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D. O'Regan, M. Abbas

NECESSARY AND SUFFICIENT CONDITIONS FOR COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACE

Abstract. The aim of this paper is to provide a necessary and sufficient condition for the existence of a common fixed point of three maps f, g and T in a complete fuzzy metric space under a general contractive condition. A common fixed point theorem for a pair of weakly biased mappings, which is more general than weakly compatible mappings is also proved.

1. Introduction and preliminaries

The evolution of fuzzy mathematics commenced with an introduction of the notion of fuzzy sets by Zadeh [18] in 1965, as a new way to represent vagueness in every day life. The concept of a fuzzy metric space has been introduced and generalized in many ways ([2], [10]). Moreover George and Veeramani ([4], [5]) modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [9]. They obtained a Hausdorff topology for this kind of fuzzy metric space which has applications in quantum particle physics, particularly in connection with both string and ϵ^{∞} theory (see, [3] and references mentioned therein). Many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces ([1], [11], [13], [15], [17]). In particular Pfeffer [14] proves that any involution r of a circle S has a fixed point iff there exists a free involution ($\neq r$) of S which commutes with r. This result shows an interdependence between commuting mappings and fixed point concepts. Jungck ([7] , [8]) further highlighted this interdependence in a more general context. This paper deals with necessary and sufficient conditions for a common fixed point of three self maps f, g and T, in which the pair $\{f, T\}$ is compatible and $\{g, T\}$ is weakly compatible.

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Moreover, this result does not require the continuity of f and g. A result proving the existence of a common fixed point for a pair of weakly biased mappings is also established in a complete fuzzy metric space for an arbitrary t-norm. An example, which illustrates the fact that the notion of weakly biased mappings is more general than that of weakly compatible, is also presented.

For sake of completeness, following [6] and [16], we recall some definitions and known results in a fuzzy metric space.

Definition 1.1. ([18]) Let X be any set. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 1.2. ([16]) A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$, for $a \leq c$, $b \leq d$. Three typical examples of t-norms are $a * b = \min\{a, b\}$ (minimum t-norm), $a * b = ab$ (product t-norm), and $a * b = \max\{a + b - 1, 0\}$ (Łukasiewicz t-norm).

Definition 1.3. The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(a) $M(x, y, t) > 0$,
(b) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(e) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous,

for each $x, y, z \in X$ and $t, s > 0$.

Note that, $M(x, y, t)$ can be thought of as the definition of nearness between x and y with respect to t. It is known that $M(x, y, .)$ is nondecreasing for all $x, y \in X$.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

The collection $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Let $(X, M, *)$ be a fuzzy metric space with the following condition

$$\lim_{t \to \infty} M(x, y, t) = 1, \text{ for all } x, y \in X. \tag{1.1}$$

For each $\mu \in (0,1)$, we know there exists $\lambda \in (0,1)$ (which may depend on n) such that

$$(1.2) \qquad \underbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}_{n \text{ copies}} \geq 1-\mu.$$

In [12] we assumed for each $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ (which does not depend on n) such that

$$(1.3) \qquad \underbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}_{n \text{ copies}} \geq 1-\mu.$$

We will make some remarks concerning (1.3) later.

We assume that a fuzzy metric space $(X, M, *)$ satisfies conditions (1.1) throughout this paper.

A sequence $\{x_n\}$ in X converges to x ([6]) if and only if for each $0 < \varepsilon < 1$, and each $t > 0$ there exists $n_0 \in N$ with

$$M(x_n, x, t) > 1 - \varepsilon$$

for all $n \geq n_0$.

A sequence $\{x_n\}$ in X is a Cauchy sequence ([6]) if and only if for each $0 < \varepsilon < 1$, and each $t > 0$ and $p \in N$ there exists $n_0 \in N$ with

$$M(x_n, x_{n+p}, t) > 1 - \varepsilon$$

for all $n \geq n_0$.

A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete. George and Veeramani [4] showed that a sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to a point x in X if and only if $\lim_{n \to \infty} M(x_n, x, t) = 1$. They also showed that $(R, M, *)$ is not complete in the sense of [6], where $M(x, y, t) = \dfrac{t}{t + d(x,y)}$ and $d(x, y) = |x - y|$. Recall that, $(R, M, *)$ is called the standard fuzzy metric space. To make R (the set of all real numbers), a complete fuzzy metric space, they presented a new definition of a Cauchy sequence: a sequence $\{x_n\}$ in a fuzzy metric space is a Cauchy sequence, if and only if for each $\varepsilon > 0$, and each $t > 0$ there exists $n_0 \in N$ with

$$M(x_n, x_m, t) > 1 - \varepsilon$$

for all $n, m \geq n_0$. This is the definition we will use in this paper.

