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# THE CONVERGENCE CLASSES OF DIVERGENT PERMUTATIONS

**Abstract.** The scope of the paper is the analysis of the convergence classes of different permutations of $\mathbb{N}$. In this paper we prove that with any divergent permutation $p$ it could be associated a family $\mathfrak{D}(p)$ of divergent permutations having the power of the continuum and such that the convergence class of $p$ is a proper subset of the convergence class of $q$ for every $q \in \mathfrak{D}(p)$. Also, the convergence classes of the countable families of divergent permutations are discussed here.

## 1. Introduction

Let us denote by $\mathfrak{P}$ the family of all permutations of $\mathbb{N}$.

Given a permutation $p \in \mathfrak{P}$ its **convergence class**, denoted by $\sum(p)$, is defined to be the family of all convergent series $\sum a_n$ of real terms such that the series $\sum a_{p(n)}$ is also convergent.

We say that two permutations $p, q \in \mathfrak{P}$ are **incomparable** if the following conditions hold true:

$$\sum(p) \setminus \sum(q) \neq \emptyset \quad \text{and} \quad \sum(q) \setminus \sum(p) \neq \emptyset.$$

A permutation $p \in \mathfrak{P}$ is called a divergent permutation if there exists a conditionally convergent series $\sum a_n$ of real terms such that the series $\sum a_{p(n)}$ is divergent. The family of all divergent permutations is denoted by $\mathfrak{D}$. We say that $p$ is a convergent permutation if $p \in \mathfrak{C} := \mathfrak{P} \setminus \mathfrak{D}$. Thus $p \in \mathfrak{P}$ is a convergent permutation if for every conditionally convergent series $\sum a_n$ of real terms the series $\sum a_{p(n)}$ is also convergent.

We introduce the notation $AB$ for the subset of $\mathfrak{P}$ defined by the following relation:

$$p \in AB \text{ if and only if } p \in A \text{ and } p^{-1} \in B$$

for every $A, B \in \{\mathfrak{C}, \mathfrak{D}\}$. We note that all these families $AB$ are nonempty.

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In this paper we prove that with any divergent permutation $p$ it could be associated a family $\mathfrak{D}(p)$ of divergent permutations having the power of the continuum and such that $\sum(p)$ is a proper subset of $\sum(q)$ for every $q \in \mathfrak{D}(p)$. Moreover, it may be assumed that any two different elements of $\mathfrak{D}(p)$ are incomparable and, additionally, that $\mathfrak{D}(p)$ is a subset of one of the following three sets: either $\mathfrak{DC}$ or $\mathfrak{DD}$ or $\Omega$ whenever $p$ also belongs to this set. For the definition of the family $\Omega$ we refer to the Section 5. Moreover, an example of the permutations $p, q \in \mathfrak{D}$ such that there is no divergent permutation $\sigma$ with the property

$$\sum(p) \cup \sum(q) \subseteq \sum(\sigma)$$

is presented in Section 4.

## 2. Notations and terminology

Let A and B be infinite subsets of $\mathbb{N}$. By $\mathfrak{D}(A,B)$ and by $\mathfrak{C}(A,B)$ we denote two subsets of $\mathfrak{P}$ defined as follows:

$$\mathfrak{D}(A,B) = \{p \in \mathfrak{P} : p(A) = B \text{ and the composition } \psi_B \circ (p\,|A) \circ \phi_A \text{ is a divergent permutation}\},$$

and

$$\mathfrak{C}(A,B) = \{p \in \mathfrak{P} : p(A) = B \text{ and the composition } \psi_B \circ (p\,|A) \circ \phi_A \text{ is a convergent permutation}\},$$

where $\phi_A$ is the increasing bijection of $\mathbb{N}$ onto $A$, $p\,|A$ is the restriction to the $A$ of $p$, and $\psi_B$ is the increasing bijection of $B$ onto $\mathbb{N}$. To simplify the notation, we write $p \in \mathfrak{U}(A)$ instead of $p \in \mathfrak{U}(A, p(A))$ where $\mathfrak{U} \in \{\mathfrak{C}, \mathfrak{D}\}$ and $A$ is an infinite subset of $\mathbb{N}$. We observe that $\mathfrak{U}(\mathbb{N},\mathbb{N}) = \mathfrak{U}$ for every $\mathfrak{U} \in \{\mathfrak{C}, \mathfrak{D}\}$.

