DEMONSTRATIO MATHEMATICA
Vol. XLII No 4 2009

Uday Chand De, Abul Kalam Gazi

# ON ALMOST PSEUDO CONFORMALLY SYMMETRIC MANIFOLDS

**Abstract.** The object of the present paper is to study a type of non-conformally flat semi-Riemannian manifolds called almost pseudo conformally symmetric manifold. The existence of an almost pseudo conformally symmetric manifold is also shown by a non-trivial example.

## 1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by E. Cartan [5], who, in particular, obtained a classification of those spaces.

Let $(M, g)$, $n = \dim M$ be a semi-Riemannian manifold, i.e. a manifold $M$ with the metric tensor $g$ with arbitrary signature and let $\nabla$ be the Levi-Civita connection of $(M, g)$. A semi-Riemannian manifold is called locally symmetric [5] if $\nabla R = 0$, where $R$ is the Riemannian curvature tensor of $(M, g)$. This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [23]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [7], recurrent manifolds introduced by A. G. Walker [32], conformally recurrent manifolds by T. Adati and T. Miyazawa [1], pseudo symmetric manifolds introduced by M. C. Chaki [6], weakly symmetric manifolds by L. Tamássy and T. Q. Binh [30], projective symmetric manifolds by G. Soós [29] etc.

---

2000 *Mathematics Subject Classification*: 53C15, 53C25, 53B20, 53B30, 53B15.
*Key words and phrases*: pseudo conformally symmetric manifold, almost pseudo coformally symmetric manifold, almost pseudo symmetric manifold.

In 1967, R. N. Sen and M. C. Chaki [28] studied certain curvature restrictions on a certain kind of conformally flat space of class one and they obtained the following expression of the covariant derivative of the curvature tensor :

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R_{lijk}\,, \tag{1.1}$$

where $R^h_{ijk}$ are the components of the curvature tensor $R$, $R_{lijk} = g_{hl}R^h_{ijk}$, $\lambda_i$ is a non-zero covariant vector and ',' denotes covariant differentiation with respect to the metric tensor $g_{ij}$.

Later in 1987, M. C. Chaki [6] called a manifold whose curvature tensor satisfies (1.1), as a pseudo symmetric manifold. In index free notation this can be stated as follows:

A non-flat semi-Riemannian manifold $(M,\ g)$, $n \geq 2$ is said to be a pseudo symmetric manifold [6] if its curvature tensor $R$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = {} & 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ & + A(Z)R(Y,X)W + A(W)R(Y,Z)X \\ & + g(R(Y,Z)W,X)\rho,\end{aligned} \tag{1.2}$$

where $A$ is a non-zero 1-form, $\rho$ is a vector field defined by

$$g(X,\rho) = A(X), \quad \text{for all } X \tag{1.3}$$

and $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. The 1-form $A$ is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of E. Cartan. An $n$-dimensional pseudo symmetric manifold is denoted by $(PS)_n$.

This is to be noted that the notion of pseudo symmetric manifold studied in particular by R. Deszcz ([3], [16], [17], [18]) is different from that of M. C. Chaki [6].

The notion of a pseudo conformally symmetric manifold was introduced by the first author and H. A. Biswas [8]. An $n$-dimensional non-conformally flat semi-Riemannian manifold $(M, g)$, $n > 3$ is called pseudo conformally symmetric manifold if the conformal curvature tensor $C$ defined by

$$\begin{aligned}C(X,Y)Z = {} & R(X,Y)Z \\ & - \frac{1}{n-2}[g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X - S(X,Z)Y] \\ & + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.4}$$

where $R$ is the curvature tensor of type $(1,3)$, $S$ is the Ricci tensor, $r$ is the scalar curvature and $L$ is the symmetric endomorphism corresponding to the

Ricci tensor $S$, that is,

$$S(X,Y) = g(LX,Y), \tag{1.5}$$

satisfies the condition

$$\begin{aligned}(\nabla_X C)(Y,Z)W = 2A(X)C(Y,Z)W &+ A(Y)C(X,Z)W \\ &+ A(Z)C(Y,X)W + A(W)C(Y,Z)X \\ &+ g(C(Y,Z)W,X)P,\end{aligned} \tag{1.6}$$

for all vector fields $X, Y, Z, W \in \chi(M)$, where $A$ is a non-zero 1-form, $\nabla$ denotes the operator of covariant differentiation with respect to the metric $g$ and $P$ is a vector field given by

$$g(X,P) = A(X), \tag{1.7}$$

for all $X$. The 1-form $A$ was called the associated 1-form of the manifold and such an $n$-dimensional manifold was denoted by $(PCS)_n$. The vector field $P$ defined by (1.7) was called the basic vector field corresponding to the 1-form $A$. If $A = 0$ on $M$ then the $(PCS)_n$ manifold is a conformally symmetric manifold [7]. For recent results on conformally symmetric manifolds we refer to [11], [12], [13], [14], [15].

Conformally recurrent manifold was introduced by T. Adati and T. Miyazawa [1] in 1967.

A semi-Riemannian manifold $(M,g)$, $n > 3$ is called conformally recurrent if the conformal curvature tensor, defined by (1.4), satisfies the condition

$$(\nabla_X C)(Y,Z)W = E(X)C(Y,Z)W, \tag{1.8}$$

where $E$ is a non-zero 1-form. If in particular $E = 0$, then the manifold reduces to a conformally symmetric manifold [7].

M. Prvanović called a non-flat semi-Riemannian manifold $(M,g)$, $n > 3$ as a conformally quasi-recurrent manifold ([24], [25]) if its conformal curvature tensor satisfies the condition (1.6).

