DEMONSTRATIO MATHEMATICA
Vol. XLII No 4 2009

Ewa Fabińska

COVERING A REDUCED POLYGON BY A DISK

Abstract. A convex body R in Euclidean d-space E^d is *reduced* if every convex body $K \subset R$ different from R has thickness smaller than the thickness $\Delta(R)$ of R. We prove that every reduced polygon $P \subset E^2$ is contained in a disk of radius $\Delta(P)$ centered at a boundary point of P.

The minimum width of a convex body C in Euclidean d-space E^d is called the *thickness* of C and it is denoted by $\Delta(C)$. A convex body $R \subset E^d$ *reduced* if $\Delta(K) < \Delta(R)$ for every convex body $K \subset R$ different from R. The class of reduced bodies is larger than the class of bodies of constant width (for $d \geq 2$). In particular, the regular odd-gons are examples of planar reduced bodies. Various properties of reduced bodies are derived in [1], [2], [4–6] and [8]. Lassak [6] conjectured that every reduced body $R \subset E^2$ is a subset of a disk of radius $\Delta(R)$ centered at a boundary point of R. We prove this conjecture to be true for the reduced polygons.

The notation of our paper is consistent with this of [5]. The diameter of a convex body C is denoted by $\mathrm{diam}(C)$ and its boundary by $\mathrm{bd}(C)$. The closed segment jointing points x and y is denoted by xy, and the distance of x and y by $|xy|$. We take the positive orientation of $\mathrm{bd}(C)$. If $x, y \in \mathrm{bd}(C)$, by $\widehat{xy}$ we mean the arc of $\mathrm{bd}(C)$ from x to y, according to the positive orientation. Let $P = v_1 v_2 \dots v_n$ be a reduced n-gon (the vertices are numbered according to the positive orientation). We identify vertices v_k and v_m whenever $m = k \pmod n$. From Theorem 7 of [5] it follows that the orthogonal projection t_i of v_i on the straight line containing the side $v_{i+(n-1)/2} v_{i+(n+1)/2}$ is strictly between the end-points of this side. Denote by β_i the angle $\angle v_i v_{i+(n+1)/2} t_{i+(n+1)/2}$. Let D_i be the disk of radius $\Delta(P)$ centered at t_i, and O_i the disk of radius $\Delta(P)$ centered at v_i. Let L_i be the line passing through v_i and $t_{i+(n+1)/2}$, and M_i the line passing through t_i and v_i.

2000 *Mathematics Subject Classification*: 52A10; 52A40; 52C15.
Key words and phrases: reduced convex body, reduced polygon, width.

We omit an easy proof of the following claim.

Claim. *Let S be a disk with center c and let $|w_1w_3| = \operatorname{diam}(S)$. If $w_2 \in cw_3$, then for every point $d \in \operatorname{bd}(S)$ we have $|w_2w_3| \le |w_2d| \le |w_2w_1|$.*

Theorem. *For every reduced polygon $P \subset E^2$ there exists a disk D of radius $\Delta(P)$ centered at a boundary point of P such that $P \subset D$.*

Proof. Without any loss of generality we may assume that $\Delta(P) = 1$. According to Theorem 4 of [5] the only point on the side $v_{i+(n-1)/2}v_{i+(n+1)/2}$ in the distance 1 from v_i is t_i. Thus, the only candidates for D are the disks D_i for $i = 1, \dots, n$, that is, $D = D_i$ for some $i \in \{1, \dots, n\}$. Hence, it suffices to prove that $P \subset D_i$ for some i. Suppose, to the contrary, that

$$P \not\subset D_i \quad \text{for} \quad i = 1, \dots, n. \tag{1}$$

Consequently, we assume that for every disk D_i there exists at least one vertex of P which does not belong to this disk.

For the convenience of the reader we divide the proof into Parts 1-5.

