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FIXED POINT THEORY ON SPACES WITH VECTOR-VALUED b-METRICS

Abstract. The purpose of this paper is to present some fixed point results for generalized singlevalued and multivalued contractions on a set endowed with one or two vector-valued b-metrics.

1. Introduction

To our best knowledge, the concept of b-metric space was introduced by Bakhtin in [1] and the extensively used by Czerwik in [4]. Since then several papers deal with fixed point theory for singelvalued and multivalued operators in b-metric spaces (see [2], [4], [10]). The purpose of this paper is to present some fixed point results for generalized singlevalued and multivalued contractions on generalized b-metrics spaces. The starting point of this work was the article D. O'Regan, R. Precup [6]. In the first section we present some results on a generalized b-metric space and in the second section we will use two generalized b-metrics. For this second part we used the results from [7], [8], [9].

2. Notations and auxiliary results

The aim of this section is to present some notions and symbols used in the paper.

We will first give the definition of a vector-valued b-metric.

Definition 2.1. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^n_+$ is said to be a vector-valued b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;

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2. $d(x,y) = d(y,x)$, for all $x, y \in X$;
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$, for all $x, y, z \in X$.

A pair (X, d) is called a generalized b-metric space.

Remark 2.1. (a) If $n = 1$ in the previous definition, then we get the concept of b-metric introduced by Bakhtin.

(b) A b-metric is a particular case of the semimetric in the sense of M. Cicchese. Actually, Cicchese in [3] replaced the third condition in the definition of the metric by

(3′) There exists $A \subset \mathbb{R}_+$, $[0, a) \subset A, (a > 0)$ and there exists $k \geq 1$ and $\varphi : A \to \mathbb{R}_+$ such that for all $x, y, z \in X$ we have $d(x,z) \leq \varphi(d(x,y)) + kd(y,z)$.

Cicchese proved in [3] a fixed point theorem for a singlevalued contraction.

(c) On the other hand, there exists a general notion of distance function, see [5]. Jachymski, Matkowski, Swiatkowski in [5] presented a fixed point theorem for a singlevalued generalized contraction in a semimetric space in the sense that d satisfy a more general condition than (3′), namely:

(3″) $d(x,y) \leq \varepsilon(max\{d(x,z), d(z,y)\})$, for $x, y, z \in X$, where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$, $\lim\limits_{t\to 0} \varepsilon(t) = 0$.

The advantage for using a b-metric is that it allows us to obtain concrete results, not only regarding the existence of the fixed point, but also regarding to the convergence of sequences of successive approximations, data dependence of the fixed point set, the study of well-posedness of the fixed point problem.

Remark 2.2. If $\alpha, \beta \in \mathbb{R}^n$ with $\alpha = (\alpha_1, \dots \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ and $c \in \mathbb{R}$, by $\alpha \leq \beta$ we mean $\alpha_i \leq \beta_i$, for all $i \in \mathbb{N}^*$ and by $\alpha \leq c$ we mean $\alpha_i \leq c$, for all $i \in \mathbb{N}^*$.

We continue by presenting the notions of convergence, compactness, closedness and completeness in a generalized b-metric space.

Definition 2.2. Let (X, d) be a generalized b-metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is called:

(a) Cauchy if and only if for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$,

(b) convergent if and only if there exists $x \in X$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$. In this case we write $\lim\limits_{n\to\infty} x_n = x$.

DEFINITION 2.3. 1. Let (X, d) be a generalized b-metric space. Then a subset $Y \subset X$ is called

(i) compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y,

(ii) closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges to an element x, we have $x \in Y$.

