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CONTACT HORIZONTALLY CONFORMAL SUBMERSIONS

Abstract. Using the notion of horizontally conformal submersion, we generalize the contact metric submersions and obtain classification theorems for this submersion when the total manifold has some special almost contact structures.

1. Introduction

Let (M, g_M) and (B, g_B) be Riemannian manifolds and $F : M \longrightarrow B$ be a smooth submersion. Then F is called a Riemannian submersion if

$$g_M(X, Y) = g_B(F_*X, F_*Y)$$

for every $X, Y \in \Gamma((kerF_*)^{\perp})$, where $*$ is symbol for the tangent map. The theory of Riemannian submersions was initiated by O'Neill in [12] and it has been used widely in differential geometry to investigate the geometry of manifolds. In [7] (see also, [5], [6] and [8]), Chinea introduced almost contact metric submersion between two almost contact manifolds with compatible metrics as a Riemannian submersion which is in addition an almost contact map. Then he showed that various properties of the total space are preserved. For Riemannian submersions between various manifolds, see: [5], [9] and [14].

On the other hand, as a generalization of Riemannian submersions, horizontally conformal submersions are defined as follows [1]: suppose that (M, g_M) and (B, g_B) are Riemannian manifolds and $F : M \longrightarrow B$ is a smooth submersion, then F is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_M(X, Y) = g_B(F_*X, F_*Y) \tag{1.1}$$

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for every $X, Y \in \Gamma((kerF_*)^\perp)$, where g_M and g_B are the Riemannian metrics of M and B, respectively. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We note that horizontally conformal submersions are special horizontally conformal maps which were introduced independently by Fuglede [10] and Ishihara [11]. We also note that a horizontally conformal submersion $F : M \longrightarrow B$ is said to be horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0 \tag{1.2}$$

at $p \in M$, where $\mathcal{H}$ is the projection on the horizontal space $(kerF_{*p})^\perp$. A vector field X on M is said to be projectable if there exists a vector field X' on B such that $F_* X_p = X'_{F(p)}$, for all $p \in M$. In this case X' and X are called $F-$ related. As it is well known, the vector field X is called a basic vector field.

In this paper, we consider horizontally conformal submersion between almost contact metric manifolds and show that vertical $kerF_*$ and horizontal $(kerF_*)^\perp$ spaces of a contact horizontally conformal submersion are invariant with respect to the almost contact structure of the total manifold M. Also we obtain that if M is a normal almost contact metric manifold and B is an almost metric manifold, then B is also normal if and only if F is a special horizontally homothetic map. Moreover, we investigate the contact character of the base manifold when the total manifold is almost Sasakian, cosymplectic or Kenmotsu.

We have seen from above results that the geometry of contact horizontally conformal submersions is quite different from the geometry of almost contact metric submersions. For example, if M is a Sasakian manifold and B is an almost contact metric manifold, then the almost contact metric submersion $F : M \longrightarrow B$ implies that B is also a Sasakian manifold. But in the contact horizontally conformal situation, this is not true even for additional condition.

2. Preliminaries

In this section, we give brief information for almost contact manifolds. Our main reference is Blair's book [2]. We also mention the second fundamental form of a map only as much as we need to carry out our work on contact horizontally conformal submersions.

An odd dimensional Riemannian manifold (M, g) is called almost contact metric manifold if there is a $(1,1)$ tensor field ϕ, a vector field ξ, called the characteristic vector field and its 1-form η such that

$$\phi^2(X) = -X + \eta(X)\xi \tag{2.1}$$

$$\eta(\xi) = 1 \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for $X, Y \in \Gamma(TM)$. It follows that $\phi\xi = 0$ and $\eta \circ \phi = 0$. An almost contact metric manifold M is said to have a normal contact structure if $N_\phi + d\eta \otimes \xi = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ and it is defined by

$$N_\phi(X,Y) = [\phi X, \phi Y] - [X,Y] - \phi[X, \phi Y] - \phi[\phi X, Y] \tag{2.4}$$

for $X, Y \in \Gamma(TM)$. The Sasakian form of an almost contact metric manifold is given by $\Phi(X,Y) = g(X, \phi Y)$. An almost contact metric manifold M is called almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$. A normal almost cosymplectic manifold is called cosymplectic. Let M be an almost contact metric manifold, if $\Phi = d\eta$, then M is called a contact metric manifold. A normal contact metric manifold is called a Sasakian manifold. Equivalently an almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.5}$$