LEMMA 1.4. ([11]) *If, for all $x, y \in X, t > 0$, and for a number $q \in (0,1)$,*

$$M(x, y, qt) \geq M(x, y, t),$$

then $x = y$.

Lemma 1.5. ([1]) *Let $(X, M, *)$ be a fuzzy metric space. If, for each $\lambda \in (0,1)$, we define $E_{\lambda,M} : X^2 \to R^+ \cup \{0\}$ as*

$$E_{\lambda,M}(x,y) = \inf\{t > 0 : M(x,y,t) > 1-\lambda\},$$

then for each $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2) + E_{\lambda,M}(x_2,x_3) + \cdots + E_{\lambda,M}(x_{n-1},x_n)$$

for any $x_1, x_2, \ldots, x_n \in X$. Also the sequence $\{x_n\}$ is convergent in a fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also the sequence $\{x_n\}$ is Cauchy in fuzzy metric space $(X, M, *)$ if and only if it is Cauchy in $E_{\lambda,M}$.
(note if (1.3) holds then the λ in Lemma 1.5 can be chosen independent of n).

The following is the special case of Lemma 1.14 in [12].

Lemma 1.6. *Let $(X, M, *)$ be a fuzzy metric space, and $\phi : [0,\infty) \to [0,\infty)$ be an onto and strictly increasing function. Then*

$$\inf\{\phi^n(t) > 0 : M(x,y,t) > 1-\lambda\} \leq \phi^n(\inf\{t > 0 : M(x,y,t) > 1-\lambda\})$$

for every $x, y \in X$, $\lambda \in (0,1)$, and $n \in N$.

Definition 1.7. ([11]) Let f and g be self maps on a fuzzy metric space $(X, M, *)$. They are compatible (or asymptotically commuting) if,

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

Definition 1.8. The mappings f and g from a fuzzy metric space $(X, M, *)$ into itself are weakly compatible if they commute at their coincidence point, that is $fx = gx$ implies that $fgx = gfx$.

It is known that a pair $\{f, g\}$ of compatible maps is weakly compatible but converse is not true in general.

2. Common fixed point theorems

Let $\phi : [0,\infty) \to [0,\infty)$ be an onto and strictly increasing function satisfying $\sum_{j=1}^{\infty} \phi^j(t) < \infty$ for $t > 0$. We note that $\lim_{n\to\infty} \phi^n(t) = 0$, for each $t > 0$, $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$.

The following result provides necessary and sufficient condition for the existence of common fixed point of three maps in a fuzzy metric space for any t-norm.

THEOREM 2.1. *Let f and g be maps from a fuzzy metric space $(X, M, *)$ into itself.*

(a) *Suppose (1.2) holds. Then f and g have a common fixed point in X if and only if there exists a continuous mapping $T : X \to fX \cap gX$ such that the pair $\{f, T\}$ is compatible and $\{g, T\}$ is weakly compatible, a point x_0 in X with $E_M(fx_0, Tx_0) = \sup\{E_{\lambda,M}(fx_0, Tx_0) : \lambda \in (0, 1)\} < \infty$, and*

$$M(Tx, Ty, \phi(t)) \geq M(fx, gy, t) \tag{2.1}$$

for all $x, y \in X$, and $t > 0$. Indeed, f, g and T have a unique common fixed point.

(b) *Suppose (1.3) holds. Then f and g have a common fixed point in X if and only if there exists a continuous mapping $T : X \to fX \cap gX$ such that the pair $\{f, T\}$ is compatible and $\{g, T\}$ is weakly compatible, and (2.1) is satisfied. Indeed, f, g and T have a unique common fixed point.*