Part 1. Recall that $\operatorname{diam}(P) = \max\{\sec\beta_i; i = 1, \dots, n\}$ (see (16) in [5]). Without lost of generality we may assume that $\operatorname{diam}(P) = \sec\beta_1$. Then from the triangle $v_1t_{1+(n+1)/2}v_{1+(n+1)/2}$ we see that $|v_1v_{1+(n+1)/2}| = \operatorname{diam}(P)$, see Figure 1. Consider the disk B_1 of radius $\operatorname{diam}(P)$ centered at the point v_1. Denote by w this intersection point of $\operatorname{bd}(B_1)$ with $L_{1+(n+1)/2}$ for which w and v_2 are on one side of M_1. Applying Claim to disk B_1 we obtain $|t_{1+(n+1)/2}w| \le |t_{1+(n+1)/2}v_{1+(n+1)/2}|$. Since $|t_{1+(n+1)/2}v_{1+(n+1)/2}| = 1$ and the radius of $D_{1+(n+1)/2}$ is $\Delta(P) = 1$, it follows that $\widehat{wv}_{1+(n+1)/2} \subset D_{1+(n+1)/2}$. Moreover, observe that by Theorem 8 of [5] the equality $|v_1t_{1+(n+1)/2}| = \tan\beta_1$ implies $|v_1t_{1+(n+1)/2}| \le \frac{\sqrt{3}}{3}$. Hence

$$v_1, v_2, \dots, v_{1+(n+1)/2} \in D_{1+(n+1)/2}. \tag{2}$$

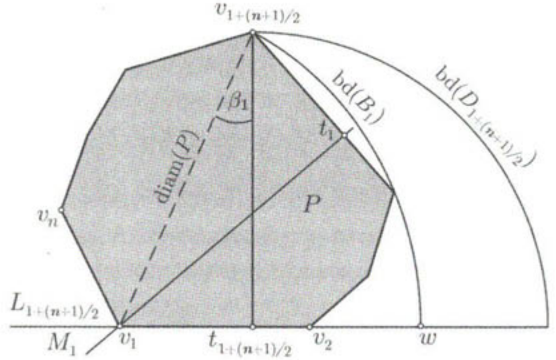

Fig. 1.

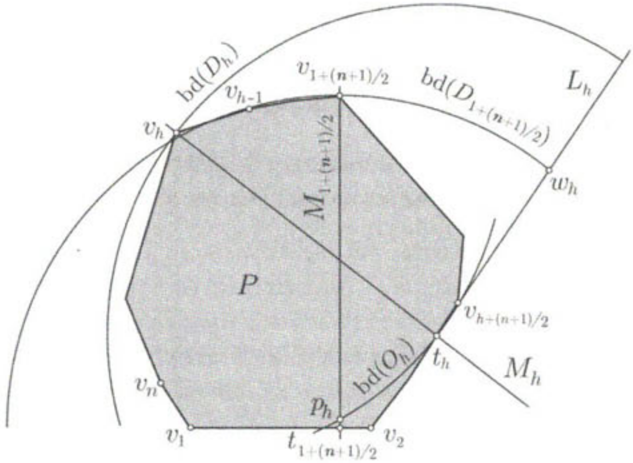

Fig. 2.

Part 2. Let $B_{1+(n+1)/2}$ be the disk of radius diam(P) centered at $v_{1+(n+1)/2}$. In the reasoning used in Part 1, we may replace the pair of disks $(D_{1+(n+1)/2}$, $B_1)$ by the pair $(D_1$, $B_{1+(n+1)/2})$. Then we obtain

(3) $$v_1, v_2, \ldots, v_{1+(n+1)/2} \in D_1.$$

Part 3. Let h be the smallest index such that $v_h \notin D_{1+(n+1)/2}$. Its existence is ensured by the assumption (1). By (2) we have $1 + (n+1)/2 < h \leq n$ (see Figure 2). Hence we see that $v_{1+(n+1)/2}, \ldots, v_{h-1} \in D_{1+(n+1)/2}$ (we do not assume $v_{1+(n+1)/2} \neq v_{h-1}$). Since the chords $t_1v_1, \ldots, t_nv_n$ pairwise intersect, t_h and v_h are on the opposite sides of $M_{1+(n+1)/2}$. Denote by p_h this intersection point of bd(O_h) with $M_{1+(n+1)/2}$ for which p_h and v_1 are on one side of M_h. Since $v_h \notin D_{1+(n+1)/2}$, we have $|t_{1+(n+1)/2}v_h| > 1$. Thus

(4) $$p_h \in t_{1+(n+1)/2}v_{1+(n+1)/2}.$$

For every $j = \{1 + (n+1)/2, \ldots, h-1\}$ the following holds true. Since $v_j \in D_{1+(n+1)/2}$, we have $|t_{1+(n+1)/2}v_j| \leq 1$. Moreover, by (4) we see that $|p_hv_j| < |t_{1+(n+1)/2}v_j| \leq 1$. By Claim applied to disk O_h we conclude that $|v_jt_h| \leq |v_jp_h| < 1$. Thus by $|t_hv_h| = 1$ we have