The object of the present paper is to study a type of non-conformally flat semi-Riemannian manifold $(M,g)$, $n > 3$ whose conformal curvature tensor $C$ satisfies the condition

$$\begin{aligned}(\nabla_X C)(Y,Z)W = [A(X) + B(X)]C(Y,Z)W &+ A(Y)C(X,Z)W \\ &+ A(Z)C(Y,X)W + A(W)C(Y,Z)X \\ &+ g(C(Y,Z)W,X)P,\end{aligned} \tag{1.9}$$

where $A$ and $B$ are two non-zero 1-forms, called the associated 1-forms, defined by

$$g(X,P) = A(X), \quad g(X,Q) = B(X), \tag{1.10}$$

for all vector fields $X$ and $\nabla$ has the meaning already mentioned. Here the vector field $P$ and $Q$ shall be called the basic vector fields of the manifold corresponding to the associated 1-forms $A$ and $B$ respectively. Such a manifold shall be called an almost pseudo conformally symmetric manifold and an $n$-dimensional manifold of this kind shall be denoted by $(APCS)_n$. Clearly, every conformally recurrent manifold is a $(APCS)_n$.

If in (1.9) $A = B$, then the manifold reduces to a pseudo conformally symmetric manifold defined by (1.6). This justifies the name "almost pseudo conformally symmetric manifold" and the use of the symbol $(APCS)_n$. In this connection it may be mentioned that in 1989 Tamássy and Binh [30] introduced weakly symmetric and weakly projectively symmetric Riemannian manifolds. A semi-Riemannian manifold $(M, g)$ is called weakly symmetric and denoted by $(WS)_n$ if there exist 1-forms $A$, $B$, $D$, $E$ and a vector field $P$ such that

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W + D(Z)R(Y,X)W \\ + E(W)R(Y,Z)X + g(R(Y,Z)W,X)P,$$

where $R$ is the curvature tensor of $(M, g)$.

On the analogy of $(WS)_n$ Tamássy and Binh [31] introduced the notion of weakly Ricci symmetric manifolds $(WRS)_n$. A semi-Riemannian manifold $(M, g)$ is called weakly Ricci symmetric if there exist 1-forms $A$, $B$, and $D$ such that

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(Y,X),$$

where $S$ is the non-zero Ricci tensor of type $(0,2)$ of the manifold.

In a subsequent paper [9] the first author and S. Bandyopadhyay introduced weakly conformally symmetric manifold. A non-conformally flat semi-Riemannian manifold $(M, g)$, $n > 3$ is called weakly conformally symmetric if its conformal curvature tensor $C$ satisfies the condition

$$(1.11)\qquad (\nabla_X C)(Y,Z)W = A(X)C(Y,Z)W + B(Y)C(X,Z)W \\ + D(Z)C(Y,X)W + E(W)C(Y,Z)X \\ + g(C(Y,Z)W,X)P,$$

where $A, B, D, E$ are 1-forms not simultaneously zero and $\nabla$ and $P$ have the meaning already mentioned. Such a manifold was denoted by $(WCS)_n$. It is to be noted that $(APCS)_n$ is not a particular case of $(WCS)_n$.

Also it may be mentioned that in a recent paper [10] the authors studied a type of non-flat semi-Riemannian manifold $(M, g)$, $n \geq 2$ whose curvature tensor $R$ of type $(1,3)$ satisfies the condition

$$(1.12)\quad (\nabla_X R)(Y,Z)W = [A(X)+B(X)]R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P,$$

where $A, B, P$ and $\nabla$ have the meaning already mentioned. Such a manifold was called an almost pseudo symmetric manifold and was denoted by $(APS)_n$.

The paper is organized as follows:

In section 2 some properties of an $(APCS)_n$, $n>3$ have been studied under certain condition. In the next section it is firstly shown that if the Ricci tensor vanishes, then $(APCS)_n$, $n>3$ reduces to a $(APS)_n$ and the relation $A(R(Y,Z)W)+B(R(Y,Z)W)=0$ holds. It is also proved that if the vector field $Q$ defined by (1.10) is a parallel vector field in an Einstein $(APCS)_n$, then $(APCS)_n$ reduces to an $(APS)_n$ provided the vector fields $P$ and $Q$ corresponding to the associated 1-forms $A$ and $B$ are not co-directional. In section 4 it is shown that if the Weyl conformal curvature tensor of an $(APCS)_n$ satisfies Bianchi's 2nd identity, then the manifold reduces to a $(PCS)_n$. It is also proved that an $(APCS)_n$ whose conformal curvature tensor satisfies Bianchi's 2nd identity can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is almost pseudo conformally symmetric. Section 5 consists of the proof of the existence of an $(APCS)_n$, $n>3$. Finally, a non-trivial example of an $(APCS)_n$ has been constructed.

## 2. $(APCS)_n$ satisfying $B(C(X,Y)Z)=0$

Contracting (1.9) over $W$ we get

$$(2.1)\quad A(C(Y,Z)X)=0,$$

for all vector fields $X, Y, Z$. Now contracting (1.9) over $X$ we get

$$(2.2)\quad (divC)(Y,Z)W = 2A(C(Y,Z)W)+B(C(Y,Z)W).$$

A Riemannian or semi-Riemannian manifold $(M,g)$ is said to be of harmonic conformal curvature [4] if $n \geq 4$ and the condition $(divC)(X,Y)Z=0$ holds. From (2.1) and (2.2) we see that $(divC)(X,Y)Z=0$ if and only if $B(C(X,Y)Z)=0$. Thus we have the following theorem:

**Theorem 2.1.** *Every $(APCS)_n$, $n>3$ is of harmonic conformal curvature if and only if $B(C(X,Y)Z)=0$ holds.*

In the rest of this section we assume that in an $(APCS)_n$ the relation

$$(2.3)\quad B(C(X,Y)Z)=0,$$

holds. It is known [19] that in a semi-Riemannian manifold $(M,g)$, $n>3$

$$(divC)(X,Y)Z = \left(\frac{n-3}{n-2}\right)[(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) + \frac{1}{2(n-1)}(g(X,Y)dr(Z) - g(Y,Z)dr(X))], \tag{2.4}$$

where $C$ denotes the conformal curvature tensor defined in (1.4). Since $B(C(X,Y)Z) = 0$ we get from Theorem 2.1 that $(divC)(X,Y)Z = 0$. Hence from (2.4) we get

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) = \frac{1}{2(n-1)}[g(Y,Z)dr(X) - g(X,Y)dr(Z)]. \tag{2.5}$$

First suppose that the Ricci tensor is a Codazzi tensor [20], that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(X,Y).$$

Then from (2.5) it follows that $r = constant$. Conversely, if $r = constant$ it follows from (2.5) that Ricci tensor is a Codazzi tensor. This leads to the following theorem:

**Theorem 2.2.** *In a $(APCS)_n$ with $B(C(X,Y)Z) = 0$, the scalar curvature is constant if and only if the Ricci tensor is a Codazzi tensor.*

If $B(C(X,Y)Z) = 0$, then from (2.4) and Theorem 2.1, we get

$$\left[(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) + \frac{1}{2(n-1)}(g(X,Y)dr(Z) - g(Y,Z)dr(X))\right] = 0, \tag{2.6}$$

which can be written as

$$(\nabla_X F)(Y,Z) - (\nabla_Z F)(X,Y) = 0,$$

where

$$F(X,Y) = S(X,Y) - \frac{r}{2(n-1)}g(X,Y). \tag{2.7}$$

Thus we can state the following theorem:

**Theorem 2.3.** *A $(APCS)_n$, $n>3$ satisfies $(\nabla_X F)(Y,Z) = (\nabla_Z F)(X,Y)$ if $B(C(X,Y)Z) = 0$.*

**Remark.** The above theorem is true for a $(PCS)_n$, $n>3$ without assuming $B(C(X,Y)Z) = 0$, because in a $(PCS)_n$ there is only one 1-form $A$ and the relation $divC = 0$ holds without any assumption.

## 3. Einstein $(APCS)_n$, $n > 3$

In this section we assume that an $(APCS)_n$ defined by (1.9) is an Einstein manifold.

We remark that manifolds with vanishing Ricci tensor (i.e. Ricci flat manifolds) are certain special Einstein manifolds. Let us assume that a $(APCS)_n$ satisfies $S(X,Y) = 0$, then with the help of (1.4) the equation (1.9) can be written as

$$(3.1)\qquad \begin{aligned}(\nabla_X R)(Y,Z)W = {} & [A(X) + B(X)]R(Y,Z)W + A(Y)R(X,Z)W \\ & + A(Z)R(Y,X)W + A(W)R(Y,Z)X \\ & + g(R(Y,Z)W,X)P.\end{aligned}$$

Thus the manifold is a $(APS)_n$. Making use of Bianchi's 2nd identity and $S(X,Y) = 0$, we get

$$(3.2)\qquad (divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0,$$

where '*div*' denotes divergence. Now contracting (3.1) over $X$ and using (3.2) and $S(X,Y) = 0$ we get

$$(3.3)\qquad A(R(Y,Z)W) + B(R(Y,Z)W) = 0.$$

Thus we can state the following theorem:

**Theorem 3.1.** *If a $(APCS)_n$ satisfies $S(X,Y) = 0$, then the manifold is a $(APS)_n$ and the relation* (3.3) *holds.*

In an Einstein manifold the Ricci tensor satisfies

$$(3.4)\qquad S(X,Y) = \frac{r}{n}g(X,Y),$$

from which it follows that

$$(3.5)\qquad dr(X) = 0 \text{ and } (\nabla_X S)(Y,Z) = 0.$$

From $(\nabla_X S)(Y,Z) = 0$ we get by contraction

$$(3.6)\qquad (\nabla_X L)(Y) = 0,$$

where $L$ is defined by (1.5). By using (1.4), (3.4), (3.5) and (3.6) we get from (1.9)

$$(3.7)\quad \begin{aligned}&(\nabla_X R)(Y,Z)W\\ &= [A(X)+B(X)]\Big[R(Y,Z)W - \frac{r}{n(n-1)}(g(Z,W)Y - g(Y,W)Z)\Big]\\ &+ A(Y)\Big[R(X,Z)W - \frac{r}{n(n-1)}(g(Z,W)X - g(X,W)Z)\Big]\\ &+ A(Z)\Big[R(Y,X)W - \frac{r}{n(n-1)}(g(X,W)Y - g(Y,W)X)\Big]\\ &+ A(W)\Big[R(Y,Z)X - \frac{r}{n(n-1)}(g(Z,X)Y - g(Y,X)Z)\Big]\\ &+ g\Big(\Big[R(Y,Z)W - \frac{r}{n(n-1)}(g(Z,W)Y - g(Y,W)Z)\Big], X\Big)P.\end{aligned}$$

Now we suppose that in an Einstein $(APCS)_n$ the vector field $Q$ defined by (1.10) is a parallel vector field ([22], p.-124; [27], p.-322). Then

$$(3.8)\qquad (\nabla_X Q) = 0,$$

for all $X \in \chi((APCS)_n)$. Applying Ricci identity we get

$$(3.9)\qquad R(X,Y)Q = 0.$$

From (3.9) it follows that

$$(3.10)\qquad 'R(X,Y,Z,Q) = 0,$$

where $'R(X,Y,Z,Q) = g(R(X,Y)Z,Q)$. In virtue of (3.10) we get by contraction

$$(3.11)\qquad S(X,Q) = 0.$$

Now by (3.8) and (3.11) we have

$$(3.12)\qquad (\nabla_X S)(Y,Q) = \nabla_X S(Y,Q) - S(\nabla_X Y,Q) - S(Y,\nabla_X Q) = 0.$$