Proof (a). Suppose $fz = z = gz$, for some z in X. Define a mapping T by $Tx = z$, for all x in X. Obviously, T is a continuous mapping of X into $fX \cap gX$ and T commutes with f and g and hence $\{f, T\}$ is compatible and $\{g, T\}$ is weakly compatible. Take, $x_0 = z$, and note $E_M(fx_0, Tx_0) < \infty$. Further, for any $t > 0$, we have

$$M(Tx, Ty, \phi(t)) = M(z, z, \phi(t)) = 1 \geq M(fx, gy, t),$$

for all $x, y \in X$. This proves the necessity of the condition. Now, conversely suppose we have a mapping T and a point x_0 as described in the statement of Theorem 2.1. Define $y_0 = fx_0$. Since $TX \subseteq gX$, we can choose a point x_1 in X such that $gx_1 = Tx_0 = y_1$. Also $TX \subseteq fX$ gives a point x_2 in X such that $fx_2 = Tx_1 = y_2$. In general, having chosen the point x_{2n-2}, we choose a point x_{2n-1} such that $gx_{2n-1} = Tx_{2n-2} = y_{2n-1}$, and a point x_{2n} such that $fx_{2n} = Tx_{2n-1} = y_{2n}$. From (2.1), we obtain

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, \phi(t)) &= M(Tx_{2n}, Tx_{2n+1}, \phi(t)) \\ &\geq M(fx_{2n}, gx_{2n+1}, t) = M(y_{2n+1}, y_{2n}, t) \end{aligned}$$

which implies that,

$$M(y_{2n+1}, y_{2n+2}, \phi(t)) \geq M(y_{2n}, y_{2n+1}, t). \tag{2.2}$$

Similarly,

$$\begin{aligned} M(y_{2n+2}, y_{2n+3}, \phi(t)) &= M(Tx_{2n+2}, Tx_{2n+1}, \phi(t)) \\ &\geq M(y_{2n+1}, y_{2n+2}, t). \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we arrive at

$$M(y_n, y_{n+1}, \phi(t)) \geq M(y_{n-1}, y_n, t).$$

Consequently,

$$\begin{aligned} M(y_n, y_{n+1}, \phi^n(t)) &\geq M(y_{n-1}, y_n, \phi^{n-1}(t)) \geq M(y_{n-2}, y_{n-1}, \phi^{n-2}(t)) \\ &\geq \cdots \geq M(y_1, y_2, \phi(t)) \geq M(y_0, y_1, t). \end{aligned}$$

Now for each $\lambda \in (0,1)$,

$$\begin{aligned} E_{\lambda,M}(y_n, y_{n+1}) &= \inf\{\phi^n(t) > 0 : M(y_n, y_{n+1}, \phi^n(t)) > 1-\lambda\} \\ &\leq \inf\{\phi^n(t) > 0 : M(y_0, y_1, t) > 1-\lambda\} \\ &\leq \phi^n(\inf\{t > 0 : M(y_0, y_1, t) > 1-\lambda\}) \\ &= \phi^n(E_{\lambda,M}(y_0, y_1)) \leq \phi^n(E_M(y_0, y_1)). \end{aligned}$$

Thus $E_M(y_n, y_{n+1}) \leq \phi^n(E_M(y_0, y_1))$. From Lemma 1.5, for each $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that for $m > n$, we have

$$\begin{aligned} E_{\mu,M}(y_n, y_m) &\leq E_{\lambda,M}(y_n, y_{n+1}) + E_{\lambda,M}(y_{n+1}, y_{n+2}) + \cdots + E_{\lambda,M}(y_{m-1}, y_m) \\ &\leq E_M(y_n, y_{n+1}) + E_M(y_{n+1}, y_{n+2}) + \cdots + E_M(y_{m-1}, y_m) \\ &\leq \phi^n(E_M(y_0, y_1)) + \phi^{n+1}(E_M(y_0, y_1)) + \cdots + \phi^{m-1}(E_M(y_0, y_1)) \\ &\leq \sum_{j=n}^{m-1} \phi^j(E_M(y_0, y_1)), \end{aligned}$$

thus $E_M(y_n, y_m) \leq \sum_{j=n}^{\infty} \phi^j(E_M(y_0, y_1))$. Since, $\sum_{j=1}^{\infty} \phi^j(t) < \infty$ for $t > 0$, we have that $\{y_n\}$ is a Cauchy sequence in X. Now since X is complete there exists a point z in X such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n-1} = z$. Since T is continuous and the pair $\{f, T\}$ is compatible, we obtain $Tz = \lim_{n\to\infty} fTx_{2n}$. Next we show that $Tz = z$. Now for any $t > 0$,

$$M(TTx_{2n}, Tx_{2n-1}, \phi^{k+1}(t)) \geq M(fTx_{2n}, gx_{2n-1}, \phi^k(t)),\ k \in \{0, 1, 2, \dots\}.$$