2. The b-metric space is complete if every Cauchy sequence converges.

DEFINITION 2.4. A matrix $C \in \mathcal{M}_{\mathrm{n}\times\mathrm{n}}(\mathbb{R}_+)$ is said to be convergent to zero if

$$C^k \to 0, \text{ as } k \to \infty.$$

LEMMA 2.1. (Rus [9]) *A matrix $C \in \mathcal{M}_{\mathrm{n}\times\mathrm{n}}(\mathbb{R}_+)$ is convergent towards zero if and only if $I - C$ is nonsingular and*

$$(I - C)^{-1} = I + C + C^2 + \dots$$

The following result is useful for some of the proofs in the paper.

LEMMA 2.2. *Let (X, d) be a generalized b-metric space and let $\{x_k\}_{k=0}^n \subset X$. Then:*

$$d(x_n, x_0) \le sd(x_0, x_1) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n).$$

3. Fixed point theorems for operators on generalized b-metric space

The first main result of this paper is the following:

THEOREM 3.1. *Let (X, d) be a complete generalized b-metric space. Assume that the operator $f : X \to X$ satisfies the following conditions:*

(a) *f is continuous;*

(b) *there exists matrices $M, N, P \in \mathcal{M}_{\mathrm{n}\times\mathrm{n}}(\mathbb{R}_+)$ with:*

  (i) *$(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{\mathrm{n}\times\mathrm{n}}(\mathbb{R}_+)$;*

  (ii) *sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;*

  (iii) *$d(f(x), f(y)) \le Md(x, y) + N[d(x, f(x)) + d(y, f(y))] + P[d(x, f(y)) + d(y, f(x))]$, for all $x, y \in X$.*

Then:

1. *f has a fixed point x^*inX.*
2. *If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{\mathrm{n}\times\mathrm{n}}(\mathbb{R}_+)$, then x^* is unique.*

Proof.

1. Let $x_0 \in X$. Consider the sequence of successive approximations for f starting from x_0, i.e. $x_{n+1} = f(x_n)$. We have that:

$$\begin{aligned} d(x_1,x_2) &= d(f(x_0), f(x_1)) \\ &\leq Md(x_0,x_1) + N[d(x_0,f(x_0)) + d(x_1,f(x_1))] \\ &\quad + P[d(x_0,f(x_1)) + d(x_1,f(x_0))] \\ &= Md(x_0,x_1) + N[d(x_0,x_1) + d(x_1,x_2)] + Pd(x_0,x_2) \\ &\leq Md(x_0,x_1) + N[d(x_0,x_1) + d(x_1,x_2)] \\ &\quad + Ps[d(x_0,x_1) + d(x_1,x_2)]. \end{aligned}$$

Thus

$$d(x_1,x_2) \leq (I - N - Ps)^{-1}(M + N + Ps)d(x_0,x_1) = Cd(x_0,x_1).$$

For the next step we have:

$$\begin{aligned} d(x_2,x_3) &= d(f(x_1), f(x_2)) \\ &\leq Md(x_1,x_2) + N[d(x_1,f(x_1)) + d(x_2,f(x_2))] \\ &\quad + P[d(x_1,f(x_2)) + d(x_2,f(x_1))] \\ &\quad = Md(x_1,x_2) + N[d(x_1,x_2) + d(x_2,x_3)] + Pd(x_1,x_3) \\ &\leq Md(x_1,x_2) + N[d(x_1,x_2) + d(x_2,x_3)] \\ &\quad + Ps[d(x_1,x_2) + d(x_2,x_3)]. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_2,x_3) &\leq (I - N - Ps)^{-1}(M + N + Ps)d(x_1,x_2) = Cd(x_1,x_2) \\ &\leq C^2 d(x_0,x_1). \end{aligned}$$