Moreover, a $c-$Sasakian manifold [9], $c \in \mathbf{R}$, is an almost contact metric manifold which is normal and satisfies $d\eta = c\Phi$. An almost contact metric manifold is $c-$ Sasakian if and only if the following formula holds

$$(\nabla_X \phi)Y = c\{g(X,Y)\xi - \eta(Y)X\}. \tag{2.6}$$

Besides Sasakian manifolds, another well known almost contact metric manifolds are Kenmotsu manifolds and they are characterized by the following tensor equation

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.7}$$

As a generalization of Sasakian and Kenmotsu manifolds, an almost contact metric manifold M is called trans-Sasakian manifold of type (α, β) ([3], [13]) if and only if

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}. \tag{2.8}$$

From (2.8), it is easy to see that a trans-Sasakian manifold is Sasakian, $c-$Sasakian or Kenmotsu according as $\beta = 0$, $\alpha = 1$; or $\beta = 0$, $\alpha = c$; or $\alpha = 0$, $\beta = 1$, respectively.

Finally, we recall the second fundamental form of a map [1]. Let (M, g_M) and (B, g_B) be Riemannian manifolds and suppose that $F : M \longrightarrow B$ is a smooth mapping between them. Then the differential F_* of F can be viewed a section of the bundle $Hom(TM, F^{-1}TB) \longrightarrow M$, where $F^{-1}TB$ is the pullback bundle which has fibres $(F^{-1}TB)_p = T_{F(p)}B, p \in M$.

$Hom(TM, F^{-1}TB)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of F is given by

$$(\nabla F_*)(X,Y) = \overset{F}{\nabla}_X F_* Y - F_*(\overset{M}{\nabla}_X Y) \tag{2.9}$$

for $X, Y \in \Gamma(TM)$, where ∇^F is the pullback connection along F. It is known that the second fundamental form is symmetric.

3. Contact horizontally conformal submersions

In this section, we consider contact horizontally conformal submersion between almost contact metric manifolds and check the contact structure of the base manifold when the total manifold has a special contact structure. First recall that a submersion F (or a map) between almost contact manifolds $(M, \phi_M, \xi_M, \eta_M)$ and $(B, \phi_B, \xi_B, \eta_B)$ is called the $(\phi_M, \phi_B)-$ holomorphic if

$$F_* \circ \phi_M = \phi_B \circ F_*. \tag{3.1}$$

It is easy to see [4] that ξ_M belongs to horizontal distribution $(kerF_*)^\perp$ when F is a submersion.

Definition 1. Let M^{2m+1} and B^{2n+1} be manifolds carrying the almost contact metric structures $(\phi_M, \xi_M, \eta_M, g_M)$ and $(\phi_B, \xi_B, \eta_B, g_B)$, respectively. A $(\phi_M, \phi_B)-$ holomorphic horizontally conformal submersion $F : M^{2m+1} \longrightarrow B^{2n+1}$ is called contact horizontally conformal submersion if the following is satisfied:

$$F_* \xi_M = \lambda \xi_B. \tag{3.2}$$

It is clear that every contact submersion is a special contact horizontally conformal submersion with $\lambda = 1$.

Remark 1. We note that contact horizontally conformal submersions were already studied in [4] by Burel under the name of semi-conformal $(\phi_M, \phi_B)-$ holomorphic submersion. He investigates the harmonicity of this map in that paper. Our objective is to obtain classification theorems when the total space has some geometric structures. For the notations, we follow [1] and [9].

Let $kerF_{*p}$ be the kernel space of F_* and denote its orthogonal complementary space in T_pM by $(kerF_*)^\perp_p$ at $p \in M$. Then one can observe that vertical distribution $kerF_*$ is ϕ_M- invariant, see [4]. Then invariant $kerF_*$ implies that $g_M(\phi_M X, V) = -g_M(X, \phi_M V) = 0$ for $X \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma(kerF_*)$. This shows that $(kerF_*)^\perp$ is also invariant and any fibre of the contact horizontally conformal submersion is an invariant submanifold. Thus, we have the following result.