Let $n \to \infty$ to obtain

$$M(Tz, z, \phi^{k+1}(t)) \geq M(Tz, z, \phi^k(t)),$$

and so for any $k \in \{0, 1, 2, \dots\}$ we have

$$M(Tz, z, \phi^{k+1}(t)) \geq M(Tz, z, t). \tag{2.4}$$

Since $\phi(t) < t$ for $t > 0$ and $M(x, y, t)$ is a nondecreasing function with respect to t, for all $x, y \in X$, we have for any $t > 0$ that

$$M(Tz, z, \phi^{k+1}(t)) \leq M(Tz, z, t). \tag{2.5}$$

From (2.4) and (2.5), $M(Tz, z, t) = C$. Taking $t \to \infty$, we obtain $C = 1$ and hence $Tz = z$. Since $TX \subseteq fX$, there exists $t \in X$ such that $ft = Tz$. Next, we claim that $Tt = z$. From (2.1),

$$M(Tt, Tx_{2n-1}, \phi(t)) \geq M(ft, gx_{2n-1}, t).$$

Let $n \to \infty$ to obtain

$$M(Tt, z, \phi(t)) \geq M(ft, z, t) = M(z, z, t) = 1.$$

Now since, $M(Tt, z, t) \geq M(Tt, z, \phi(t))$, we have $M(Tt, z, t) = 1$ for every $t > 0$, and hence, $Tt = z = Tz = ft$. Now $\{f, T\}$ is compatible, $Tt = ft$ implies that $Tft = fTt$ and hence $fz = z$. As $TX \subseteq gX$, there exists a $u \in X$ such that $Tz = gu$. Now we show that $Tu = z$. From (2.1), for every $t > 0$, we have

$$M(Tz, Tu, \phi(t)) \geq M(fz, gu, t) = M(z, z, t) = 1,$$

so,

$$M(Tz, Tu, t) \geq M(Tz, Tu, \phi(t)) \geq 1.$$

Thus $Tu = Tz$. Since $\{T, g\}$ is weakly compatible, $Tu = gu$ implies that $gTu = Tgu$, and hence $gz = gTu = Tgu = Tz = z$, thus z is a common fixed point of f, g and T which establishes the sufficiency of the condition. Next, suppose that u_1 and u_2 are two common fixed points of f, g and T. Now, for any $k \in \{0, 1, 2, \dots\}$, and $t > 0$ we have

$$\begin{aligned} M(u_1, u_2, \phi^{k+1}(t)) &= M(Tu_1, Tu_2, \phi^{k+1}(t)) \\ &\geq M(fu_1, gu_2, \phi^k(t)) = M(u_1, u_2, \phi^k(t)). \end{aligned}$$

Thus,

$$M(u_1, u_2, \phi^{k+1}(t)) \geq M(u_1, u_2, \phi^k(t)) \geq \cdots \geq M(u_1, u_2, t).$$

On the other hand we know for any $t > 0$ that we have,

$$M(u_1, u_2, t) \geq M(u_1, u_2, \phi^{k+1}(t)),$$

and so we have $M(u_1, u_2, t) = C$. Let $t \to \infty$, and we obtain $C = 1$ and hence $u_1 = u_2$.

Proof (b). The argument is as in case (a) except we use the last two lines of Lemma 1.5.

Corollary 2.2. *Let f and g be maps from a fuzzy metric space $(X, M, *)$ into itself, where $*$ is a minimum t-norm. Then f and g have a common fixed point in X if and only if there exists a continuous mapping $T : X \to fX \cap gX$ such that the pair $\{f, T\}$ is compatible and $\{g, T\}$ is weakly compatible, and*

$$M(Tx, Ty, \phi(t)) \geq M(fx, gy, t)$$

for all $x, y \in X$, and $t > 0$. Indeed, f, g and T have a unique common fixed point.