By an inductively procedure we obtain that:

$$d(x_k,x_{k+1}) \leq C^k d(x_0,x_1), \text{ for each } k \in \mathbb{N}.$$

We will prove next that $(x_k)_{k\in\mathbb{N}}$ is Cauchy, by estimating $d(x_k, x_{k+p})$. We have:

$$\begin{aligned} d(x_k,x_{k+p}) &\leq sd(x_k,x_{k+1}) + s^2 d(x_{k+1},x_{k+2}) + \cdots + s^{p-2}d(x_{k+p-3},x_{k+p-2}) \\ &\quad + s^{p-1}d(x_{k+p-2},x_{k+p-1}) + s^{p-1}d(x_{k+p-1},x_{k+p}) \\ &\leq sC^k d(x_0,x_1) + s^2 C^{k+1} d(x_0,x_1) + \cdots + s^{p-2}C^{k+p-3}d(x_0,x_1) \\ &\quad + s^{p-1}C^{k+p-2}d(x_0,x_1) + s^{p-1}C^{k+p-1}d(x_0,x_1) \\ &= sC^k d(x_0,x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-2}C^{p-1}] \\ &\leq sC^k d(x_0,x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-1}C^{p-1}] \\ &\leq sC^k d(x_0,x_1)(I - sC)^{-1} \leq (sC)^k d(x_0,x_1)(I - sC)^{-1}. \end{aligned}$$

Note that $(I - sC)$ is nonsingular since sC is convergent to zero. This implies that the sequence $(x_k)_{k\in\mathbb{N}}$ is Cauchy. From the fact that (X, d) is complete we have that there exists $x^* \in X$ such that $d(x_k, x^*) \to 0$, as $k \to \infty$.

By (a) we have that $d(f(x_{k-1}), f(x^*)) \to 0$, as $k \to \infty$. But $d(f(x_{k-1}), f(x^*)) = d(x_k, f(x^*))$. Hence we have $x^* = f(x^*)$, so x^* is a fixed point for f.

2. For uniqueness we suppose that there exists $y \in X$ such that $y = f(y)$. We have that:

$$\begin{aligned} d(x^*, y) &= d(f(x^*), f(y)) \\ &\leq Md(x^*, y) + N[d(x^*, f(x^*)) + d(y, f(y))] \\ &\quad + P[d(x^*, f(y)) + d(y, f(x^*))] \\ &= Md(x^*, y) + 2Pd(x^*, y). \end{aligned}$$

This implies that

$$(I - M - 2P)d(x^*, y) \leq 0,$$

so $x^* = y$. ∎

The next result is the multivalued variant of the previous theorem.

Theorem 3.2. *Let (X, d) be a complete generalized b-metric space and $F : X \to P_{cl}(X)$ be a multivalued operator such that there exists matrices $M, N, P \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ with:*

(i) *$(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$,$(I - Ps)$ is nonsingular and $(I - Ps)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and $[I - s(I - Ps)^{-1}N]$ is nonsingular and $[I - s(I - Ps)^{-1}N]^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$;*
(ii) *sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;*
(iii) *for each $x, y \in X$ and each $u \in F(x)$ there exists $v \in F(y)$ such that:*

$$d(u, v) \leq Md(x, y) + N[d(x, u) + d(y, v)] + P[d(x, v) + d(y, u)].$$

Then F has a fixed point x^* in X.

Proof. Let $x_0 \in X$ and $x_1 \in F(x_0)$. there exists $x_2 \in F(x_1)$ such that

$$\begin{aligned} d(x_1, x_2) &\leq Md(x_0, x_1) + N[d(x_0, x_1) + d(x_1, x_2)] + P[d(x_0, x_2) + d(x_1, x_1)] \\ &= (M + N + Ps)d(x_0, x_1) + (N + Ps)d(x_1, x_2). \end{aligned}$$

So we have:

$$d(x_1, x_2) \leq (I - N - Ps)^{-1}(M + N + Ps)d(x_0, x_1) = Cd(x_0, x_1).$$

There exists $x_3 \in F(x_2)$ such that

$$\begin{aligned} d(x_2, x_3) &\leq Md(x_1, x_2) + N[d(x_1, x_2) + d(x_2, x_3)] + P[d(x_1, x_3) + d(x_2, x_2)] \\ &= (M + N + Ps)d(x_1, x_2) + (N + Ps)d(x_2, x_3). \end{aligned}$$