PROPOSITION 3.1. *Let F be a contact horizontally conformal submersion between almost contact metric manifolds. Then the distribution $ker F_*$ and $(ker F_*)^{\perp}$ are invariant with respect to the almost contact structure of the total manifold and the fibres of F are invariant submanifolds.*

For any submersion $F : M \longrightarrow B$ between Riemannian manifolds, the restriction of the differential F_{*p} to the horizontal space $(ker F_{*p})^{\perp}$ maps that space isomorphically on to $T_{F(p)}B$. Denote its inverse by $\hat{}$, then for any vector $\hat{Z} \in (ker F_{*p})^{\perp}$ is called the horizontal lift of Z. If Z is a vector field on an open subset V of B, then the horizontal lift of Z is horizontal vector field $\hat{Z}$ on $F^{-1}(V)$ such that $F_*(\hat{Z}) = Z \circ F$, [1].

We denote the space $(ker F_*)^{\perp} - span\{\xi_M\}$ by $\tilde{\mathcal{D}}$. Then, we say that a contact horizontally conformal submersion is $\tilde{\mathcal{D}}-$ homothetic if $X(\lambda) = 0$ for every $X \in \Gamma(\tilde{\mathcal{D}})$. Now, we can state and prove our first classification theorem for contact horizontally conformal submersions.

THEOREM 3.1. *Let $(M, \phi_M, \xi_M, \eta_M, g_M)$ be a normal almost contact metric manifold and $(B, \phi_B, \xi_B, \eta_B)$ be an almost contact metric manifold. Suppose that $F : M \longrightarrow B$ is a contact horizontally conformal submersion. Then B is normal if and only if F is $\tilde{\mathcal{D}}-$ homothetic.*

Proof. Let $\bar{X}$ and $\bar{Y}$ be vector fields on an open subset of B, and X, Y their horizontal lifts to M. First, using (3.1) and $F_*[X,Y] = [F_*X, F_*Y]$, it is easy to see that $F_*N_{\phi_M}(X,Y) = N_{\phi_B}(F_*X, F_*Y)$ for basic vector fields $X, Y \in \Gamma(TM)$. Now, for $X, Y \in \Gamma((ker F_*)^{\perp})$, we have

$$g_M(X,Y) = \frac{1}{\lambda^2} g_B(F_*X, F_*Y). \tag{3.3}$$

Using (3.3), we derive

$$2d\eta_M(X,Y) = X\left[\frac{1}{\lambda^2} g_B(F_*Y, F_*\xi_M)\right] - Y\left[\frac{1}{\lambda^2} g_B(F_*X, F_*\xi_M)\right] - \frac{1}{\lambda^2} g_B([F_*X, F_*Y], F_*\xi_M).$$

Thus we get

$$\begin{aligned} 2d\eta_M(X,Y) = {} & X\left(\frac{1}{\lambda^2}\right) g_B(F_*Y, F_*\xi_M) + \frac{1}{\lambda^2} \bar{X} g_B(F_*Y, F_*\xi_M) \\ & - Y\left(\frac{1}{\lambda^2}\right) g_B(F_*X, F_*\xi_M) - \frac{1}{\lambda^2} \bar{Y} g_B(F_*X, F_*\xi_M) \\ & - \frac{1}{\lambda^2} g_B([F_*X, F_*Y], F_*\xi_M). \end{aligned}$$

Hence, we have

$$2d\eta_M(X,Y) = -\frac{2}{\lambda^3}X(\lambda)g_B(F_*Y, F_*\xi_M) + \frac{1}{\lambda^2}\bar{X}g_B(F_*Y, F_*\xi_M) + \frac{2}{\lambda^3}Y(\lambda)g_B(F_*X, F_*\xi_M) - \frac{1}{\lambda^2}\bar{Y}g_B(F_*X, F_*\xi_M) - \frac{1}{\lambda^2}g_B([F_*X, F_*Y], F_*\xi_M).$$

Since F is horizontally conformal, using (1.1) in the above equation, we arrive at

$$2d\eta_M(X,Y) = -\frac{2X(\lambda)}{\lambda}g_M(Y, \xi_M) + \frac{1}{\lambda^2}\bar{X}g_B(F_*Y, F_*\xi_M) + \frac{2Y(\lambda)}{\lambda}g_M(X, \xi_M) - \frac{1}{\lambda^2}\bar{Y}g_B(F_*X, F_*\xi_M) - \frac{1}{\lambda^2}g_B([F_*X, F_*Y], F_*\xi_M).$$