Proof. Note that the result follows from Theorem 2.1(b), since (1.3) holds because

$$\underbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}_{n \text{ copies}} = 1-\lambda.$$

Example 2.3. Let $X = [0,1]$ and $*$ be a minimum norm. Let $a > b > 1$ with a sufficiently larger than b. Let M be the fuzzy metric defined by

$$M(x,y,t) = \left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1}, \text{ for all } x,y \in X, t > 0.$$

Take $\phi(t) = kt$, where $k \in [\frac{1}{b}, 1)$, $fx = \dfrac{x}{b}$ and $gx = \dfrac{x}{a}$. Now define a continuous map $Tx = \dfrac{x}{ab}$. Note that the pair $\{f,T\}$ is compatible and $\{g,T\}$ is weakly compatible, and

$$M(Tx,Ty,kt) = \left[\exp\left(\frac{|x-y|}{abkt}\right)\right]^{-1} \geq \left[\exp\left(\frac{|\frac{a}{b}x-y|}{at}\right)\right]^{-1} = M(fx,gy,t).$$

Thus T satisfies the conditions of Cor. 2.2 and hence f, g and T have a unique common fixed point.

Corollary 2.4. *Let f be a map from a fuzzy metric space $(X,M,*)$ into itself.*

(a) *Suppose (1.2) holds. Then f has a fixed point in X if and only if there exists a continuous mapping $g : X \to fX$ which commutes with f, a point x_0 in X such that $E_M(fx_0,gx_0) = \sup\{E_{\lambda,M}(fx_0,gx_0) : \lambda \in (0,1)\} < \infty$, and g satisfies*

$$M(gx,gy,\phi(t)) \geq M(fx,fy,t),$$

for all $x,y \in X$, and $t > 0$. Indeed, f and g have a unique common fixed point.

(b) *Suppose (1.3) holds. Then f has a fixed point in X if and only if there exists a continuous mapping $g : X \to fX$ which commutes with f, and g satisfies*

$$M(gx,gy,\phi(t)) \geq M(fx,fy,t),$$

for all $x,y \in X$, and $t > 0$. Indeed, f and g have a unique common fixed point.

Corollary 2.5. *Let f and g be commuting maps from a complete fuzzy metric space $(X,M,*)$ into itself.*

(a) *Suppose (1.2) holds, f is continuous and $fX \subseteq gX$. Assume there exists $k \in N$ with,*

$$M(f^k x, f^k y, \phi(t)) \geq M(gx,gy,t),$$

for all $x, y \in X$, $t > 0$, and there exists a point x_0 in X such that $E_M(f^k x_0, gx_0) = \sup\{E_{\lambda,M}(f^k x_0, gx_0) : \lambda \in (0,1)\} < \infty$. Then f and g have a unique common fixed point.

(b) *Suppose* (1.3) *holds, f is continuous and $fX \subseteq gX$. Assume there exists $k \in N$ with,*

$$M(f^k x, f^k y, \phi(t)) \geq M(gx, gy, t),$$

for all $x, y \in X$, $t > 0$. Then f and g have a unique common fixed point.

DEFINITION 2.6. Let f and g be maps from a fuzzy metric space $(X, M, *)$ into itself. The pair $\{f, g\}$ is said to be weakly f-biased if whenever $fx = gx$ for some x in X then for all $t > 0$, one has

$$M(fgx, fx, t) \geq M(gfx, gx, t).$$

Obviously any weakly compatible pair $\{f, g\}$ is weakly f-biased and weakly g-biased. The following is the example of a pair of a maps $\{f, g\}$ which is weakly f-biased but not weakly g-biased. Also the pair $\{f, g\}$ is not weakly compatible.

EXAMPLE 2.7. Let $X = [0,1]$ and $a * b = \min\{a, b\}$. Let M be the standard fuzzy metric induced by d, where $d(x, y) = |x - y|$ for $x, y \in X$. Then $(X, M, *)$ is a complete fuzzy metric space. Let $fx = \frac{1}{2}(x+1)$ and $gx = 1 - x$. Now, $f\frac{1}{3} = g\frac{1}{3}$, and $M\big(fg\frac{1}{3}, f\frac{1}{3}, t\big) > M\big(gf\frac{1}{3}, g\frac{1}{3}, t\big)$ for all $t > 0$. Note that the pair $\{f, g\}$ is not weakly compatible.

EXAMPLE 2.8. Let $([0,1], M, *)$ be a complete fuzzy metric space. Let $f1 = 0$ and $fx = 1$ when $x \neq 1$ and $gx = 1$ for all $x \in [0,1]$. Here $f0 = g0$ but $M(fg0, f0, t) \not> M(gf0, g0, t)$ for all $t > 0$ so the pair $\{f, g\}$ is not weakly f-biased.