So we have:

$$d(x_2, x_3) \leq (I - N - Ps)^{-1}(M + N + Ps)d(x_1, x_2) = Cd(x_1, x_2) \\ \leq C^2 d(x_0, x_1).$$

By an inductively procedure we can construct a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that:

$$x_k \in F(x_{k-1}), k \in \mathbb{N}^*$$

and

$$d(x_k, x_{k+1}) \leq C^k d(x_0, x_1), k \in \mathbb{N}.$$

We will prove next that $(x_k)_{k\in\mathbb{N}}$ is Cauchy, estimating $d(x_k, x_{k+p})$. Thus we have:

$$\begin{aligned} d(x_k, x_{k+p}) &\leq sd(x_k, x_{k+1}) + s^2 d(x_{k+1}, x_{k+2}) + \cdots + s^{p-2} d(x_{k+p-3}, x_{k+p-2}) \\ &\quad + s^{p-1} d(x_{k+p-2}, x_{k+p-1}) + s^{p-1} d(x_{k+p-1}, x_{k+p}) \\ &\leq sC^k d(x_0, x_1) + s^2 C^{k+1} d(x_0, x_1) + \cdots + s^{p-2} C^{k+p-3} d(x_0, x_1) \\ &\quad + s^{p-1} C^{k+p-2} d(x_0, x_1) + s^{p-1} C^{k+p-1} d(x_0, x_1) \\ &= sC^k d(x_0, x_1)[I + sC + \cdots + s^{p-2} C^{p-2} + s^{p-2} C^{p-1}] \\ &\leq sC^k d(x_0, x_1)[I + sC + \cdots + s^{p-2} C^{p-2} + s^{p-1} C^{p-1}] \\ &\leq sC^k d(x_0, x_1)(I - sC)^{-1} \leq (sC)^k d(x_0, x_1)(I - sC)^{-1}. \end{aligned}$$

Note that $(I - sC)$ is nonsingular since sC is convergent to zero. This implies that the sequence $(x_k)_{k\in\mathbb{N}}$ is Cauchy. From the fact that (X, d) is complete we have that there exists $x^* \in X$ such that $d(x_k, x^*) \to 0$, as $k \to \infty$.

For $x_k \in F(x_{k-1})$ there exists $u_k \in F(x^*)$ such that

$$\begin{aligned} d(x_k, u_k) &\leq Md(x_{k-1}, x^*) + N[d(x_{k-1}, x_k) + d(x^*, u_k)] \\ &\quad + P[d(x_{k-1}, u_k) + d(x_k, x^*)] \\ &\leq Md(x_{k-1}, x^*) + N[d(x_{k-1}, x_k) + d(x^*, u_k)] \\ &\quad + P[sd(x_{k-1}, x_k) + sd(x_k, u_k) + d(x_k, x^*)]. \end{aligned}$$

So we have:

$$\begin{aligned} (I - Ps)d(x_k, u_k) &\leq Md(x_{k-1}, x^*) + N[d(x_{k-1}, x_k) + d(x^*, u_k)] \\ &\quad + Psd(x_{k-1}, x_k) + Pd(x_k, x^*). \end{aligned}$$

We will next estimate $d(x^*, u_k)$ and obtain

$$\begin{aligned} &d(x^*, u_k) \leq s[d(x^*, x_k) + d(x_k, u_k)] \leq s\{d(x^*, x_k) + (I - Ps)^{-1} \\ &\quad \cdot [Md(x_{k-1}, x^*) + N[d(x_{k-1}, x_k) + d(x^*, u_k)] + Psd(x_{k-1}, x_k) + Pd(x_k, x^*)]\} \\ &= s(I - Ps)^{-1} N d(x^*, u_k) + s\{d(x^*, x_k) + (I - Ps)^{-1} \\ &\quad \cdot [Md(x_{k-1}, x^*) + Nd(x_{k-1}, x_k) + Psd(x_{k-1}, x_k) + Pd(x_k, x^*)]\}. \end{aligned}$$