(5) $$v_{1+(n+1)/2}, \ldots, v_h \in D_h.$$

Denote by w_h this intersection point of bd$(D_{1+(n+1)/2})$ with L_h for which w_h and t_h are on one side of $M_{1+(n+1)/2}$. By (6) we have $|t_hv_{1+(n+1)/2}| < 1$. Thus applying Claim to disk $D_{1+(n+1)/2}$, we get $|t_hw_h| \leq |t_hv_{1+(n+1)/2}| < 1$. Consequently, $\widehat{w_hv}_{1+(n+1)/2} \subset D_h$. Thus by (2) and (5) we obtain

(6) $$v_{h+(n+1)/2}, \ldots, v_h \in D_h.$$

Part 4. In the consideration applied to Part 3, we may replace the pair of disks $(D_{1+(n+1)/2},\ D_h)$ by the pair of arbitrary disks $(D_r,\ D_s)$ such that $h \le r < s \le n$, where s is the smallest index for which $v_s \notin D_r$. Then we obtain

$$v_{s+(n+1)/2}, \dots, v_s \in D_s. \tag{7}$$

Part 5. Since in Part 4 we follow the steps from Part 3 for every pair of disks D_r and D_s, there is an index $s \in \{h+1, \dots, n\}$ such that

$$v_s, \dots, v_n \in D_s. \tag{8}$$

Consider two possibilities. If $v_1 \notin D_s$, then similarly as in Part 3 we show that $v_{1+(n+1)/2}, \dots, v_n, v_1 \in D_1$. Thus by (3) we conclude that $P \subset D_1$. This contradicts the assumption (1).

Now consider the opposite situation, when $v_1 \in D_s$. Let w_s be this intersection point of $\mathrm{bd}(D_1)$ with L_s for which w_s and t_s are on one side of M_1. Since $|t_s v_1| \le 1$, applying Claim to disk D_1, we get $|t_s w_s| \le |t_s v_1| \le 1$. Hence, analogously like in Part 3, we get $\widehat{v_1 w_s} \subset D_s$ and $v_1, \dots, v_{s+(n-1)/2} \in D_s$. Thus by (7) and (8) we obtain $P \subset D_s$. Again this contradicts the assumption (1).

This completes the proof. ∎

Remarks. In [6] there is also shown that every reduced polygon P is contained in a disk of radius $\frac{2}{3} \cdot \Delta(P)$ (with center not necessarily at the boundary of P). The author conjectures that *every reduced polygon P can be covered by a disk of radius* $\frac{1}{2}\mathrm{diam}^2(P)/\Delta(P)$. Observe that this estimate is sharp for regular odd-gons. Let us mention the unpublished conjecture of M. Lassak that also every reduced convex body K of any normed plane M^2 is a subset of a disk of M^2 of radius $\Delta(K)$ centered at a boundary point of K. For some results on reduced bodies in normed spaces see [3] and [7].

References

[1] B. V. Dekster, *Reduced, strictly convex plane figure is of constant width*, J. Geom. 26 (1986), 77–81.

[2] B. V. Dekster, *On reduced convex bodies*, Israel J. Math. 56 (1986), 247–256.

[3] E. Fabińska, M. Lassak, *Reduced bodies in normed planes*, Israel J. Math. 161 (2007), 75–87.

[4] H. Groemer, *Extremal convex sets*, Monatsh. Math. 96 (1983), 29–39.

[5] M. Lassak, *Reduced convex bodies in the plane*, Israel J. Math. 70 (1990), 365–379.

[6] M. Lassak, *On the smallest disk containing a planar reduced convex body*, Arch. Math. 80 (2003), 553–560.

[7] M. Lassak, H. Martini, *Reduced bodies in Minkowski space*, Acta Math. Hungar. 106 (2005), 17-26.

[8] H. Martini, K. J. Swanepoel, *Non-planar simplices are not reduced*, Publ. Math. Debrecen 64 (2004), 101-106.

UNIVERSITY OF TECHNOLOGY
Kaliskiego 7
85-796 BYDGOSZCZ, POLAND
E-mail: fabinska@utp.edu.pl

Received December 13, 2008; revised version March, 2009.