In the following theorem we assume that $\phi : [0, \infty) \to [0, \infty)$ is an onto and strictly increasing function with $\lim_{n\to\infty} \phi^n(t) = 0$, for $t > 0$. Note we do not assume that $\sum_{j=1}^{\infty} \phi^j(t) < \infty$ for $t > 0$.

THEOREM 2.9. *Let f be a map from a fuzzy metric space $(X, M, *)$ into itself.*

(a) *Suppose* (1.2) *holds. Then f has a fixed point in X if and only if there exists a continuous mapping $g : X \to fX$, $g(X)$ is complete in X, the pair $\{f, g\}$ is weakly f-biased, a point x_0 in X with $E_M(fx_0, gx_0) = \sup\{E_{\lambda,M}(fx_0, gx_0) : \lambda \in (0,1)\} < \infty$, and*

$$M(gx, gy, \phi(t)) \geq M(fx, fy, t), \tag{2.6}$$

for all $x, y \in X$, and $t > 0$. Indeed, f and g have a unique common fixed point.

(b) *Suppose* (1.3) *holds. Then f has a fixed point in X if and only if there exists a continuous mapping $g : X \to fX$, $g(X)$ is complete in X, the pair $\{f, g\}$ is weakly f-biased, and $M(gx, gy, \phi(t)) \geq M(fx, fy, t)$, for all $x, y \in X$, and $t > 0$. Indeed, f and g have a unique common fixed point.*

Proof (a). Suppose $fz = z$ for some z in X. Define a mapping g of X into itself by $gx = z$ for all x in X. Obviously, g is a continuous mapping of X into fX and $M(fgx, fx, t) = M(gfx, gx, t)$ for every $t > 0$ whenever $fx = gx$ for some x in X. Further, for any $t > 0$, we have

$$M(gx, gy, \phi(t)) = M(z, z, \phi(t)) = 1 \geq M(fx, fy, t),$$

for all $x, y \in X$. Take, $x_0 = z$, then obviously, $E_M(fx_0, gx_0) < \infty$. This proves the necessity of the condition. Now, conversely suppose that we have a mapping g and a point x_0 as described in the statement of Theorem 2.9. Define $y_0 = fx_0$. Since $gX \subseteq fX$, we can choose a point x_1 in X such that $fx_1 = gx_0 = y_1$. In general having chosen the point x_{n-1}, we choose a point x_n such that $fx_n = gx_{n-1} = y_n$. From (2.6), we obtain

$$\begin{aligned} M(y_n, y_{n+1}, \phi^n(t)) &= M(gx_{n-1}, gx_n, \phi^n(t)) \geq M(fx_{n-1}, fx_n, \phi^{n-1}(t)) \\ &= M(gx_{n-2}, gx_{n-1}, \phi^{n-1}(t)) \geq M(fx_{n-2}, fx_{n-1}, \phi^{n-2}(t)) \\ &\geq \cdots \geq M(fx_0, fx_1, t) = M(y_0, y_1, t). \end{aligned}$$

Now for each $\lambda \in (0, 1)$,

$$\begin{aligned} E_{\lambda,M}(y_n, y_{n+1}) &= \inf\{\phi^n(t) > 0 : M(y_n, y_{n+1}, \phi^n(t)) > 1 - \lambda\} \\ &\leq \inf\{\phi^n(t) > 0 : M(y_0, y_1, t) > 1 - \lambda\} \\ &\leq \phi^n(\inf\{t > 0 : M(y_0, y_1, t) > 1 - \lambda\}) \\ &= \phi^n(E_{\lambda,M}(y_0, y_1)) \leq \phi^n(E_M(y_0, y_1)). \end{aligned}$$

Thus $E_{\lambda,M}(y_n, y_{n+1}) \leq \phi^n(E_M(y_0, y_1))$ for each $\lambda \in (0, 1)$, and so

$$E_M(y_n, y_{n+1}) \leq \phi^n(E_M(y_0, y_1)).$$

Let $\varepsilon > 0$ be fixed. Then there exists a $n \in N$ so that

$$E_M(y_n, y_{n+1}) < \varepsilon - \phi(\varepsilon). \tag{2.7}$$

Now,

$$\begin{aligned} M(y_{n+1}, y_{n+2}, \phi(t)) &= M(gx_n, gx_{n+1}, \phi(t)) \\ &\geq M(fx_n, fx_{n+1}, t) = M(y_n, y_{n+1}, t) \end{aligned}$$

implies that

$$E_{\lambda,M}(y_{n+1}, y_{n+2}) \leq \phi(E_{\lambda,M}(y_n, y_{n+1})), \text{ for each } \lambda \in (0, 1). \tag{2.8}$$

