots,p} \dim_\eta \overline{\varphi^{\{n\}}(W_i)} < n,$$

a contradiction. So obtained W_j is not algebraic vertical, and it is geometric vertical, since $\dim_\eta \varphi^{\{n\}}(W_j) \leq \dim_\eta S < n$. ∎

The proposition yields a necessary and sufficient condition for $X^{\{n\}}_{\xi\{n\}}$ to have purely geometric vertical components in the case of dominating mappings:

COROLLARY 5.2. ([1, Cor. 3.2]) *Let* $\varphi_\xi : X_\xi \to Y_\eta$ *be a morphism of germs of complex-analytic spaces with* X_ξ *of pure dimension* m *and* Y_η *irreducible of dimension* n. *Assume that* φ_ξ *is dominating; i.e.,* $m - l = n$ *(where* l *is the minimal fibre dimension). Then* $X^{\{n\}}_{\xi\{n\}}$ *has an isolated purely geometric vertical component if and only if (the germ at* η *of)* Y *equals (the germ at* η *of) the Zariski closure of* S.

Proof. Let Y and X be as before. If $\dim_\eta \overline{S} < n$, then there exists a proper locally analytic subset $Z \subset Y$ containing S and such that $\dim_\eta Z < n$.

Then, for any isolated component W of $X^{\{n\}}$, either $\varphi^{\{n\}}(W) \subset S$ or $W \cap (\varphi^{\{n\}})^{-1}(Y \setminus S) \neq \varnothing$. In the first case $\varphi^{\{n\}}(W) \subset Z$, so W is algebraic vertical. In the second case there exists a point $z = (x_1, \ldots, x_n) \in W$ such that $\mathrm{fbd}_{x_i}\varphi = l$, $i = 1, \ldots, n$, hence the image under φ of an arbitrarily small neighbourhood of x_i has nonempty interior in Y (by Remmert's Rank Theorem and since $m - l = n$). Therefore, the image under $\varphi^{\{n\}}$ of any sufficiently small neighbourhood of z has nonempty interior in Y; i.e., W is not a vertical component in neither sense. ∎

Next, we show an example of a Gabrielov regular mapping $\varphi : X \to Y$, with smooth X and Y, and such that the Zariski closure of S equals Y. By Proposition 5.1 above, the top fibred power of X over Y has an isolated irreducible purely geometric vertical component. This shows that a fibred power of a Gabrielov regular map may itself be irregular.

Example 5.3. Let $X = Y = \mathbb{C}^4$ and define φ as follows

$$(x,\ y,\ s,\ t) \mapsto (x,\ (x+s)y,\ x^2y^2e^y,\ x^2y^2e^{y(1+xe^y)} + st).$$

It is easy to check that the generic fibre dimension of φ equals 0, hence the generic rank of φ is 4, and so φ is Gabrielov regular.

On the other hand, S contains the image of the set $\mathbb{C}^2 \times \{0\} \times \mathbb{C}$, consisting of points (z_1, z_2, z_3, z_4) of the form $(x,\ xy,\ x^2y^2e^y,\ x^2y^2e^{y(1+xe^y)})$, whose Zariski closure equals $\mathbb{C}^4$ (cf. [10, Ch. II, § 5]). Hence also $\overline{S} = \mathbb{C}^4$.

Finally, we shall prove the Gabrielov regularity criterion:

Proof of Theorem 1.12. Suppose first that $\varphi|_{A_j}$ ($j = l, \ldots, k$) are regular. Fix a positive integer d and let W be an isolated irreducible component of $X^{\{d\}}$. Since the components of $X^{\{d\}}$ are precisely the representatives of those of $X^{\{d\}}_{\xi^{\{d\}}}$, it suffices to show that $\varphi^{\{d\}}|W$ is regular.

Let q be the greatest integer for which a generic fibre $F = F_1 \times \cdots \times F_d$ of $\varphi^{\{d\}}|W$ contains a component F_s of dimension q. Then all the generic fibres of $\varphi^{\{d\}}|_W$ have a q-dimensional component, and $\varphi^{\{d\}}(W) \subset B_q = \varphi(A_q)$. Indeed, by the upper semi-continuity, the fibre dimension can only drop in a small open neighbourhood of a given point. On the other hand, the sum $\mathrm{fbd}_{x_1}\varphi + \cdots + \mathrm{fbd}_{x_d}\varphi$ is already minimal along F, so all its summands must remain constant. The property of being a fibre of dimension q is an open condition on A_q. Hence W is induced by an irreducible component V of A_q with the generic fibre dimension of $\varphi|_V$ equal to q, in the sense that there exists a component V of A_q such that $\varphi^{\{d\}}(W) = \varphi(V)$. By assumption, $\dim_\eta \overline{\varphi(V)} = \dim_\eta \varphi(V)$ (where closure is in the Zariski topology in Y), hence also $\dim_\eta \overline{\varphi^{\{d\}}(W)} = \dim_\eta \varphi^{\{d\}}(W)$; i.e., $\varphi^{\{d\}}|_W$ is Gabrielov regular.