From (3.7) we get

$$(3.13)\quad \begin{aligned}&(\nabla_X S)(Z,W)\\ &= A(R(X,Z)W) - \frac{r}{n(n-1)}[g(Z,W)A(X) - g(X,W)A(Z)].\end{aligned}$$

Putting $W = Q$ in (3.13) and applying (3.9), (3.12) and (1.10) we obtain

$$\frac{r}{n(n-1)}[B(Z)A(X) - B(X)A(Z)] = 0.$$

If $B(Z)A(X) \neq B(X)A(Z)$, we get $r = 0$ and then from (3.7) we see that the manifold becomes an $(APS)_n$. Thus we have the following theorem:

**THEOREM 3.2.** *If the vector field $Q$ is a parallel vector field in an Einstein $(APCS)_n$, then $(APCS)_n$ reduces to a $(APS)_n$ provided the vector fields $P$ and $Q$ corresponding to the associated 1-forms $A$ and $B$ are not codirectional.*

## 4. Semi-symmetric metric connection

It is well known that the conformal curvature tensor satisfies the conditions

$$C(X,Y)Z + C(Y,Z)X + C(Z,X)Y = 0, \tag{4.1}$$

$$C(X,Y)Z = -C(Y,X)Z. \tag{4.2}$$

But, in general, the Weyl conformal curvature tensor $C(X,Y)Z$ does not satisfy Bianchi's 2nd identity

$$(\nabla_X C)(Y,Z)W + (\nabla_Y C)(Z,X)W + (\nabla_Z C)(X,Y)W = 0. \tag{4.3}$$

In this section we suppose that the condition (4.3) holds in the investigated $A(PCS)_n$. Now by using (4.1), (4.2) and (4.3) we get from (1.9) that

$$G(X)C(Y,Z)W + G(Y)C(Z,X)W + G(Z)C(X,Y)W = 0, \tag{4.4}$$

where $G(X) = B(X) - A(X)$ and $\rho$ is a vector field defined by

$$g(X,\rho) = G(x). \tag{4.5}$$

Contracting $X$ in (4.4) we get

$$G(C(Y,Z)W) = 0. \tag{4.6}$$

Now putting $X = \rho$ in (4.4) and using (4.6) we have

$$G(\rho)C(Y,Z)W = 0. \tag{4.7}$$

Hence either the manifold is conformally flat or $G(\rho) = 0$. But by hypothesis $C \neq 0$. So $G(\rho) = 0$. Then $\rho$ is a null vector and from the definition of $G$ we obtain $B = A$ which implies with the help of (1.9) that the manifold reduces to a $(PCS)_n$. Thus we conclude the following theorem:

**THEOREM 4.1.** *If the Weyl conformal curvature tensor of a $(APCS)_n$ satisfies Bianchi's 2nd identity then the manifold reduces to a $(PCS)_n$.*

A linear connection $\tilde{\nabla}$ on a semi-Riemannian manifold $(M,g)$ is said to be a semi-symmetric metric connection [34] if the torsion tensor $T$ of the connection $\tilde{\nabla}$ is given by

$$T(X,Y) = \pi(Y)X - \pi(X)Y,$$

for all $X, Y \in \chi(M)$ and $\tilde{\nabla} g = 0$, where $\pi$ is a 1-form on $M$. If $\pi$ is a 1-form on $M$, then its associated vector field $V$ is related by

$$g(X,V) = \pi(X),$$

for all $X \in \chi(M)$.

If $\tilde{\nabla}$ and $\nabla$ are the semi-symmetric metric connection and the Levi-Civita connection respectively on a semi-Riemannian manifold $(M, g)$, then for all $X, Y \in \chi(M)$ we have

$$\tilde{\nabla}_X Y - \nabla_X Y = \pi(Y)X - g(X,Y)V. \tag{4.8}$$

We shall denote the tensors determined with respect to $\tilde{\nabla}$ by a tilde above. For example $\tilde{R}(X,Y)Z$, $\tilde{S}(X,Y)$, $\tilde{r}$ are the curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$ and the scalar curvature respectively, while the conformal curvature tensor $\tilde{C}$ of type $(1,3)$ is

$$\begin{aligned}\tilde{C}(X,Y)Z &= \tilde{R}(X,Y)Z \\ &\quad - \frac{1}{n-2}[g(Y,Z)\tilde{L}X - g(X,Z)\tilde{L}Y + \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] \\ &\quad + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],\end{aligned}$$

where $\tilde{L}$ is the symmetric endomorphism corresponding to the Ricci tensor $\tilde{S}$, that is, $\tilde{S}(X,Y) = g(\tilde{L}X, Y)$ for all $X, Y \in \chi(M)$. It is known [2] that

$$\tilde{C}(X,Y)Z = C(X,Y)Z. \tag{4.9}$$

Applying the operator $\tilde{\nabla}$ to (4.9) and using (1.9) and (4.8), we get

$$\begin{aligned}(\tilde{\nabla}_X\tilde{C})(Y,Z)W &= [A(X) + B(X)]C(Y,Z)W \\ &\quad + [A(Y) - \pi(Y)]C(X,Z)W \\ &\quad + [A(Z) - \pi(Z)]C(Y,X)W \\ &\quad + [A(W) - \pi(W)]C(Y,Z)X \\ &\quad + g(C(Y,Z)W,X)(P - V) + \pi(C(Y,Z)W)X \\ &\quad + g(X,Y)C(V,Z)W + g(X,Z)C(Y,V)W \\ &\quad + g(X,W)C(Y,Z)V.\end{aligned} \tag{4.10}$$