Let $B$ be a nonempty subset of $\mathbb{N}$. Then a subset $I$ of $B$ is said to be an interval of $B$ either if $I = \emptyset$ or when it can be expressed in the form $I = B \cap \{n, n+1, \ldots, n+m-1\}$ for some $n, m \in \mathbb{N}$. For abbreviation, we write "$I$ is an interval" instead of "$I$ is an interval of $\mathbb{N}$". Only intervals of the subsets of $\mathbb{N}$ are discussed in this paper. We will use the symbols $[n,m]$, $[n,m)$, $(n,m]$ and $(n,m)$ with $n,m \in \mathbb{N}$, $n \leq m$, to denote the intervals $\{n, n+1, \ldots, m\}$, $\{n, n+1, \ldots, m-1\}$, $\{n+1, n+2, \ldots, m\}$ and $\{n+1, n+2, \ldots, m-1\}$ respectively.

We say $A \subset \mathbb{N}$ is a union of $n$ (or at least $n$ or at most $n$) mutually separated intervals (abbrev.: MSIs) if there exists a family $\mathfrak{J}$ of nonempty intervals with $\bigcup \mathfrak{J} = A$, card $\mathfrak{J} = n$ (or card $\mathfrak{J} \geq n$ or card $\mathfrak{J} \leq n$, resp.) and such that $\operatorname{dist}(I,J) \geq 2$ for any two different members $I$ and $J$ of $\mathfrak{J}$.

We say that two sequences $\{x_i : i = 1, 2, \ldots, n\}$ and $\{y_i : i = 1, 2, \ldots, n\}$ of positive integers are spliced when they are both one-one, have no common values and satisfy one of the following conditions: either

$$x_{p(i)} < y_{q(i)} < x_{p(i+1)} < y_{q(n)} \text{ for every } i = 1, 2, \ldots, n-1,$$

or

$$y_{q(i)} < x_{p(i)} < y_{q(i+1)} < x_{p(n)} \text{ for every } i = 1, 2, \ldots, n-1.$$

Here $p$ and $q$ denote the permutations of the set $\{1, 2, \ldots, n\}$ chosen in such a way that the sequences $\{x_{p(i)} : i = 1, 2, \ldots, n\}$ and $\{y_{q(i)} : i = 1, 2, \ldots, n\}$ are both increasing. In other words, two finite sequences $x$ and $y$ are spliced if they have the same cardinality, are both one-one, have no common values and if the increasing sequence in which the elements of $x$ and $y$ alternate could be created from all elements of the sequences $x$ and $y$.

A family $S$ of increasing sequences of positive integers having the power of the continuum is said to be a *Sierpiński's* family if any two different members $\{a_n\}$ and $\{b_n\}$ of $S$ are almost disjoint. This means that the sets of values of the sequences $\{a_n\}$ and $\{b_n\}$ have only a finite number of common elements.

We use the symbol $c(p|A)$, where $p \in \mathfrak{C}$ and $A$ is a nonempty subset of $\mathbb{N}$, to denote the following positive integer:

$$c(p\,|A) = \max\{k \in \mathbb{N} : \text{ there exists an interval } I \text{ of the set } A \text{ such that the } p(I) \text{ is a union of } k \text{ MSIs of the set } p(A)\ \}.$$

For abbreviation, we write $c(p)$ instead of $c(p \mid \mathbb{N})$ for every $p \in \mathfrak{C}$.

We will denote inclusion by $\subseteq$. The use of the sign $\subset$ will be reserved for cases when the subset is proper.

We shall write $K < L$ for any two subsets $K$ and $L$ of $\mathbb{N}$ whenever $K = \emptyset$ or $K$ and $L$ are nonempty and $k < l$ for any $k \in K$ and $l \in L$.

In this paper we will often identify a given sequence with its set of values. Moreover, we will consider only series $\sum a_n$ of real terms.

## 3. Countable families of divergent permutations

In this section we consider some basic properties of convergence classes of a given countable families of divergent permutations. First we remark that the convergence class of any divergent permutation is wide in the meaning that the intersection of any countable family of the convergence classes of divergent permutations is nonempty. This is expressed in the following theorem, which is taken from [8].