Suppose now that there exists $j \in \{l, \dots, k\}$ for which $\varphi|_{A_j}$ is not regular. We shall show that then regularity of $\varphi^{\{n\}}_{\xi\{n\}} : X^{\{n\}}_{\xi\{n\}} \to Y_\eta$ fails, where $n = \dim Y$.

Fix $j \in \{l, \dots, k\}$ such that $\varphi|_{A_j}$ is not regular. Pick $y \in B_j$ with $\dim \varphi^{-1}(y) = j$, and let Z be an isolated irreducible component of the fibre $(\varphi^{\{n\}})^{-1}(y)$ of dimension nj. Let W be an isolated irreducible component of $X^{\{n\}}$ containing Z. Then $\varphi^{\{n\}}(W) \subset B_j$, by Proposition 4.1(2). Moreover, $\varphi^{\{n\}}|_W$ has no fibres of dimension less than or equal to $n(j-1)$. Indeed, otherwise the generic fibre dimension of $\varphi^{\{n\}}|W$ would be at most $n(j-1)$, so that $\dim W \leq n(j-1)+n = nj = \dim Z$, and hence $W = Z$, a contradiction (see the proof of Proposition 4.1). Thus, a generic fibre $F = F_1 \times \cdots \times F_n$ of $\varphi^{\{n\}}|_W$ contains a component F_s of dimension j.

Now, there is an isolated irreducible component V of A_j such that $\dim_\eta \overline{\varphi(V)} > \dim_\eta \varphi(V)$ and the generic fibre dimension of $\varphi|_V$ is j. Our $y \in B_j$ can then be chosen from $\varphi(V)$, and Z a component of $\left((\varphi|_V)^{-1}(y)\right)^n$. Since being a j-dimensional fibre is an open condition on A_j, then (as in the first part of the proof) we find that $\overline{\varphi^{\{n\}}(W)} = \overline{\varphi(V)}$, so that $\dim_\eta \overline{\varphi^{\{n\}}(W)} > \dim_\eta \varphi^{\{n\}}(W)$. Thus $\varphi^{\{n\}}_{\xi\{n\}} : X^{\{n\}}_{\xi\{n\}} \to Y_\eta$ is not Gabrielov regular. ∎

6. Homological algebra apparatus

This section is devoted to establishing the algebraic tools for the proof of our flatness criterion. First, we recall some homological properties of almost finitely generated modules (see Definition 1.6), established by Galligo and Kwieciński [9], that generalize the corresponding properties of finite modules used in Auslander's [5]. We then generalize a lemma of Auslander [5, Lemma 3.1] to almost finitely generated modules (Lemma 6.3 below). Finally, we recall a lemma of [4] concerning zero-divisors in analytic tensor products.

Let $R = \mathbb{C}\{y_1, \dots, y_n\}$ denote a regular local analytic $\mathbb{C}$-algebra of dimension n. Let $\tilde{\otimes}_R$ denote the analytic tensor product over R, and let $\widetilde{\mathrm{Tor}}^R$ be the corresponding derived functor.

Let F denote an almost finitely generated R-module. We define the *flat dimension* $\mathrm{fd}_R(F)$ of F over R as the minimal length of an R-flat resolution of F (i.e., a resolution by R-flat modules). It is easy to see that

$$\text{(6.1)} \qquad \mathrm{fd}_R(F) = \max\{i \in \mathbb{N} : \widetilde{\mathrm{Tor}}_i^R(F, N) \neq 0 \ \text{ for some } N\}.$$

Indeed, if M is an almost finitely generated R-module, then

$$\text{(6.2)} \qquad M \text{ is } R\text{–flat} \Leftrightarrow \widetilde{\mathrm{Tor}}_1^R(M, R/\mathfrak{m}_R) = 0,$$

where $\mathfrak{m}_R$ is the maximal ideal of R (cf. [14, Prop. 6.2]). Let $(A, \mathfrak{m}_A)$ be a regular local R-algebra such that F is a finite A-module. Then (6.1) follows from (6.2) applied to the kernels of a minimal A-free (hence R-flat) resolution

$$\mathcal{F}_* : \quad \dots \xrightarrow{\alpha_{i+1}} F_{i+1} \xrightarrow{\alpha_i} F_i \xrightarrow{\alpha_{i-1}} \dots \xrightarrow{\alpha_1} F_1 \xrightarrow{\alpha_0} F_0 \to F$$

of F. ($\mathcal{F}_*$ *minimal* means that $\alpha_i(F_{i+1}) \subset \mathfrak{m}_A F_i$, for all $i \in \mathbb{N}$).

The *depth* $\operatorname{depth}_R(F)$ of F as an R-module is defined as the length of a maximal F-sequence in R (i.e., a sequence $a_1, \dots, a_s \in \mathfrak{m}_R$ such that a_j is not a zero-divisor in $F/(a_1, \dots, a_{j-1})F$, for $j = 1, \dots, s$). Since all the maximal F-sequences in R have the same length, depth is well defined: As observed in [9, Lemma 2.4], the classical proof of Northcott-Rees for finitely generated modules (see, e.g., [16, §VI, Prop. 3.1]), carries over to the case of almost finitely generated modules.