So we have:

$[I - s(I - Ps)^{-1}N]d(x^*, u_k) \leq s\{d(x^*, x_k) + (I - Ps)^{-1}[Md(x_{k-1}, x^*) + Nd(x_{k-1}, x_k) + Psd(x_{k-1}, x_k) + Pd(x_k, x^*)]\} \to 0$, as $k \to \infty$. Hence $u_k \to x^*$. But $u_k \in F(x^*)$ and $F(x^*)$ is closed so we obtain that $x^* \in F(x^*)$. ∎

REMARK 3.1. It is an open problem to give results of this type for singlevalued and multivalued operators on semimetric spaces, in the sense of Jachymski, Matkowski, Swiatkowski [5].

4. Fixed point theorems for operators on generalized b-metric space with two b-metrics

THEOREM 4.1. *Let (X, δ) be a complete generalized b-metric space and d another vector-valued b-metric on X. Assume that the operator $f : X \to X$ satisfies the following conditions:*

(a) *f is (d, δ)-uniformly continuous or there exists a matrix $U \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that $\delta(x, y) \leq U \cdot d(x, y)$, for all $x, y \in X$;*

(b) *f is (δ, δ)-continuous;*

(c) *there exist matrices $M, N, P \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ with:*

  (i) *$(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$;*

  (ii) *sC is convergent towards zero, where $C = (I - N - Ps)^{-1}(M + N + Ps)$;*

  (iii) *$d(f(x), f(y)) \leq Md(x, y) + N[d(x, f(x)) + d(y, f(y))] + P[d(x, f(y)) + d(y, f(x))]$, for all $x, y \in X$.*

Then:

1. *For any $x_0 \in X$ we have $\delta(f^k(x_0), x^*) \to 0$, as $k \to \infty$, where x^* is a fixed point for f.*

2. *If, in addition, $(I - M - 2P)$ is nonsingular and $(I - M - 2P)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ then x^* is unique.*

Proof.

1. Let $x_0 \in X$. Consider the sequence of successive approximations for f starting from x_0, i.e. $x_{k+1} = f(x_k)$. We have that:

$$\begin{aligned} d(x_1, x_2) &= d(f(x_0), f(x_1)) \\ &\leq Md(x_0, x_1) + N[d(x_0, f(x_0)) + d(x_1, f(x_1))] \\ &\quad + P[d(x_0, f(x_1)) + d(x_1, f(x_0))] \\ &= Md(x_0, x_1) + N[d(x_0, x_1) + d(x_1, x_2)] + Pd(x_0, x_2) \\ &\leq Md(x_0, x_1) + N[d(x_0, x_1) + d(x_1, x_2)] \\ &\quad + Ps[d(x_0, x_1) + d(x_1, x_2)]. \end{aligned}$$

So we have:

$$d(x_1, x_2) \leq (M + N + Ps)d(x_0, x_1) + (N + Ps)d(x_1, x_2).$$

Thus

$$d(x_1,x_2) \le (I-N-Ps)^{-1}(M+N+Ps)d(x_0,x_1) = Cd(x_0,x_1).$$

For the next step we have:

$$\begin{aligned} d(x_2,x_3) &= d(f(x_1),f(x_2)) \\ &\le Md(x_1,x_2) + N[d(x_1,f(x_1)) + d(x_2,f(x_2))] \\ &\quad + P[d(x_1,f(x_2)) + d(x_2,f(x_1))] \\ &= Md(x_1,x_2) + N[d(x_1,x_2) + d(x_2,x_3)] + Pd(x_1,x_3) \\ &\le Md(x_1,x_2) + N[d(x_1,x_2) + d(x_2,x_3)] \\ &\quad + Ps[d(x_1,x_2) + d(x_2,x_3)]. \end{aligned}$$