Hence we derive

$$2d\eta_M(X,Y) = -\frac{2X(\lambda)}{\lambda}g_M(Y, \xi_M) + \frac{1}{\lambda^2}Xg_B(\bar{Y}\circ F, \lambda\xi_B \circ F) + \frac{2Y(\lambda)}{\lambda}g_M(X, \xi_M) - \frac{1}{\lambda^2}Yg_B(\bar{X}\circ F, \lambda\xi_B \circ F) - \frac{1}{\lambda^2}g_B([F_*X, F_*Y], F_*\xi_M).$$

Then using (3.2), we obtain

$$2d\eta_M(X,Y) = -\frac{2X(\lambda)}{\lambda}g_M(Y, \xi_M) + \frac{1}{\lambda^2}X(\lambda)g_B(F_*Y, \xi_B) + \frac{1}{\lambda}\bar{X}g_B(F_*Y, \xi_B) + \frac{2Y(\lambda)}{\lambda}g_M(X, \xi_M) - \frac{1}{\lambda^2}Y(\lambda)g_B(F_*X, \xi_B) - \frac{1}{\lambda}\bar{Y}g_B(F_*X, \xi_B) - \frac{1}{\lambda^2}g_B([F_*X, F_*Y], F_*\xi_M).$$

Thus from (1.1) and (3.2)we derive

$$2d\eta_M(X,Y) = -\frac{2X(\lambda)}{\lambda}g_M(Y, \xi_M) + \frac{X(\lambda)}{\lambda}g_M(Y, \xi_M) + \frac{1}{\lambda}\bar{X}g_B(F_*Y, \xi_B) + \frac{2Y(\lambda)}{\lambda}g_M(X, \xi_M) - \frac{Y(\lambda)}{\lambda}g_B(X, \xi_M) - \frac{1}{\lambda}\bar{Y}g_B(F_*X, \xi_B) - \frac{1}{\lambda}g_B([F_*X, F_*Y], \xi_B).$$

Hence, we have

$$2d\eta_M(X,Y) = -X(ln\lambda)\eta_M(Y) + Y(ln\lambda)\eta_M(X) + \frac{2}{\lambda}d\eta_B(F_*X, F_*Y). \tag{3.4}$$

On the other hand, since M is normal, we have

$$F_*(N_{\phi_M}(X,Y) + d\eta_M(X,Y)\xi_M) = 0.$$

Hence, we derive

$$N_{\phi_B}(F_*X, F_*Y) + d\eta_M(X,Y)F_*(\xi_M) = 0.$$

Then (3.4) implies that

$$N_{\phi_B}(F_*X, F_*Y) + \left\{-\frac{1}{2}\{X(ln\lambda)\eta_M(Y) - Y(ln\lambda)\eta_M(X)\right\} + \frac{1}{\lambda}d\eta_B(F_*X, F_*Y)\}\lambda\xi_B = 0. \tag{3.5}$$

Now, if B is also normal, then we have

$$\{-X(\lambda)\eta_M(Y) + Y(\lambda)\eta_M(X)\}\xi_B = 0.$$

Since B is an almost contact metric manifold, we have $\xi_B \neq 0$. Then, for $X = \xi_M$ and $Y \in \Gamma(\tilde{\mathcal{D}})$, above equation gives $Y(\lambda) = 0$, which shows that F is $\tilde{\mathcal{D}}-$ homothetic. Conversely, suppose that F is $\tilde{\mathcal{D}}-$ homothetic, then for $X, Y \in \Gamma(\tilde{\mathcal{D}})$ we have $N_{\phi_B}(F_*X, F_*Y) + d\eta_B(F_*X, F_*Y)\xi_B = 0$. For $X \in \Gamma(\tilde{\mathcal{D}})$ and $Y = \frac{1}{\lambda}\xi_M$, from (3.5) we also have $N_{\phi_B}(F_*X, \xi_B) + d\eta_B(F_*X, \xi_B)\xi_B = 0$. Thus proof is complete.