**THEOREM 3.1.** *For each sequence $\{p_n\}$ of divergent permutations there exist two conditionally convergent series $\sum a_n$ and $\sum b_n$ of real terms such* *that the series* $\sum_{k=1}^{\infty} a_{p_n(k)}$ *is convergent to zero for every* $n \in \mathbb{N}$ *and the set of the limit points of the series* $\sum_{k=1}^{\infty} b_{p_n(k)}$ *is equal to* $\mathbb{R}$ *for every* $n \in \mathbb{N}$.

**Corollary 3.2.** *For each sequence* $\{p_n\}$ *of divergent permutations we have* $\bigcap_{n\in\mathbb{N}} \sum(p_n) \neq \emptyset$.

**Theorem 3.3.** *Let* $p_n, q_n \in \mathfrak{P}$, $n \in \mathbb{N}$. *If there exists an infinite subset* $A$ *of* $\mathbb{N}$ *such that* $p_n \in \mathfrak{C}(p_n^{-1}(A))$ *and* $q_n \in \mathfrak{D}(q_n^{-1}(A))$ *for every positive integer* $n$ *then*

$$\bigcap_{n=1}^{\infty} \sum(p_n) \setminus \bigcup_{n=1}^{\infty} \sum(q_n) \neq \emptyset.$$

**Proof.** Let for every $n \in \mathbb{N}$, $s_n$ and $\sigma_n$ be the increasing mappings of $\mathbb{N}$ onto the sets $p_n^{-1}(A)$ and $q_n^{-1}(A)$, respectively, and let $\delta$ be the increasing mapping of $A$ onto $\mathbb{N}$. Then, by the hypothesis, the permutation $\phi_n$ defined by $\phi_n = \delta q_n \sigma_n$ is divergent for each $n \in \mathbb{N}$. Hence, by Theorem 3.1, there exists a conditionally convergent series $\sum b_n$ such that all series of the form $\sum_{k=1}^{\infty} b_{\phi_n(k)}$, $n \in \mathbb{N}$, are divergent. Then the series $\sum_{k=1}^{\infty} b_{\psi_n(k)}$, $n \in \mathbb{N}$, are certainly convergent because all permutation $\psi_n := \delta p_n s_n$, $n \in \mathbb{N}$, are convergent.

Let $\{a(n)\}$ be the increasing sequence of all members of the set $A$. We define a new series $\sum d_n$ by setting $d_{a(n)} = b_n$ for every $n \in \mathbb{N}$ and $d_n = 0$ for the remaining indices $n \in \mathbb{N}$.

It can be readily checked that the series $\sum d_n$ and any of its $p_n$ – rearrangements i.e. the series $\sum_{k=1}^{\infty} d_{p_n(k)}$, $n \in \mathbb{N}$, are convergent. This follows immediately from the relation

$$\sum_{i\in I} d_{p_n(i)} = \sum_{i\in B} d_{p_n(i)} = \sum_{i\in B} b_{\delta p_n(i)} = \sum_{j\in J} b_{\psi_n(j)},$$

which holds for every nonempty interval $I$, where $B = I \cap p_n^{-1}(A)$ and $J = [\min s_n^{-1}(I \cap p_n^{-1}(A)), \max s_n^{-1}(I \cap p_n^{-1}(A))]$.

On the other hand, all series of the form $\sum_{k=1}^{\infty} d_{q_n(k)}$, $n \in \mathbb{N}$, are divergent. This assertion follows from the following relation which holds for every nonempty interval $J$:

$$\sum_{i\in I} d_{q_n(i)} = \sum_{i\in B} d_{q_n(i)} = \sum_{i\in B} b_{\delta q_n(i)} = \sum_{j\in J} b_{\phi_n(j)},$$

where $B = I \cap q_n^{-1}(A)$ and $I = [\min \sigma_n(J), \max \sigma_n(J)]$. The proof is completed. ■