This implies that

$$\begin{aligned} d(x_2,x_3) &\le (I-N-Ps)^{-1}(M+N+Ps)d(x_1,x_2) = Cd(x_1,x_2) \\ &\le C^2 d(x_0,x_1). \end{aligned}$$

By an inductively procedure we obtain that:

$$d(x_k,x_{k+1}) \le C^k d(x_0,x_1), \text{ for each } k \in \mathbb{N}.$$

We will prove next that $(x_k)_{k\in\mathbb{N}}$ is Cauchy, by estimating $d(x_k,x_{k+p})$. We have:

$$\begin{aligned} d(x_k,x_{k+p}) &\le sd(x_k,x_{k+1}) + s^2 d(x_{k+1},x_{k+2}) + \cdots + s^{p-2} d(x_{k+p-3},x_{k+p-2}) \\ &\quad + s^{p-1} d(x_{k+p-2},x_{k+p-1}) + s^{p-1} d(x_{k+p-1},x_{k+p}) \\ &\le sC^k d(x_0,x_1) + s^2 C^{k+1} d(x_0,x_1) + \cdots + s^{p-2} C^{k+p-3} d(x_0,x_1) \\ &\quad + s^{p-1} C^{k+p-2} d(x_0,x_1) + s^{p-1} C^{k+p-1} d(x_0,x_1) \\ &= sC^k d(x_0,x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-2}C^{p-1}] \\ &\le sC^k d(x_0,x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-1}C^{p-1}] \\ &\le sC^k d(x_0,x_1)(I-sC)^{-1} \le (sC)^k d(x_0,x_1)(I-sC)^{-1}. \end{aligned}$$

Note that $(I-sC)$ is nonsingular since sC is convergent to zero. This implies that the sequence $(x_k)_{k\in\mathbb{N}}$ is d-Cauchy. It follows from (a), that $(x_k)_{k\in\mathbb{N}}$ is a δ-Cauchy sequence. Since (X,δ) is a complete generalized metric space, there exists $x^* \in X$ such that $\delta(x_k,x^*) \to 0$, as $k \to \infty$.

By (b) we have that $\delta(f(x_{k-1}),f(x^*)) \to 0$, as $k \to \infty$. But $\delta(f(x_{k-1}),f(x^*)) = \delta(x_k,f(x^*))$. Hence we have $x^* = f(x^*)$. Thus x^* is a fixed point for f and

$$\delta(f^k(x_0),x^*) \to 0, \text{ as } k \to \infty.$$

2. For uniqueness we suppose that there exists $y \in X$ such that $y = f(y)$. We have that:

$$\begin{aligned} d(x^*, y) &= d(f(x^*), f(y)) \\ &\leq Md(x^*, y) + N[d(x^*, f(x^*)) + d(y, f(y))] \\ &\quad + P[d(x^*, f(y)) + d(y, f(x^*))] \\ &= Md(x^*, y) + 2Pd(x^*, y). \end{aligned}$$

This implies that

$$(I - M - 2P)d(x^*, y) \leq 0,$$

so $x^* = y$.

Since $\delta(f(x_{k-1}), f(x^*)) \to 0$, as $k \to \infty$ and x^* is a unique fixed point of f, we have

$$\delta(f^k(x_0), x^*) \to 0, \text{ as } k \to \infty. \blacksquare$$

The next result is the multivalued variant of the previous theorem.