Lemma 6.1. *Let M and N be almost finitely generated R-modules. Then the following hold:*

(1) Rigidity of $\widetilde{\operatorname{Tor}}^R$ *[9, Prop. 2.2(4)]. If $\widetilde{\operatorname{Tor}}_{i_0}^R(M,N) = 0$ for some $i_0 \in \mathbb{N}$, then $\widetilde{\operatorname{Tor}}_i^R(M,N) = 0$ for all $i \geq i_0$.*
(2) Auslander–Buchsbaum-type formula *[9, Thm. 2.7].*
$$\operatorname{fd}_R(M) + \operatorname{depth}_R(M) = n.$$
(3) Additivity of flat dimension *[9, Prop. 2.10]. If $\widetilde{\operatorname{Tor}}_i^R(M,N) = 0$ for all $i \geq 1$, then*
$$\operatorname{fd}_R(M) + \operatorname{fd}_R(N) = \operatorname{fd}_R(M \tilde{\otimes}_R N).$$
(4) Verticality of $\widetilde{\operatorname{Tor}}^R$ *(cf. [9, Prop. 4.5]). For all $i \geq 1$, $\widetilde{\operatorname{Tor}}_i^R(M,N)$ is an almost finitely generated R-module, and every element of $\widetilde{\operatorname{Tor}}_i^R(M,N)$ is vertical over R.*

Remark 6.2. In regard to property (4) above, notice that, contrary to the finitely generated case, the analytic $\widetilde{\operatorname{Tor}}_i^R$ need not be torsion R-modules. It is an open problem whether they necessarily contain R-zero-divisors.

Finally, recall the following two lemmas of [4]. The proofs of these results are purely algebraic, and therefore we skip them here (detailed arguments may be found in [4]). Having said that, it is interesting to note the geometric meaning of Lemma 6.3(1) below: If $\varphi_\xi : X_\xi \to Y_\eta$ has no geometric vertical components in the n-fold fibred power, then there are no such components in any d-fold power, for $d = 1, \dots, n$.

Lemma 6.3. ([4, Lemma 3.3], cf. [5, Lemma 3.1]) *Let $A = R\{x\}$ denote a regular local analytic R-algebra, $x = (x_1, \dots, x_m)$. Let F be a finitely generated A-torsion-free module, and let N be a module which is finitely* *generated over* $B = A^{\tilde{\otimes}^j_R}$, *for some* $j \geq 1$. *Suppose that* $F\tilde{\otimes}_R N$ *has no vertical elements over* R. *Then:*

(1) N *has no vertical elements over* R;

(2) $\widetilde{\mathrm{Tor}}_i^R(F, N) = 0$, *for all* $i \geq 1$;

(3) $\mathrm{fd}_R(F) + \mathrm{fd}_R(N) = \mathrm{fd}_R(F\tilde{\otimes}_R N)$.

Lemma 6.4. ([4, Lemma 5.1]) *Let* $R = \mathbb{C}\{y_1, \dots, y_n\}$, *and let* A *and* B *be regular local analytic* R*-algebras. Suppose that* M *and* N *are finite* A*- and* B*-modules (respectively). Let* $g \in A$, $h \in B$, *and* $m \in gM\tilde{\otimes}_R hN$ *all be nonzero elements. If* $m = 0$ *as an element of* $M\tilde{\otimes}_R N$, *then* $g\tilde{\otimes}_R h$ *is a zero-divisor of* $gM\tilde{\otimes}_R hN$.

7. Vertical components and flatness

Our flatness criterion 1.8 follows immediately from a more general Theorem 7.1 below.

Theorem 7.1. ([4, Thm. 1.10]) *Let* R *be a regular local analytic* $\mathbb{C}$*-algebra and let* F *denote an almost finitely generated* R*-module. Let* $n = \dim R$. *Then* F *is* R*-flat if and only if the* n*-fold analytic tensor power* $F^{\tilde{\otimes}^n_R}$ *has no vertical elements over* R.

Proof of Theorem 1.8. Let $R = \mathbb{C}\{y_1, \dots, y_n\}$. Then $\mathcal{O}_{Y,\eta} \cong R$ and $\mathcal{O}_{X,\xi}$ can be viewed as an almost finitely generated R-module. (We can embed (a representative of) X_ξ in $\mathbb{C}^m$ and identify X_ξ with the graph of φ_ξ in $\mathbb{C}^m_\xi \times Y_\eta$ to realize $\mathcal{O}_{X,\xi}$ as a quotient of $R\{x_1, \dots, x_m\}$.) Since $\mathcal{O}_{X,\xi}\tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi} \cong \mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$, the left-hand side has no vertical elements over $\mathcal{O}_{Y,\eta}$ if and only if the n-fold fibred power of X_ξ over Y_η has no geometric vertical components (by Remark 1.5). ∎

We prove Theorem 7.1 by combining the analysis of vertical components with the homological algebraic tools of Auslander [5] (as extended in the previous section). The proof proceeds by induction on the fibre dimension m of A over R. The general phenomenon at the heart of the proof is that one can decompose F into two submodules, such that the homological structure of F as an R-module is inherited from that of one of the submodules.