From Lemma 1.5, for $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ with

$$E_{\mu,M}(y_n, y_{n+2}) \le E_{\lambda,M}(y_n, y_{n+1}) + E_{\lambda,M}(y_{n+1}, y_{n+2}),$$

which from (2.7) and (2.8) further implies that

$$\begin{aligned} E_{\mu,M}(y_n, y_{n+2}) &\le E_{\lambda,M}(y_n, y_{n+1}) + \phi(E_{\lambda,M}(y_n, y_{n+1})) \\ &\le E_M(y_n, y_{n+1}) + \phi(E_M(y_n, y_{n+1})) \\ &< \varepsilon - \phi(\varepsilon) + \phi(\varepsilon - \phi(\varepsilon)) < \varepsilon. \end{aligned}$$

Thus $E_M(y_n, y_{n+2}) < \varepsilon$. Now again for each $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} E_{\mu,M}(y_n, y_{n+3}) &\le E_{\lambda,M}(y_n, y_{n+1}) + E_{\lambda,M}(y_{n+1}, y_{n+3}) \\ &\le E_{\lambda,M}(y_n, y_{n+1}) + \phi(E_{\lambda,M}(y_n, y_{n+2})) \\ &\le E_M(y_n, y_{n+1}) + \phi(E_M(y_n, y_{n+2})) \\ &< \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon; \end{aligned}$$

note, in the proof we used the fact that

$$\begin{aligned} M(y_{n+1}, y_{n+3}, \phi(t)) &= M(gx_n, gx_{n+2}, \phi(t)) \\ &\ge M(fx_n, fx_{n+2}, t) = M(y_n, y_{n+2}, t), \end{aligned}$$

so $E_{\lambda,M}(y_{n+1}, y_{n+3}) \le \phi(E_{\lambda,M}(y_n, y_{n+2}))$. Thus $E_M(y_n, y_{n+3}) < \varepsilon$. By induction, $E_M(y_n, y_{n+k}) < \varepsilon$ for $k \in N$. Thus $\{y_n\}$ is a Cauchy sequence in X. Now since gX is complete, there exists a point z in gX such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = z$. Since $z \in gX$, we obtain a point u in X such that $fu = z$. Moreover,

$$M(gx_n, gu, \phi(t)) \ge M(fx_n, fu, t) \to 1, \text{ as } n \to \infty.$$

Hence $gx_n \to gu$, which implies that $fu = gu$. Now, we claim that $gu = u_1$(say) is a fixed point of g. Since the pair $\{f, g\}$ is weakly f-biased, therefore for any $k \in \{0,1,2,\dots\}$ and $t > 0$, we obtain

$$\begin{aligned} M(g^2u, gu, \phi^{k+1}(t)) &\ge M(fgu, fu, \phi^k(t)) \\ &\ge M(gfu, gu, \phi^k(t)) = M(g^2u, gu, \phi^k(t)). \end{aligned}$$

Thus for any $k \in \{0,1,2,\dots\}$ and $t > 0$ we have

$$M(g^2u, gu, \phi^{k+1}(t)) \ge M(g^2u, gu, t). \tag{2.9}$$

Since $\phi(t) < t$ for $t > 0$ and $M(x,y,t)$ is a nondecreasing function with respect to t, for all $x, y \in X$, we have for any $t > 0$ that

$$M(g^2u, gu, t) \ge M(g^2u, gu, \phi^{k+1}(t)). \tag{2.10}$$

From (2.9) and (2.10) we have $M(g^2u, gu, t) = C$ for $t > 0$. Taking $t \to \infty$, we obtain $C = 1$ and hence u_1 is a fixed point of g. Now, again for any

$k \in \{0, 1, 2, \dots\}$ and $t > 0$, we obtain

$$\begin{aligned} M(fu_1, u_1, \phi^{k+1}(t)) &= M(fgu, fu, \phi^{k+1}(t)) \geq M(gfu, gu, \phi^{k+1}(t)) \\ &\geq M(ffu, fu, \phi^k(t)) = M(fu_1, u_1, \phi^k(t)). \end{aligned}$$