Theorem 4.2. *Let (X, δ) be a complete generalized b-metric space and d be another vector-valued b-metric on X. Assume that for the multivalued operator $F : X \to P_{cl}(X)$ the following conditions are satisfied:*

(a) *there exists*

$$U \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$$

such that $\delta(x, y) \leq U \cdot d(x, y)$, for all $x, y \in X$;

(b) *$F : (X, \delta) \to (P(X), H_\delta)$ is closed (i.e. it has closed graph);*

(c) *there exist matrices $M, N, P \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ with:*

 (i) *$(I - N - Ps)$ is nonsingular and $(I - N - Ps)^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$;*

 (ii) *sC is convergent towards zero, where $C = (I-N-Ps)^{-1}(M+N+Ps)$;*

 (iii) *for each $x, y \in X$ and each $u \in F(x)$ there exists $v \in F(y)$ such that:*

$$d(u, v) \leq Md(x, y) + N[d(x, u) + d(y, v)] + P[d(x, v) + d(y, u)].$$

Then F has a fixed point x^* in X.

Proof. Let $x_0 \in X$ and $x_1 \in F(x_0)$. There exists $x_2 \in F(x_1)$ such that

$$\begin{aligned} d(x_1, x_2) &\leq Md(x_0, x_1) + N[d(x_0, x_1) + d(x_1, x_2)] + P[d(x_0, x_2) + d(x_1, x_1)] \\ &= (M + N + Ps)d(x_0, x_1) + (N + Ps)d(x_1, x_2). \end{aligned}$$

So we have:

$$d(x_1, x_2) \leq (I - N - Ps)^{-1}(M + N + Ps)d(x_0, x_1) = Cd(x_0, x_1).$$

There exists $x_3 \in F(x_2)$ such that

$$\begin{aligned} d(x_2,x_3) &\leq Md(x_1,x_2) + N[d(x_1,x_2) + d(x_2,x_3)] + P[d(x_1,x_3) + d(x_2,x_2)] \\ &= (M+N+Ps)d(x_1,x_2) + (N+Ps)d(x_2,x_3). \end{aligned}$$

So we have:

$$\begin{aligned} d(x_2,x_3) &\leq (I-N-Ps)^{-1}(M+N+Ps)d(x_1,x_2) = Cd(x_1,x_2) \\ &\leq C^2 d(x_0,x_1). \end{aligned}$$

By an inductively procedure we can construct a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that:

$$x_k \in F(x_{k-1}), k \in \mathbb{N}$$

and

$$d(x_k,x_{k+1}) \leq C^k d(x_0,x_1).$$

We will prove next that $(x_k)_{k\in\mathbb{N}}$ is Cauchy, estimating $d(x_k,x_{k+p})$. So we have:

$$\begin{aligned} d(x_k,x_{k+p}) &\leq sd(x_k,x_{k+1}) + s^2 d(x_{k+1},x_{k+2}) + \cdots + s^{p-2}d(x_{k+p-3},x_{k+p-2}) \\ &\quad + s^{p-1}d(x_{k+p-2},x_{k+p-1}) + s^{p-1}d(x_{k+p-1},x_{k+p}) \\ &\leq sC^k d(x_0,x_1) + s^2C^{k+1}d(x_0,x_1) + \cdots + s^{p-2}C^{k+p-3}d(x_0,x_1) \\ &\quad + s^{p-1}C^{k+p-2}d(x_0,x_1) + s^{p-1}C^{k+p-1}d(x_0,x_1) \\ &= sC^k d(x_0,x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-2}C^{p-1}] \\ &\leq sC^k d(x_0,x_1)[I + sC + \cdots + s^{p-2}C^{p-2} + s^{p-1}C^{p-1}] \\ &\leq sC^k d(x_0,x_1)(I-sC)^{-1} \leq (sC)^k d(x_0,x_1)(I-sC)^{-1}. \end{aligned}$$

Note that $(I-sC)$ is nonsingular, since sC is convergent to zero. This implies that the sequence $(x_k)_{k\in\mathbb{N}}$ is d-Cauchy. It follows from (a), that $(x_k)_{k\in\mathbb{N}}$ is a δ-Cauchy sequence. Since (X,δ) is a complete generalized metric space, there exists $x^* \in X$ such that $\delta(x_k,x^*) \to 0$ as $k \to \infty$. From (b) we get that $x^* \in F(x^*)$. ∎

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