We have a short exact sequence $0 \to K \to F \to N \to 0$, where K is the A-torsion submodule of F, and N is a suitably chosen A-torsion-free submodule. Then N can be treated by Auslander's techniques of Section 6. The A-torsion module K, in turn, is supported over a subalgebra of A of strictly smaller fibre dimension over R, so it can be treated by induction. The last ingredient of the proof is to show that the vertical elements coming either from the powers of K or those of N, must embed into the corresponding powers of F.

7.1. Proof of Theorem 7.1

We can assume that F is an almost finitely generated module over $R := \mathbb{C}\{y_1, \ldots, y_n\}$. By Remark 1.7, there exists $m \geq 0$ such that F is finitely generated as a module over $A = R\{x\}$, where $x = (x_1, \ldots, x_m)$. Let X and Y be connected open neighbourhoods of the origins in $\mathbb{C}^{m+n}$ and $\mathbb{C}^n$ (respectively), and let $\varphi : X \to Y$ be the canonical coordinate projection. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules whose stalk at the origin in X equals F, and let $\mathcal{G}$ be a coherent $\mathcal{O}_{X^{\{n\}}}$-module whose stalk at the origin $0^{\{n\}}$ in $X^{\{n\}}$ equals $F^{\tilde{\otimes}^n_R}$. We can identify R with $\mathcal{O}_{Y,0}$ and A with $\mathcal{O}_{X,0}$. Then F is R-flat if and only if $\mathcal{F}_0$ is $\mathcal{O}_{Y,0}$-flat.

We first prove the "only if" direction of Theorem 7.1, by contradiction. Assume that F is R-flat. Since flatness is an open condition, by Douady's theorem [6], we can assume that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_Y$-flat. Suppose that $F^{\tilde{\otimes}^n_R}$ has a vertical element over $\mathcal{O}_{Y,0}$. In other words, (after shrinking X and Y if necessary) there exist a nonzero section $\tilde{m} \in \mathcal{G}$ and an analytic subset $Z \subset X^{\{n\}}$, such that $Z_0 = \mathcal{V}(\mathrm{Ann}_{\mathcal{O}_{X^{\{n\}},0^{\{n\}}}}(\tilde{m}_0))$ and the image $\varphi^{\{n\}}(Z)$ has empty interior in Y. Let $\tilde{\varphi}$ denote the restriction $\varphi^{\{n\}}|_Z : Z \to Y$. Consider $\xi \in Z$ such that the fibre dimension of $\tilde{\varphi}$ at ξ is minimal. Then the fibre dimension $\mathrm{fbd}_x\tilde{\varphi}$ is constant on some open neighbourhood U of ξ in Z. By the Remmert Rank Theorem, $\tilde{\varphi}(U)$ is locally analytic in Y near $\eta = \tilde{\varphi}(\xi)$. Since $\tilde{\varphi}(Z)$ has empty interior in Y, it follows that there is a holomorphic function g in a neighbourhood of η in Y, such that $(\tilde{\varphi}(U))_\eta \subset \mathcal{V}(g_\eta)$. Therefore, $\tilde{\varphi}^*_\xi(g_\eta)\cdot\tilde{m}_\xi = 0$ in $\mathcal{G}_\xi$; i.e., $\mathcal{G}_\xi$ has a (nonzero) zero-divisor in $\mathcal{O}_{Y,\eta}$, contradicting flatness.

We will now prove the more difficult "if" direction of the theorem, by induction on m. If $m = 0$, then F is finitely generated over R, and the result follows from Auslander's theorem 1.1 (because flatness of finitely generated modules over a local ring is equivalent to freeness, the analytic tensor product equals the ordinary tensor product for finite modules, and vertical elements in finite modules are just zero-divisors).

The inductive step will be divided into three cases:

(1) F is torsion-free over A;
(2) F is a torsion A-module;
(3) F is neither A-torsion-free nor a torsion A-module.

Case (1). We prove this case independently of the inductive hypothesis. We essentially repeat the argument of Galligo and Kwieciński [9], which itself is an adaptation of Auslander [5] to the almost finitely generated context.

Suppose that $F^{\tilde{\otimes}^n_R}$ has no vertical elements over R. Then it follows from Lemma 6.3(1) that $F^{\tilde{\otimes}^i_R}$ has no vertical elements, for $i = 1, \ldots, n$. By

Lemma 6.3(3),

$$\mathrm{fd}_R(F^{\tilde{\otimes}^n_R}) = \mathrm{fd}_R(F) + \mathrm{fd}_R(F^{\tilde{\otimes}^{n-1}_R}) = \cdots = n \cdot \mathrm{fd}_R(F).$$

On the other hand, since $F^{\tilde{\otimes}^n_R}$ has no vertical elements over R, it has no zero-divisors over R, so that $\mathrm{depth}_R(F^{\tilde{\otimes}^n_R}) \geq 1$. It follows from Lemma 6.1(2) that $\mathrm{fd}_R(F^{\tilde{\otimes}^n_R}) < n$. Hence $n \cdot \mathrm{fd}_R(F) < n$. This is possible only if $\mathrm{fd}_R(F) = 0$; i.e., F is R-flat.