Thus for any $k \in \{0, 1, 2, \dots\}$ and $t > 0$ we have

$$M(fu_1, u_1, \phi^{k+1}(t)) \geq M(fu_1, u_1, t).$$

Also, for any $t > 0$ and $k \in \{0, 1, 2, \dots\}$

$$M(fu_1, u_1, \phi^{k+1}(t)) \leq M(fu_1, u_1, t).$$

Hence $M(fu_1, u_1, t) = C$. Taking $t \to \infty$ we obtain $C = 1$, and u_1 is a fixed point of f. Thus u_1 is a common fixed point of f and g, and this establishes the sufficiency of the condition. Now, if we suppose that u_2 is another common fixed point of f and g, then for any $k \in \{0, 1, 2, \dots\}$, and $t > 0$ we have

$$\begin{aligned} M(u_1, u_2, \phi^{k+1}(t)) &= M(gu_1, gu_2, \phi^{k+1}(t)) \\ &\geq M(fu_1, fu_2, \phi^k(t)) = M(u_1, u_2, \phi^k(t)). \end{aligned}$$

Thus, for any $k \in \{0, 1, 2, \dots\}$ and $t > 0$ we have

$$M(u_1, u_2, \phi^{k+1}(t)) \geq M(u_1, u_2, t).$$

On the other hand we know for any $t > 0$ that we have,

$$M(u_1, u_2, t) \geq M(u_1, u_2, \phi^{k+1}(t)),$$

and so we have $M(u_1, u_2, t) = C$. Let $t \to \infty$, and we obtain $C = 1$ and hence $u_1 = u_2$.

Proof (b). The argument is as in case (a) except we use the last two lines of Lemma 1.5.

COROLLARY 2.10. *Let f be a map from a fuzzy metric space $(X, M, *)$ into itself, where $*$ is a minimum t-norm. Then f has a fixed point in X if and only if there exists a continuous mapping $g : X \to fX$ such that the pair $\{f, g\}$ is weakly f-biased, and*

$$M(gx, gy, \phi(t)) \geq M(fx, fy, t)$$

for all $x, y \in X$, and $t > 0$. Indeed, f and g have a unique common fixed point.

EXAMPLE 2.11. Let $([0, 1], M, *)$ be a complete fuzzy metric space. Let $fx = 1 - x$, when $0 \leq x < 1$ and $f1 = 1$. Define $gx = 1$ for all $x \in [0, 1]$. Take $x_0 = 0$, then $E_M(fx_0, gx_0) < \infty$, $f(0) = g(0)$ implies that $M(fg0, f0, t) = M(gf0, g0, t)$, and $M(gx, gy, \phi(t)) \geq M(fx, fy, t)$ for every $t > 0$. Thus the pair $\{f, g\}$ satisfies every condition of Theorem 2.9 (case (a)) and hence have a common fixed point. Note that the pair $\{f, g\}$ has two coincidence points but has a unique common fixed point.

EXAMPLE 2.12. Let $X = [0,1]$ and $a * b = \min\{a, b\}$. Let M be the standard fuzzy metric induced by d, where $d(x, y) = |x - y|$ for $x, y \in X$. Then $(X, M, *)$ is a complete fuzzy metric space. Let

$$fx = \begin{cases} \frac{1}{2}(1-x), x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ \frac{3}{4}, & x = \frac{1}{2} \end{cases}$$

and $gx = \frac{1}{3}$ for all $x \in [0,1]$. Here, $f\frac{1}{3} = g\frac{1}{3}$ implies that $M(fg\frac{1}{3}, f\frac{1}{3}, t) = M(gf\frac{1}{3}, g\frac{1}{3}, t)$, and $M(gx, gy, \phi(t)) \geq M(fx, fy, t)$ for every $t > 0$. Thus the pair $\{f, g\}$ satisfies the conditions of Cor. 2.10 and hence have a common fixed point.

REMARK 2.13. The results presented in this paper can be extended to $\mathcal{L}$-fuzzy metric spaces.

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D. O' Regan
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF IRELAND
GALWAY, IRELAND
R-mail: donal.oregan@nuigalway.ie

Mujahid Abbas
DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, IN 47405-7106. U.S.A
Permanent address:
DEPARTMENT OF MATHEMATICS
LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
54792-LAHORE, PAKISTAN
E-mail: mujahid@lums.edu.pk

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