Therefore, if we take $\pi(X) = G(X) = B(X) - A(X)$ then (4.10) reduces with the help of (4.6) and (4.9) to

$$\begin{aligned}(\tilde{\nabla}_X C)(Y,Z)W &= [A(X) + B(X)]C(Y,Z)W \\ &\quad + [2A(Y) - B(Y)]C(X,Z)W \\ &\quad + [2A(Z) - B(Z)]C(Y,X)W \\ &\quad + [2A(W) - B(W)]C(Y,Z)X \\ &\quad + g(C(Y,Z)W,X)(2P - Q).\end{aligned} \tag{4.11}$$

Now if we put $2A(X) - B(X) = J(X)$ and $2B(X) - A(X) = K(X)$ in (4.11) we get

$$
\begin{aligned}
(4.12)\quad (\tilde{\nabla}_X C)(Y,Z)W = {} & [J(X)+K(X)]C(Y,Z)W + J(Y)C(X,Z)W \\
& + J(Z)C(Y,X)W + J(W)C(Y,Z)X \\
& + g(C(Y,Z)W,X)\tilde{P},
\end{aligned}
$$

where $\tilde{P}$ is a vector field defined by $g(X,\tilde{P}) = J(X)$. Thus the conformal curvature tensor is almost pseudo conformally symmetric with respect to the semi-symmetric connection $\tilde{\nabla}$. This leads to the following theorem:

**Theorem 4.2.** *A $(APCS)_n$ can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is almost pseudo conformally symmetric.*

If, in particular, $G(X) = A(X)$ then (4.12) reduces to

$$(\tilde{\nabla}_X C)(Y,Z)W = 3A(X)C(Y,Z)W.$$

Hence we can state the following:

**Corollary 1.** *A $(APCS)_n$ can be endowed with a unique determined semi-symmetric metric connection with respect to which the conformal curvature tensor is recurrent if the associated 1-form $\pi(X)$ of the semi-symmetric metric connection is equal to the associated 1-form $A(X)$ of the $(APCS)_n$.*

## 5. Existence of an $(APCS)_n$

Let $(M,g)$, $n > 3$ be a semi-Riemannian manifold with the fundamental metric tensor $g$. The change of the metric

$$\overset{*}{g} = \mu^2 g, \tag{5.1}$$

where $\mu$ is a certain positive function, does not change the angle between two vectors at a point and is called conformal deformation of the metric.

If $\mu$ is a positive constant, then the conformal deformation is said to be homothetic.

If $\overset{*}{\nabla}$ and $\nabla$ denote the operator of covariant differentiation with respect to $\overset{*}{g}$ and $g$ respectively, we have [33]

$$\overset{*}{\nabla}_X Y - \nabla_X Y = \omega(X)Y + \omega(Y)X - g(X,Y)U \tag{5.2}$$

for any vector fields $X$, $Y$, where $\omega$ is a 1-form defined by

$$\omega = d\, log\mu \tag{5.3}$$

and $U$ is a vector field defined by

$$g(X,U) = \omega(X)\,. \tag{5.4}$$

By conformal deformation (5.1), as is well-known we have

$$\overset{*}{C}(X,Y)Z = C(X,Y)Z, \tag{5.5}$$

where the symbol $*$ denotes the quantities of $(M, \overset{*}{g})$. Differentiating (5.5) covariantly and making use of the relation (5.2) we get

$$(5.6)\quad (\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = (\nabla_X C)(Y,Z)W - 2\omega(X)C(Y,Z)W - [\omega(Y)C(X,Z)W + \omega(Z)C(Y,X)W + \omega(W)C(Y,Z)X + g(C(Y,Z)W,X)U] + \omega(C(Y,Z)W)X + g(X,Y)C(U,Z)W + g(X,Z)C(Y,U)W + g(X,W)C(Y,Z)U.$$

Now we assume that $M$ is a conformally recurrent manifold, then

$$(5.7)\quad (\nabla_X C)(Y,Z)W = E(X)C(Y,Z)W,$$

where $E$ is a non-zero 1-form. From (5.5), (5.6) and (5.7) we get

$$(5.8)\quad (\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = [E(X) - 2\omega(X)]\overset{*}{C}(Y,Z)W - [\omega(Y)\overset{*}{C}(X,Z)W + \omega(Z)\overset{*}{C}(Y,X)W + \omega(W)\overset{*}{C}(Y,Z)X + g(\overset{*}{C}(Y,Z)W,X)U] + \omega(\overset{*}{C}(Y,Z)W)X + g(X,Y)\overset{*}{C}(U,Z)W + g(X,Z)\overset{*}{C}(Y,U)W + g(X,W)\overset{*}{C}(Y,Z)U.$$

Contracting $Y$ in (5.8) and after some simple calculations we get

$$(5.9)\quad \omega(\overset{*}{C}(X,Z)W) = 0.$$

Using (5.9) we get from (5.8)

$$(5.10)\quad (\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = [E(X) - 2\omega(X)]\overset{*}{C}(Y,Z)W - [\omega(Y)\overset{*}{C}(X,Z)W + \omega(Z)\overset{*}{C}(Y,X)W + \omega(W)\overset{*}{C}(Y,Z)X + g(\overset{*}{C}(Y,Z)W,X)U].$$

Now let us suppose that $\overset{*}{A}(X) = -\omega(X)$ and $\overset{*}{B}(X) = E(X) - \omega(X)$ for all $X$. Then (5.10) reduces to

$$(5.11)\quad (\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = [\overset{*}{A}(X) + \overset{*}{B}(X)]\overset{*}{C}(Y,Z)W + \overset{*}{A}(Y)\overset{*}{C}(X,Z)W + \overset{*}{A}(Z)\overset{*}{C}(Y,X)W + \overset{*}{A}(W)\overset{*}{C}(Y,Z)X + g(\overset{*}{C}(Y,Z)W,X)\overset{*}{P}$$

which implies that $(M, \overset{*}{g})$ is a $(APCS)_n$. Thus we have the following theorem:

**THEOREM 5.1.** *A conformal deformation of every conformally recurrent metric is a $(APCS)_n$ metric, provided that $n > 3$.*

If the 1-form of recurrence $E = 0$ in (5.7), then $(M, g)$ is conformally symmetric and (5.10) reduces to

$$(\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = -2\omega(X)\overset{*}{C}(Y,Z)W - [\omega(Y)\overset{*}{C}(X,Z)W + \omega(Z)\overset{*}{C}(Y,X)W + \omega(W)\overset{*}{C}(Y,Z)X + g(\overset{*}{C}(Y,Z)W,X)U]. \tag{5.12}$$

Now if we put $\overset{*}{D}(X) = -\omega(X)$ for all $X$ then (5.12) implies that $(M, \overset{*}{g})$ is a $(PCS)_n$. Hence we have the following corollary :

**COROLLARY 2.** *A conformal deformation of every conformally symmetric metric is a $(PCS)_n$ metric, provided that $n > 3$.*

Now we enquire whether a $(APCS)_n$ metric under a conformal deformation becomes a $(APCS)_n$ metric or not.

Let us suppose that $(M, g)$ is a $(APCS)_n$ manifold. Then from (1.9), (5.5) and (5.6) we get

$$\begin{aligned}(\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = {} & [A(X) + B(X) - 2\omega(X)]\overset{*}{C}(Y,Z)W + [A(Y) \\ & - \omega(Y)]\overset{*}{C}(X,Z)W + [A(Z) - \omega(Z)]\overset{*}{C}(Y,X)W \\ & + [A(W) - \omega(W)]\overset{*}{C}(Y,Z)X \\ & + g(\overset{*}{C}(Y,Z)W,X)(P-U) + \omega(\overset{*}{C}(Y,Z)W)X \\ & + g(X,Y)\overset{*}{C}(U,Z)W + g(X,Z)\overset{*}{C}(Y,U)W \\ & + g(X,W)\overset{*}{C}(Y,Z)U.\end{aligned} \tag{5.13}$$

Contracting $Y$ in (5.13) and after some simple calculations we get

$$\omega(\overset{*}{C}(X,Z)W) = 0. \tag{5.14}$$

Using (5.14) we get from (5.13)

$$\begin{aligned}(\overset{*}{\nabla}_X \overset{*}{C})(Y,Z)W = {} & [A(X) + B(X) - 2\omega(X)]\overset{*}{C}(Y,Z)W \\ & + [A(Y) - \omega(Y)]\overset{*}{C}(X,Z)W \\ & + [A(Z) - \omega(Z)]\overset{*}{C}(Y,X)W \\ & + [A(W) - \omega(W)]\overset{*}{C}(Y,Z)X \\ & + g(\overset{*}{C}(Y,Z)W,X)(P-U).\end{aligned} \tag{5.15}$$

Let us suppose that $\overset{*}{F}(X) = A(X) - \omega(X)$ and $\overset{*}{G}(X) = B(X) - \omega(X)$ for all $X$ and $\overset{*}{P}$ is a vector field defined by $g(X, \overset{*}{P}) = \overset{*}{F}(X)$. Then $\overset{*}{P} = P - U$ and from (5.15) we see that $(M, \overset{*}{g})$ is a $(APCS)_n$. Hence the following theorem holds.

**Theorem 5.2.** *A conformal deformation of every $(APCS)_n$ metric is a $(APCS)_n$ metric, provided that $n > 3$.*

Next if we put $A(X) = \omega(X)$ for all $X$ in (5.15) we immediately see that $(M, \overset{*}{g})$ becomes a conformally recurrent manifold. So we can state the following corollary:

**Corollary 3.** *A conformal deformation of every $(APCS)_n$ metric is a conformally recurrent metric, provided that $n > 3$ and $A(X) = \omega(X)$ for all $X$.*

## 6. Example of an $(APCS)_n$

In this section we want to construct an example of an almost pseudo conformally symmetric manifold.

On coordinate space $\mathbb{R}^n$ (with coordinates $x^1, x^2, \ldots, x^n$) we define a Riemannian space $V_n$. We calculate the components of the curvature tensor, the Ricci tensor, the conformal curvature tensor and of its covariant derivatives and then we verify the relation (1.9).

Let each Latin index runs over $1, 2, \ldots, n$ and each Greek index over $2, 3, \ldots, (n-1)$. We define a Riemannian metric on the $\mathbb{R}^n (n \geq 4)$ by the formula

$$ds^2 = \phi (dx^1)^2 + K_{\alpha\beta} dx^\alpha dx^\beta + 2 dx^1 dx^n, \tag{6.1}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant and $\phi$ is a function of $x^1, x^2, \ldots, x^{n-1}$ and independent of $x^n$. In the metric considered, the only non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are [26]

$$\begin{gathered} \Gamma^\beta_{11} = -\frac{1}{2} K^{\alpha\beta} \phi_{.\alpha}, \quad \Gamma^n_{11} = \frac{1}{2}\phi_{.1}, \quad \Gamma^n_{1\alpha} = \frac{1}{2}\phi_{.\alpha}, \\ R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta}, \quad R_{11} = \frac{1}{2} K^{\alpha\beta}\phi_{.\alpha\beta}, \end{gathered} \tag{6.2}$$

where '.' denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$.