Case (2). Suppose that F is not R-flat. We will show that then $F^{\tilde{\otimes}^n_R}$ contains vertical elements over R. Let $I = \mathrm{Ann}_A(F)$. Since every element of F is annihilated by some nonzero element of A, and F is finitely generated over A, then I is a nonzero ideal in A. Put $B = A/I$; then F is finitely generated over B. Let $I(0)$ denote the *evaluation* of I *at* $y = 0$ (i.e., $I(0)$ is the ideal generated by I in $A(0) := A\tilde{\otimes}_R R/\mathfrak{m}_R \cong \mathbb{C}\{x_1, \ldots, x_m\}$).

First suppose that $I(0) \neq (0)$. Then there exists $g \in I$ such that $g(0) := g(0, x) \neq 0$, and F is a finite $A/(g)A$-module. It follows that (after an appropriate linear change in the x-coordinates) g is regular in x_m and hence, by the Weierstrass Preparation Theorem, that F is finite over $R\{x_1, \ldots, x_{m-1}\}$. Therefore, $F^{\tilde{\otimes}^n_R}$ has a vertical element over R, by the inductive hypothesis.

On the other hand, suppose that $I(0) = (0)$. Then $B\tilde{\otimes}_R R/\mathfrak{m}_R = A(0)/I(0)$ equals $\mathbb{C}\{x_1, \ldots, x_m\}$. Let Z be a closed analytic subspace of X such that $\mathcal{O}_{Z,0} \cong B$, and let $\tilde{\varphi} := \varphi|_Z$. It follows that the fibre $\tilde{\varphi}^{-1}(0)$ equals $\mathbb{C}^m$. Of course, m is not the generic fibre dimension of $\tilde{\varphi}$ restricted to any component, because otherwise all its fibres would equal $\mathbb{C}^m$, so we would have $B = A$ and $I = (0)$, contrary to the choice of I. Therefore, by Proposition 4.3, there is an isolated algebraic vertical component in the n-fold fibred power of $\tilde{\varphi}_0$; i.e., $B^{\tilde{\otimes}^n_R}$ has a zero-divisor in R. But $F^{\tilde{\otimes}^n_R}$ is a finitely generated $B^{\tilde{\otimes}^n_R}$-module, so itself it has a zero-divisor (hence a vertical element) over R.

Case (3). Suppose that F is not R-flat, F has zero-divisors in A, but $\mathrm{Ann}_A(F) = (0)$. Let

$$K := \{f \in F : af = 0 \text{ for some nonzero } a \in A\};$$

i.e., K is the A-torsion submodule of F. Since K is a submodule of a finitely generated module over a Noetherian ring, K is finitely generated; say $K = \sum_{i=1}^{s} A\cdot f_i$. Take $a_i \in A \setminus \{0\}$ such that $a_i f_i = 0$, and put $g = a_1 \ldots a_s$. Then the sequence of A-modules

$$0 \to K \to F \xrightarrow{\cdot g} gF \to 0 \tag{7.1}$$

is exact, and gF is a torsion-free A-module.

First suppose that gF is R-flat. By applying $\tilde{\otimes}_R K$ and $F\tilde{\otimes}_R$ to (7.1), we get short exact sequences

$$0 \to K\tilde{\otimes}_R K \to F\tilde{\otimes}_R K \to gF\tilde{\otimes}_R K \to 0,$$
$$0 \to F\tilde{\otimes}_R K \to F\tilde{\otimes}_R F \to F\tilde{\otimes}_R gF \to 0.$$

So we have injections

$$K\tilde{\otimes}_R K \hookrightarrow F\tilde{\otimes}_R K \hookrightarrow F\tilde{\otimes}_R F,$$

and by induction, an injection $K^{\tilde{\otimes}_R^i} \hookrightarrow F^{\tilde{\otimes}_R^i}$, for all $i \geq 1$. In particular, $K^{\tilde{\otimes}_R^n}$ is a submodule of $F^{\tilde{\otimes}_R^n}$. Since gF is R-flat and F is not R-flat, it follows that K is not R-flat. Therefore, by Case (2), $K^{\tilde{\otimes}_R^n}$ (and hence $F^{\tilde{\otimes}_R^n}$) has a vertical element over R.

Now suppose that gF is not R-flat. Then $(gF)^{\tilde{\otimes}_R^n}$ has a vertical element over R, by Case (1). We will show that $(gF)^{\tilde{\otimes}_R^n}$ embeds into $F^{\tilde{\otimes}_R^n}$, and hence so do its vertical elements. By Lemma 6.4, in order for $(gF)^{\tilde{\otimes}_R^n}$ to embed into $F^{\tilde{\otimes}_R^n}$, it suffices to prove that $g^{\tilde{\otimes}_R^n}$ is not a zero-divisor of $(gF)^{\tilde{\otimes}_R^n}$.