In a similar way, we have the following theorem:

Theorem 3.2. *Let $(M, \phi_M, \xi_M, \eta_M, g_M)$ be a Sasakian manifold and $(B, \phi_B, \xi_B, \eta_B)$ be an almost contact metric manifold. Suppose that $F: M \longrightarrow B$ is a contact horizontally conformal submersion. Then B is $\frac{1}{\lambda}-$ Sasakian manifold if and only if F is $\tilde{\mathcal{D}}-$ homothetic.*

Proof. Let Φ_M be the Sasakian form of M. First, since $(kerF_*)^{\perp}$ is invariant, we have $\Phi_M(X,Y) = g_M(X, \phi_M Y) = \frac{1}{\lambda^2}g_B(F_*X, F_*\phi_M Y)$ for $X, Y \in \Gamma((kerF_*)^{\perp})$. Using (3.1), we get

$$\Phi_M(X,Y) = \frac{1}{\lambda^2}\Phi_B(F_*X, F_*Y), \tag{3.6}$$

where Φ_B is the Sasakian form of B. On the other hand, Sasakian M implies that $\Phi_M(X,Y) = d\eta_M(X,Y)$ for every $X, Y \in \Gamma(TM)$. Using (3.4) and (3.6) in the above equation, we get

$$\Phi_B(F_*X, F_*Y) = \frac{\lambda}{2}\{-X(\lambda)\eta_M(Y) + Y(\lambda)\eta_M(X) + 2d\eta_B(F_*X, F_*Y)\}. \tag{3.7}$$

From Theorem 3.1, we know that B is normal if and only if F is $\tilde{\mathcal{D}}-$ homothetic. Thus it is enough to show that $\Phi_B = \lambda\, d\eta_B$ for manifold B. Since F is $\tilde{\mathcal{D}}-$ homothetic, for $X, Y \in \Gamma(\tilde{\mathcal{D}})$, from (3.7), we get

$$\Phi_B(F_*X, F_*Y) = \lambda\, d\eta_B(F_*X, F_*Y).$$

For $X \in \Gamma(\tilde{\mathcal{D}})$ and $Y = \frac{1}{\lambda}\xi_M$, we also have

$$\Phi_B(F_*X, \xi_B) = \lambda\, d\eta_B(F_*X, \xi_B)$$

which shows that B is $\frac{1}{\lambda}-$ Sasakian.

When the total manifold of a contact horizontally conformal submersion is almost cosymplectic, we have the following strong result.

THEOREM 3.3. *Let $(M, \phi_M, \xi_M, \eta_M, g_M)$ be an almost cosymplectic manifold, $(B, \phi_B, \xi_B, \eta_B)$ be an almost contact metric manifold and $F : M \longrightarrow B$ be a contact horizontally conformal submersion. Then B is also almost cosymplectic if and only if F is horizontally homothetic submersion.*

Proof. First, recall that we have

$$\begin{aligned} 3d\Phi_M(X,Y,Z) &= X\Phi_M(Y,Z) + Y\Phi_M(Z,X) + Z\Phi_M(X,Y) \\ &\quad -\Phi_M([X,Y],Z) - \Phi_M([Y,Z],X) - \Phi_M([Z,X],Y) \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$. Since F is contact horizontally conformal submersion, from (3.1) and (1.1), we get

$$\begin{aligned} 3d\Phi_M(X,Y,Z) &= X\left[\frac{1}{\lambda^2} g_B(F_*Y, \phi_B F_*Z)\right] + Y[\frac{1}{\lambda^2} g_B(F_*Z, \phi_B F_*X)] \\ &\quad + Z\left[\frac{1}{\lambda^2} g_B(F_*X, \phi_B F_*Y)\right] - \frac{1}{\lambda^2}\{g_B([F_*X, F_*Y], F_*Z) \\ &\quad + g_B([F_*Y, F_*Z], F_*X) + g_B([F_*Z, F_*X], F_*Y)\}, \end{aligned}$$

where X, Y and Z are basic vector fields. Hence we have

$$\begin{aligned} 3d\Phi_M(X,Y,Z) &= -\frac{2}{\lambda^3} X(\lambda) g_B(F_*Y, \phi_B F_*Z) + \frac{1}{\lambda^2}\bar{X} g_B(F_*Y, \phi_B F_*Z) \\ &\quad - \frac{2}{\lambda^3} Y(\lambda) g_B(F_*Z, \phi_B F_*X) + \frac{1}{\lambda^2}\bar{Y} g_B(F_*Z, \phi_B F_*X) \\ &\quad - \frac{2}{\lambda^3} Z(\lambda) g_B(F_*X, \phi_B F_*Y) + \frac{1}{\lambda^2}\bar{Z} g_B(F_*X, \phi_B F_*Y) \\ &\quad - \frac{1}{\lambda^2}\{g_B([F_*X, F_*Y], F_*Z) + g_B([F_*Y, F_*Z], F_*X) \\ &\quad + g_B([F_*Z, F_*X], F_*Y)\}. \end{aligned}$$