Here we consider $K_{\alpha\beta}$ as Kronecker symbol $\delta_{\alpha\beta}$ and

$$\phi = (M_{\alpha\beta} + \delta_{\alpha\beta}) x^\alpha x^\beta e^{(x^1)^2},$$

where $M_{\alpha\beta}$ are constant and satisfy the relations

$$M_{\alpha\beta} = 0, \text{ for } \alpha \neq \beta,$$
$$\neq 0, \text{ for } \alpha = \beta,$$

$$\sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0. \tag{6.3}$$

This is to be noted that such type of metric with different form of $\phi$ was considered by Grycak and Hotloś [21]. Now according to our consideration we have the following relations:

$$\phi_{.\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta})e^{(x^1)^2},$$
$$\delta_{\alpha\beta}\delta^{\alpha\beta} = n-2 \text{ and } \delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0.$$

Therefore,

$$\delta^{\alpha\beta}\phi_{.\alpha\beta} = 2(\delta^{\alpha\beta}M_{\alpha\beta} + \delta^{\alpha\beta}\delta_{\alpha\beta})e^{(x^1)^2} = 2(n-2)e^{(x^1)^2}.$$

Since $\phi_{.\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components for $R_{hijk}$ and $R_{ij}$ in virtue of (6.2) are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\phi_{.\alpha\alpha} = (1 + M_{\alpha\alpha})e^{(x^1)^2}$$

and

$$R_{11} = \frac{1}{2}\phi_{.\alpha\beta}\delta^{\alpha\beta} = (n-2)e^{(x^1)^2}.$$

By Lemma 1 of [21] we can easily verify that this space is a semisymmetric space, i.e., $R_{hijk,lm} = R_{hijk,ml}$ where ',' denotes covariant differentiation with respect to the metric tensor $g_{ij}$, holds for this space.

We recall that a pseudo-Riemannian manifold [semi-Riemannian manifold] $(M, g)$ is said to be Ricci-simple if

$$\operatorname{rank}(S) \leq 1$$

on $M$, where $S$ is the Ricci tensor of the manifold. Note that Ricci-simple manifolds are special quasi-Einstein manifolds. Since the manifolds determined in this section are non-Ricci flat Ricci-simple manifolds, they are non-Einstein.

Again from (6.1) we obtain $g_{ni} = g_{in} = 0$ for $i \neq 1$ which implies $g^{11} = 0$. Hence $R = g^{ij}R_{ij} = g^{11}R_{11} = 0$. Therefore, $V_n$ will be a space whose scalar curvature is zero. Hence the only non-zero components of the conformal curvature tensor $C_{hijk}$ are

$$(6.4)\qquad C_{1\alpha\alpha 1} = R_{1\alpha\alpha 1} - \frac{1}{n-2}(g_{\alpha\alpha}R_{11})$$
$$= (1+M_{\alpha\alpha})e^{(x^1)^2} - \frac{1}{n-2}(n-2)e^{(x^1)^2} = M_{\alpha\alpha}e^{(x^1)^2}$$

which never vanish. Now the only non-zero components of $C_{hijk,m}$ are

$$(6.5)\qquad C_{1\alpha\alpha 1,1} = 2x^1 M_{\alpha\alpha}e^{(x^1)^2} = 2x^1 C_{1\alpha\alpha 1} \neq 0.$$

Hence $V_n$ is neither conformally flat nor conformally symmetric [7]. We shall now show that $V_n$ is a $(APCS)_n$. Let us consider the associated 1-form as follows:

$$(6.6)\qquad A_i(x) = \begin{cases} x^1, \text{ for i=1} \\ 0, \text{ otherwise,} \end{cases}$$

$$(6.7)\qquad B_i(x) = \begin{cases} -x^1, \text{ for i=1} \\ 0, \quad \text{otherwise,} \end{cases}$$

at any point $x \in V_n$.

To verify the relation (1.9) it is sufficient to prove the followings:

$$(6.8)\qquad C_{1\alpha\alpha 1,1} = (3A_1 + B_1)C_{1\alpha\alpha 1},$$
$$(6.9)\qquad C_{11\alpha 1,\alpha} = A_1 C_{\alpha 1\alpha 1} + A_1 C_{1\alpha\alpha 1},$$
$$(6.10)\qquad C_{1\alpha 11,\alpha} = A_1 C_{1\alpha\alpha 1} + A_1 C_{1\alpha 1\alpha},$$

as for the case other than (6.8), (6.9) and (6.10) the components of each term of (1.9) vanish identically and the relation (1.9) holds trivially. Now from (6.4), (6.5), (6.6) and (6.7) we get the following relation for the right hand side (r.h.s.) and the left hand side (l.h.s.) of (6.8)

$$r.h.s.\ of\ (6.8) = (3A_1 + B_1)C_{1\alpha\alpha 1} = (3x^1 - x^1)C_{1\alpha\alpha 1}$$
$$= 2x^1 C_{1\alpha\alpha 1} = C_{1\alpha\alpha 1,1} = l.h.s.\ of\ (6.8).$$

Now

$$r.h.s. \text{ of } (6.9) = x^1(C_{\alpha 1\alpha 1} + C_{1\alpha\alpha 1})$$
$$= 0 \text{ (by skew-symmetric properties of } C_{hijk})$$
$$= l.h.s. \text{ of } (6.9).$$

By similar argument as in (6.9) it can be shown that the relation (6.10) is also true.

It is to be noted that (1.9) can be satisfied by a number of 1-forms $A$, $B$, namely, by those which fulfil (6.8), (6.9), (6.10). Thus we can state the following:

**THEOREM 6.1.** *Let $V_n (n \geq 4)$ be a Riemannian space with the metric of the form*

$$ds^2 = \phi(dx^1)^2 + \delta_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n,$$
$$\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta e^{(x^1)^2},$$

*where $M_{\alpha\beta}$ are constant defined by (6.3), then $V_n$ is an almost pseudo conformally symmetric space with zero scalar curvature which is neither conformally flat nor conformally symmetric.*

**Acknowledgement.** The authors are thankful to the referee for his valuable suggestions towards the improvement of this paper and his remarks for further studies on this manifold.