To simplify the notation, let B denote the ring $A^{\tilde{\otimes}_R^n}$, and let $h := g^{\tilde{\otimes}_R^n} \in B$. Since $(gF)^{\tilde{\otimes}_R^n}$ is a finite B-module, we can write $(gF)^{\tilde{\otimes}_R^n} = B^q/M$, where $q \geq 1$ and M is a B-submodule of B^q. Given $b \in B$, let $M : b$ denote the B-submodule of B^q consisting of those elements $m \in B^q$ for which $b{\cdot}m \in M$. Since

$$M : h \subset M : h^2 \subset \cdots \subset M : h^l \subset \dots$$

is an increasing sequence of submodules of a Noetherian module B^q, it stabilizes; i.e., there exists $k \geq 1$ such that $M : h^{k+1} = M : h^k$. In other words, there exists $k \geq 1$ such that h is not a zero-divisor in $h^k{\cdot}B^q/M$; i.e., $g^{\tilde{\otimes}_R^n}$ is not a zero-divisor in $(g^{k+1}F)^{\tilde{\otimes}_R^n}$.

Observe though that (because gF is A-torsion-free), multiplication by g induces an isomorphism $gF \to g^2F$ of A-modules, and in general, $gF \cong g^lF$, for $l \geq 1$. We thus have isomorphisms $(gF)^{\tilde{\otimes}_R^n} \cong (g^lF)^{\tilde{\otimes}_R^n}$ of B-modules, for $l \geq 1$. In particular, for every $l \geq 1$, $g^{\tilde{\otimes}_R^n}$ is a zero-divisor of $(gF)^{\tilde{\otimes}_R^n}$ if and only if it is a zero-divisor of $(g^lF)^{\tilde{\otimes}_R^n}$. Therefore, by Lemma 6.4, we have an embedding $(gF)^{\tilde{\otimes}_R^n} \hookrightarrow F^{\tilde{\otimes}_R^n}$. This completes the proof of Theorem 7.1. ∎

7.2. Proof of Theorem 1.9

In light of Remark 1.3, the "only if" implication of the theorem is automatic (without any assumptions on φ_ξ). Indeed, if $\mathcal{O}_{X,\xi}$ is $\mathcal{O}_{Y,\eta}$-flat, then so are all its analytic tensor powers (as flatness is preserved by any base change and compositions of flat mappings are flat). In particular, the n-fold power $\mathcal{O}_{X,\xi}^{\tilde{\otimes}^n_{\mathcal{O}_{Y,\eta}}} \cong \mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ is flat, hence torsion-free, over $\mathcal{O}_{Y,\eta}$.

For the proof of the "if" direction we will proceed in four cases, according to the conditions in Theorem 1.9.

Case (1). Suppose that $\dim Y_\eta = n < 3$ and $\varphi_\xi : X_\xi \to Y_\eta$ is not flat. Then, by Theorem 1.8, there exists a geometric vertical component in $X^{\{n\}}_{\xi^{\{n\}}}$. We will show that the restriction of $\varphi^{\{n\}}_{\xi^{\{n\}}}$ to this component is Gabrielov regular, and hence the component is algebraic vertical.

In fact, every morphism $\psi_\zeta : Z_\zeta \to Y_\eta$ of germs of analytic spaces with target of dimension less than 3 is regular: By the Remmert Rank Theorem the Gabrielov-non-regular locus $\mathrm{NR}(\psi)$ is contained in the locus of non-generic fibre dimension of ψ, and thus the image of $\mathrm{NR}(\psi)$ is of codimension (at least) two in the image $\psi(Z)$. If Z_ζ is geometric vertical, then the image $\psi(Z)$ is already of codimension (at least) one in Y, and hence

$$\dim \psi(\mathrm{NR}(\psi)) \leq \dim Y - 1 - 2 \leq -1.$$

Thus, $\psi(\mathrm{NR}(\psi)) = \varnothing$ and ψ is Gabrielov regular.

Case (2). Suppose that $\varphi_\xi : X_\xi \to Y_\eta$ is a non-flat Nash morphism of Nash germs. By Theorem 1.8 again, there exists a geometric vertical component in $X^{\{n\}}_{\xi^{\{n\}}}$. Such a component is necessarily a Nash germ itself, by [2, Prop. 5.1], and hence the restriction of $\varphi^{\{n\}}_{\xi^{\{n\}}}$ to this component is Gabrielov regular, by [23, Thm. 2.10] and Chevalley Theorem [20, Ch. VII, § 8.3].

Case (3). Suppose now that X_ξ is pure-dimensional, $\varphi_\xi : X_\xi \to Y_\eta$ maps the singular locus of X_ξ into a proper analytic subgerm Z_η of Y_η, and $X^{\{n\}}_{\xi^{\{n\}}}$ has no algebraic vertical components. In particular, there are no *isolated* algebraic vertical components in $X^{\{n\}}_{\xi^{\{n\}}}$, and hence φ_ξ is open, by Theorem 1.10.