Then, using (1.1), (3.1) and the formula of the exterrior derivative, we arrive at

$$(3.8)\quad 3d\Phi_M(X,Y,Z) = -2X(ln\lambda)g_M(Y,\phi_M Z) - 2Y(ln\lambda)g_M(Z,\phi_M X) -2Z(ln\lambda)g_M(X,\phi_M Y) + \frac{3}{\lambda^2}d\Phi_B(F_*X,F_*Y,F_*Z).$$

Since M is almost cosymplectic, from (3.4) and (3.8), we derive

$$(3.9)\quad \frac{3}{\lambda^2}d\Phi_B(F_*X,F_*Y,F_*Z) = 2X(ln\lambda)g_M(Y,\phi_M Z) +2Y(ln\lambda)g_M(Z,\phi_M X) +2Z(ln\lambda)g_M(X,\phi_M Y)$$

and

$$(3.10)\quad \frac{2}{\lambda}d\eta_B(F_*X,F_*Y) = X(ln\lambda)\eta_M(Y) - Y(ln\lambda)\eta_M(X).$$

Now, suppose that B is almost cosymplectic, then from (3.9), for $X = \phi_M Z \in \Gamma(\tilde{\mathcal{D}})$, we get

$$\phi_M Z(ln\lambda)g_M(Y,\phi_M Z) - Y(ln\lambda)g_M(Z,Z) + Z(ln\lambda)g_M(\phi_M Z,\phi_M Y) = 0.$$

Thus, for $Y = \xi_M$, we obtain

$$(3.11)\quad \xi_M(ln\lambda)g_M(Z,Z) = 0.$$

In similar way, from (3.10), for $X \in \Gamma(\tilde{\mathcal{D}})$ and $Y = \xi_M$, we have

$$(3.12)\quad X(ln\lambda) = 0.$$

Since $(kerF_*)^\perp = \tilde{\mathcal{D}} \oplus span\{\xi_M\}$ and g_M is a Riemannian metric, we have $T(ln\lambda) = 0$ for $T \in \Gamma((kerF_*)^\perp)$, which proves that F is horizontally homothetic. The converse is clear from (3.9) and (3.10).

Finally, we investigate contact character of the base manifold of a contact horizontally conformal submersion when the total manifold is Kenmotsu manifold.

THEOREM 3.4. *Let $F : M \longrightarrow B$ be a contact horizontally conformal submersion from a Kenmotsu manifold $(M,\phi_M,\xi_M,\eta_M,g_M)$ to an almost contact metric manifold, (B,ϕ_B,ξ_B,η_B) such that $dim(B) \neq 2$. Then the following statements are equivalent:*

(a) *B is a trans-Sasakian manifold of type $(0,\frac{1}{\lambda})$.*
(b) *For every $X,Y \in \Gamma((kerF_*)^\perp)$, $(\nabla F_*)(X,\phi_M Y) = \phi_B(\nabla F_*)(X,Y)$.*
(c) *F is horizontally homothetic.*

Proof. Since M is Kenmotsu, from (2.7), (3.1), (3.3) and (3.2) we have

$$\overset{M}{\nabla}_X\phi_M Y - \phi_M\overset{M}{\nabla}_X Y = \frac{1}{\lambda^2}g_B(\phi_B F_*X,F_*Y)\xi_M - \frac{1}{\lambda}\eta_B(F_*Y)\phi_M X$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. Hence, using again (3.2) and (3.1), we get

$$F_*(\overset{M}{\nabla}_X \phi_M Y) - F_*(\phi_M \overset{M}{\nabla}_X Y) = \frac{1}{\lambda}\{g_B(\phi_B F_*X, F_*Y)\xi_B - \eta_B(F_*Y)\phi_B F_*X\}.$$

Considering (2.9) and (3.1) we write

$$-(\nabla F_*)(X, \phi_M Y) + \overset{F}{\nabla}_X \phi_B F_*Y - \phi_B F_* \overset{M}{\nabla}_X Y = \frac{1}{\lambda}\{g_B(\phi_B F_*X, F_*Y)\xi_B - \eta_B(F_*Y)\phi_B F_*X\}.$$