## References

[1] T. Adati, T. Miyazawa, *On a Riemannian space with recurrent conformal curvature*, Tensor (N. S.) 18 (1967), 348–354.

[2] K. Amur, S. S. Pujar, *On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection*, Tensor (N. S.) 32 (1978), 35–38.

[3] M. Belkhelfa, R. Deszcz, M. Głogowska, M. Hotloś, D. Kowalczyk, L. Verstraelen, *On some type of curvature conditions*, Banach Center Publ. 57 , Inst. Math. Polish Acad. Sci. 2002, 179–194.

[4] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, (1987).

[5] E. Cartan, *Sur une classe remarquable d'espaces de Riemannian*, Bull. Soc. Math. France 54 (1926), 214–264.

[6] M. C. Chaki, *On pseudo symmetric manifolds*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi 33 (1987), 53–58.

[7] M. C. Chaki, B. Gupta, *On conformally symmetric spaces*, Indian J. Math. 5 (1963), 113–295.

[8] U. C. De, H. A. Biswas, *On pseudo conformally symmetric spaces*, Bull. Calcutta Math. Soc. 85 (1993), 479–485.

[9] U. C. De, S. Bandyopadhyay, *On weakly conformally symmetric spaces*, Publ. Math. Debrecen 57 (2000), 71–78.

[10] U. C. De, A. K. Gazi, A. Kalam, *On almost pseudo symmetric manifolds*, Ann. Univ. Sci. Budapest, Sec. Math. 2009, to appear.

[11] A. Derdziński, W. Roter, *Global properties of indefinite metrics with parallel Weyl tensor*, Pure and Applied Differential Geometry-PADGE 2007, Shaker Verlag, Aachen, 2007, 63–72.

[12] A. Derdziński, W. Roter, *Projectively flat surfaces, null parallel distributions and conformally symmetric manifolds*, Tôhoku Math. J. 59 (2007), 566–602.

[13] A. Derdziński, W. Roter, *The local structure of conformally symmetric manifolds*, Bull. Belg. Math. Soc. Simon Stevin, 16 (2009), 1–12.

[14] A. Derdziński, W. Roter, *On compact manifolds admitting indefinite metrics with parallel Weyl tensor*, ar xiv; 0705.1106.

[15] A. Derdziński, W. Roter, *Compact pseudo Riemannian manifolds with parallel Weyl tensor*, Math. DG/0702491.

[16] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belg. Math. Soc. Serie A 44(1992), 1–34.
[17] R. Deszcz, W. Grycak, *On some classes of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 19 (1987), 311–322.
[18] R. Deszcz, S. Haesen, L. Verstraelen, *On natural symmetries*, Topics in Differential Geometry, Ed. Rom. Acad. Bucharest 2008, 249–308.
[19] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1949.
[20] D. Ferus, *A remark on Codazzi tensors in constant curvature space*, Lecture note in Mathematics, 838, Global Differential Geometry and Global Analysis, Springer Verlag, New York, (1981), p. 257.
[21] W. Grycak, M. Hotloś, *On the existence of certain types of Riemannian metrics*, Colloquium Mathematicum 47 (1982), 31–37.
[22] S. Kobayashi, K. Nomizu, *Foundation of Differential Geometry*, 4 , Interscience Publishers, (1983).
[23] B. O'Neill, *Semi-Riemannian Geometry with Applications to the Relativity*, Academic Press, New York-London, 1983.
[24] M. Prvanović, *Conformally quasi-recurrent manifolds*, Proceedings of the $5^{th}$ National Seminar on Finsler space, February 10–15, 1988, Brasov, Romania, 321–328.
[25] M. Prvanović, *Some theorems on conformally quasi-recurrent manifolds*, Univ. u Novom Sadu zb. Rad. Prirod. Mat. Fak. Ser. Mat. 19, 2, (1989), 21–31.
[26] W. Roter, *On conformally symmetric Ricci-recurrent spaces*, Colloquium Mathematicum 31 (1974), 87–96.
[27] J. A. Schouten, *Ricci Calculus* (2nd Ed.), Springer Verlag, (1954).
[28] R. N. Sen, M. C. Chaki, *On curvature restrictions of a certain kind of conformally flat Riemannian space of class one*, Proc. Nat. Inst. Sci. India 33 Part A(1967), 100–102.
[29] SOOŚ, G. *Über die geodätischen Abbildungen von Riemannschen Räumen auf projektiv symmetrische Riemannsche Räume*, Acta Math. Acad. Sci. Hungar. Tom 9 (1958), 359–361.
[30] L. Tamássy, T. Q. Binh, *On weakly symmetric and weakly projectively symmetric Riemannian manifolds*, Colloq. Math. Soc. Janos Bolyai 56 (1989), 663–670.
[31] L. Tamássay, T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N. S.) 53 (1993), 140–148.
[32] A. G. Walker, *On Ruse's space of recurrent curvature*, Proc. London Math. Soc. 52 (1951), 36–64.
[33] K. Yano, *Integral Formulas in Riemannian Geometry*, Marcel Dekker, New York, 1970.
[34] K. Yano, *On semi-symmetric connection*, Rev. Roumaine Math. Pures Appl. 15 (1970), 1570–1586.

Uday Chand De
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
35, B.C. Road
KOLKATA-700019
WEST BENGAL, INDIA
E-mail: uc_de@yahoo.com

Abul Kalam Gazi
MOYNAGODI E.B.A.U HIGH MADRASAH
P.O. NOAPARA, 24-PARGS (NORTH)
KOLKATA-700125
W. B., INDIA

*Received February 5, 2008; revised version April 14, 2009.*