Openness being an open condition, we can extend $\varphi_\xi : X_\xi \to Y_\eta$ to an open holomorphic mapping $\varphi : X \to Y$ of analytic spaces, where Y is smooth of dimension n. Moreover, this can be done so that Z_η extends to a proper analytic subset Z of Y with $\varphi(X^{\mathrm{sng}}) \subset Z$. We may now conclude that φ is flat over $Y \setminus Z$, as for a mapping of smooth spaces openness is equivalent to flatness (see, e.g., [8, Prop. 3.20]). Hence also $\varphi^{\{d\}} : X^{\{d\}} \to Y$ is flat over $Y \setminus Z$ for every $d \geq 1$, because this is so locally.

For simplicity of notation, put $R := \mathcal{O}_{Y,\eta}$, $M := \mathcal{O}_{X,\xi}$, and $N := \mathcal{O}_{X^{\{n-1\}},\xi^{\{n-1\}}}$. We shall show that $\widetilde{\mathrm{Tor}}^R_i(M,N) = 0$ for all $i \geq 1$. By torsion-freeness of $M \tilde{\otimes}_R N = \mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ and injectivity of the morphism

$$M \ni m \mapsto m \tilde{\otimes}_R 1 \tilde{\otimes}_R \dots \tilde{\otimes}_R 1 \in M \tilde{\otimes}_R N,$$

we get that M is torsion-free over R. Then, by [9, Lem. 5.2], we have an exact sequence of almost finitely generated R-modules

$$0 \to M \to F \to F/M \to 0,$$

where F is R-flat. After tensoring with N, we get an exact sequence

$$0 \to \widetilde{\mathrm{Tor}}_1^R(F/M, N) \xrightarrow{\lambda} M\tilde{\otimes}_R N \to F\tilde{\otimes}_R N \to F/M\tilde{\otimes}_R N \to 0$$

and isomorphisms

$$\widetilde{\mathrm{Tor}}_{i+1}^R(F/M, N) \cong \widetilde{\mathrm{Tor}}_i^R(M, N) \quad \text{for all } i \geq 1. \tag{7.2}$$

Flatness of $\varphi^{\{n-1\}} : X^{\{n-1\}} \to Y$ over $Y \setminus Z$ implies that $\widetilde{\mathrm{Tor}}_1^R(F/M, N)$ is supported over Z_η; i.e., $m \cdot r = 0$ for every $m \in \widetilde{\mathrm{Tor}}_1^R(F/M, N)$ and $r \in \mathfrak{I}$, where $\mathfrak{I} \subset R$ is the ideal of Z_η. Since $Z_\eta \subsetneq Y_\eta$, it follows that there is a nonzero $r \in R$ such that $r \cdot \lambda(m) = 0$ for every $m \in \widetilde{\mathrm{Tor}}_1^R(F/M, N)$. Hence $\widetilde{\mathrm{Tor}}_1^R(F/M, N) = 0$, by injectivity of λ and torsion-freeness of $M\tilde{\otimes}_R N$. Consequently, $\widetilde{\mathrm{Tor}}_{i+1}^R(F/M, N) = 0$, by rigidity of $\widetilde{\mathrm{Tor}}^R$ (Lemma 6.1(1)), and so $\widetilde{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \geq 1$, by (7.2).

Therefore, as in the proof of Case (1) of Theorem 7.1 above,

$$n - 1 \geq \mathrm{fd}_R(M\tilde{\otimes}_R N) = \mathrm{fd}_R(M) + \mathrm{fd}_R(N) = \ldots = n \cdot \mathrm{fd}_R(M),$$

by Lemma 6.3(3), so that $\mathrm{fd}_R(M) = 0$; i.e., $\mathcal{O}_{X,\xi}$ is $\mathcal{O}_{Y,\eta}$-flat.

Case (4). Let $\varphi_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, where $\mathcal{O}_{X,\xi}$ is Cohen–Macaulay and $\mathcal{O}_{Y,\eta}$ is regular. Then, by [8, Prop. 3.20], φ_ξ is flat if and only if it is open. Hence, if there are no algebraic vertical components in $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$, the flatness of φ_ξ follows from Theorem 1.10. ∎

8. Algebraic case

In this final section, we present the algebraic analogues of our openness and flatness criteria (Theorems 1.10 and 7.1). Although these are not exactly within the main scope of this survey, Theorems 8.1 and 8.2 below provide important motivation for our main work in that they allow for actual *computation* of openness and flatness, as explained in Remark 8.4.

Theorem 8.1. *Let $\varphi : X \to Y$ be a polynomial mapping of affine algebraic varieties (over $\mathbb{C}$). Assume that Y is (analytically) locally irreducible, of dimension n. Then φ is open if and only if the coordinate ring $A(X^{\{n\}})$ of the n-fold fibred power $X^{\{n\}}$ is a torsion-free $A(Y)$-module.*

The following flatness criterion generalizes Vasconcelos's conjecture [24, Conj. 6.2] in the case of $\mathbb{C}$-algebras.