Hence, we get

$$-(\nabla F_*)(X, \phi_M Y) + \overset{F}{\nabla}_X \phi_B F_*Y + \phi_B(\nabla F_*)(X, Y) - \phi_B \overset{F}{\nabla}_X F_*Y = \frac{1}{\lambda}\{g_B(\phi_B F_*X, F_*Y)\xi_B - \eta_B(F_*Y)\phi_B F_*X\}.$$

Thus, we obtain

$$(\overset{F}{\nabla}_X \phi_B)F_*Y = (\nabla F_*)(X, \phi_M Y) - \phi_B(\nabla F_*)(X, Y) + \frac{1}{\lambda}\{g_B(\phi_B F_*X, F_*Y)\xi_B - \eta_B(F_*Y)\phi_B F_*X\}. \tag{3.13}$$

Then (3.13) proves (a)$\Leftrightarrow$ (b). Now suppose that (b) holds. First, recall from ([1], Lemma 4.5.1, page: 119), we have

$$(\nabla F_*)(X, Y) = X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*X - g(X, Y)F_*(grad \ln\lambda) \tag{3.14}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. Substituting (3.14) in (b), we get

$$\phi_M Y(ln\lambda)F_*X - g_M(X, \phi_M Y)F_*(grad\, ln\lambda) = Y(ln\lambda)F_*(\phi_M X) - g_M(X, Y)F_*(\phi_M gradln\lambda) \tag{3.15}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. Then for $Y = \xi_M$ and $X \in \Gamma(\tilde{\mathcal{D}})$, we derive

$$\xi_M(ln\lambda)\phi_B F_*X = 0.$$

Applying ϕ_B to this equation and using (2.1), we have

$$-\xi_M(ln\lambda)F_*X + \eta_B(F_*X)\xi_M(ln\lambda)\xi_B = 0.$$

Here, $\eta_B(F_*X) = g_B(F_*X, \xi_B) = \frac{1}{\lambda}g_B(F_*X, F_*\xi_M) = \lambda g_M(X, \xi_M)$. Hence, we obtain

$$\eta_B(F_*X) = 0. \tag{3.16}$$

Using (3.16), we arrive at

$$\xi_M(ln\lambda)F_*X = 0.$$

Then non-constant F submersion implies that

$$\xi_M(ln\lambda) = 0. \tag{3.17}$$

On the other hand, interchanging the role of X and Y in (3.15), we get

$$(3.18)\quad \phi_M X(ln\lambda)F_*Y - g_M(Y, \phi_M X)F_*(gradln\lambda) = X(ln\lambda)F_*(\phi_M Y) - g_M(X,Y)F_*(\phi_M gradln\lambda)$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. Then from (3.15) and (3.18) we have

$$(3.19)\quad \phi_M Y(ln\lambda)F_*X - \phi_M X(ln\lambda)F_*Y - 2g_M(X, \phi_M Y)F_*(gradln\lambda) = Y(ln\lambda)F_*(\phi_M X) - X(ln\lambda)F_*(\phi_M Y).$$

Taking $Y = \xi_M$ and $X \in \Gamma(\tilde{\mathcal{D}})$ in (3.19), using $\phi_M \xi_M = 0$ we obtain

$$-\phi_M X(ln\lambda)F_*(\xi_M) = \xi_M(ln\lambda)F_*(\phi_M X).$$

Using again (3.17), we arrive at

$$-\phi_M X(ln\lambda)F_*(\xi_M) = 0.$$

Then (3.2) implies that

$$-\phi_M X(\lambda)\xi_B = 0.$$

Since $\tilde{\mathcal{D}}$ is invariant with respect to ϕ_M and $\xi_B \neq 0$, we conclude that

$$(3.20)\qquad Z(\lambda) = 0$$

for any $Z \in \Gamma(\tilde{\mathcal{D}})$. Then from (3.17) and (3.20), we obtain $Y(\lambda) = 0$ for $Y \in \Gamma((kerF_*)^{\perp})$, which shows that F is horizontally homothetic. Conversely, if F is horizontally homothetic, from (3.14) and (3.1), one can obtain that (b) holds. Thus we have shown that (b)⇔ (c). This completes the proof. ∎

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