Theorem 8.2. ([4, Thm. 1.3]) *Let R be a regular $\mathbb{C}$-algebra of finite type. Let A denote an R-algebra of finite type and let F denote a finitely generated*

A-module. If $\mathfrak{p} \subset A$ is a prime ideal, then the localization $F_\mathfrak{p}$ is R-flat if and only if the n-fold tensor power $F_\mathfrak{p}^{\otimes_R^n}$ is a torsion-free R-module, where $n = \dim R$.

Equivalently, if $\mathfrak{p} \subset A$ is a prime ideal and $\mathfrak{q} = \mathfrak{p} \cap R$, then $F_\mathfrak{p}$ is $R_\mathfrak{q}$-flat if and only if $F_\mathfrak{p}^{\otimes_{R_\mathfrak{q}}^n}$ is $R_\mathfrak{q}$-torsion-free.

COROLLARY 8.3. *With the assumptions of Theorem 8.2, F is R-flat if and only if $F^{\otimes_R^n}$ is R-torsion-free.*

REMARK 8.4. By Corollary 8.3, and the prime-avoidance lemma [7, Lemma 3.3], in order to verify that F is not R-flat, it is enough to find an associated prime of $F^{\otimes_R^n}$ in $A^{\otimes_R^n}$ which contains a nonzero element $r \in R$. Similarly, by Theorem 8.1, to verify non-openness of a polynomial map $\varphi : X \to Y$, it is enough to find a *minimal* associated prime of $A(X)^{\otimes_{A(Y)}^n}$ which contains a nonzero element $r \in A(Y)$.

In other words, the study of openness and flatness in the algebraic case, boils down to showing that the contraction of a certain prime ideal to the base ring is non-trivial. Thus Theorems 8.2 and 8.1 together with Gröbner-basis algorithms for primary decomposition (see [25] or [11]) provide tools for checking flatness and openness by effective computation.

Proof of Theorem 8.1. For simplicity of notation, write $R := A(Y)$ and $S := A(X)$. First suppose that φ is open. Then it is universally open (by [12, Cor. 14.4.3]), and hence all its fibred powers are open, too. In particular, the canonical mapping $\varphi^{\{n\}} : X^{\{n\}} \to Y$ is dominating, when restricted to every *isolated* irreducible component of $X^{\{n\}}$. Consequently, the minimal associated primes of $S^{\otimes_R^n}$ contract to zero in R; i.e., the coordinate ring $A(X^{\{n\}}) = (S^{\otimes_R^n})_{\mathrm{red}}$ is a torsion-free R-module.

Now suppose that φ is not open. Then it is not open at some point $\xi \in X$, and so $\varphi_\xi : X_\xi \to Y_\eta$ is a non-open morphism of germs of analytic spaces. By Theorem 1.10 and Remark 1.11, the reduced local ring $(\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}})_{\mathrm{red}}$ has a zero-divisor in $\mathcal{O}_{Y,\eta}$, where $\eta = \varphi(\xi)$. Hence $(S^{\otimes_R^n})_{\mathrm{red}}$ is not torsion-free over R. ∎

Proof of Theorem 8.2. If $F_\mathfrak{p}$ is R-flat, then $F_\mathfrak{p}^{\otimes_R^k}$ is R-flat and therefore R-torsion-free, for all k.

On the other hand, flatness and torsion-freeness are both local properties. I.e., $F_\mathfrak{p}$ is R-flat (respectively, R-torsion-free) if and only if $F_\mathfrak{m}$ is R-flat (respectively, R-torsion-free), for every maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{p}$. Suppose that $F_\mathfrak{p}$ is not R-flat. Then there is a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$ such that $F_\mathfrak{m}$ is not R-flat, and it suffices to prove that $F_\mathfrak{m}^{\otimes_R^n}$ is not R-torsion-free. Let $\varphi : X \to Y$ be the morphism of complex-analytic spaces associated to the morphism Spec $A \to$ Spec R and let $\mathcal{F}$ be the coherent sheaf of $\mathcal{O}_X$-modules associated to F. Let $\xi \in X$ be the point corresponding to the maximal ideal $\mathfrak{m}$ of Spec A. It follows from faithful flatness of completion that $\mathcal{F}_\xi$ is not $\mathcal{O}_{Y,\eta}$-flat, where $\eta = \varphi(\xi)$. By Theorem 7.1, $\mathcal{F}_\xi^{\tilde{\otimes}^n_{\mathcal{O}_{Y,\eta}}}$ has a vertical element over $\mathcal{O}_{Y,\eta}$. Since $\varphi^{\{n\}}$ is the holomorphic map induced by the ring homomorphism $R \to A^{\otimes^n_R}$, it follows from Chevalley's Theorem that $\mathcal{F}_\xi^{\tilde{\otimes}^n_{\mathcal{O}_{Y,\eta}}}$ has a zero-divisor in $\mathcal{O}_{Y,\eta}$. Then $F_\mathfrak{m}^{\otimes^n_R}$ has a zero-divisor in R; i.e., $F_\mathfrak{m}^{\otimes^n_R}$ is not R-torsion-free. ∎

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