**Corollary 3.4.** *Let $p, q \in \mathfrak{P}$. If there exists an infinite subset $A$ of $\mathbb{N}$ such that $p \in \mathfrak{C}(p^{-1}(A))$ and $q \in \mathfrak{D}(q^{-1}(A))$ then*

$$\sum(p) \setminus \sum(q) \neq \emptyset.$$

**Corollary 3.5.** *Let $p, q \in \mathfrak{P}$. If $\sum(p) \subseteq \sum(q)$ then for any infinite subset $A$ of $\mathbb{N}$ the following implication proves true:*

$$if\ p \in \mathfrak{C}(p^{-1}(A))\ then\ q \in \mathfrak{C}(q^{-1}(A)).$$

**Corollary 3.6.** *Let $p, q \in \mathfrak{P}$. If $\sum(p) = \sum(q)$ then for any infinite subset $A$ of $\mathbb{N}$ the following relation holds:*

$$p \in \mathfrak{C}(p^{-1}(A)) \quad if\ and\ only\ if \quad q \in \mathfrak{C}(q^{-1}(A)).$$

Let $p_i \in \mathfrak{P}$ and let $A_i$ be an infinite subset of $\mathbb{N}$ for every $i = 1, 2, \ldots, n$. Assume that the sets $A_1, A_2, \ldots, A_n$, form a partition of $\mathbb{N}$ as well as the sets $p_1(A_1), p_2(A_2), \ldots, p_n(A_n)$, do.

Moreover, assume that the action of any permutation $p_i$ is concentrated on the set $A_i$ in the sense that all permutations $q_i$, $i = 1, 2, \ldots, n$, and the permutation $\phi$ are convergent. Here

$$q_i(k) := \begin{cases} p_i(k) \text{ for } k \in \mathbb{N} \setminus A_i, \\ \gamma_i(k) \text{ for } k \in A_i, \end{cases}$$

for every $i = 1, \ldots, n$ and

$$\phi(k) := \gamma_i(k),$$

for $k \in A_i$ and $i = 1, \ldots, n$, where $\gamma_i$ is the increasing bijection of the set $A_i$ onto the set $p_i(A_i)$ for every $i = 1, \ldots, n$. Then there is a following relation between the convergence classes of permutations $p_i$, $i = 1, \ldots, n$, and the convergence class of the permutation $\sigma$ defined by $\sigma(k) = p_i(k)$ for $k \in A_i$ and $i = 1, \ldots, n$.

**Lemma 3.7.** *We have $\bigcap_{i=1}^{n} \sum(p_i) \subseteq \sum(\sigma)$.*

**Proof.** Notice that for any series $\sum a_k$ and for any nonempty interval $I$ the following equality is satisfied:

$$\sum_{k \in I} a_{\sigma(k)} = \sum_{i=1}^{n} \sum_{k \in I} a_{p_i(k)} + \sum_{k \in I} a_{\phi(k)} - \sum_{i=1}^{n} \sum_{k \in I} a_{q_i(k)},$$

which, by the hypothesis, implies the desired inclusion. ■

**Remark 3.8.** There exists the possibility of replacing the weak inequality sign by the equality sign in the assertion of Lemma 3.7. Indeed, assume that there exists a positive integer $t$ such that for any interval $I$ there exists a family of pairwise disjoint subintervals $I_1, I_2, \ldots, I_k$ of $I$ with $k \le t$ satisfying the following two conditions:

(1) any interval $I_i$, $i = 1, \ldots, k$, is simultaneously a subset of some set $A_j$ with $j = j(i) \in \{1, \ldots, n\}$,

(2) the set $\sigma(I \setminus \bigcup_{i=1}^{k} I_i)$ is a union of at most $t$ MSIs.

Then we have $\bigcap_{i=1}^{n} \sum(p_i) = \sum(\sigma)$.

The following example shows that the inclusion in the assertion of Lemma 3.7 could be made strictly for every positive integer $n$. Here we consider only the case $n = 2$; the generalization to any $n \in \mathbb{N}$ is rather straightforward (and will be omitted here).

**Example 3.9.** Put

$$p_1(a_n + 2i - 1) = \begin{cases} a_n + 2i - 1 & \textit{for } i = 1, 2, \ldots, n, \\ a_n + 2(i - n - 1) & \textit{for } i = n+1, n+2, \ldots, 2n, \end{cases}$$

$$p_1(a_n + 2i) = a_n + 2n + i \textit{ for } i = 0, 1, \ldots, 2n - 1,$$

$$p_2(a_n + 2i) = 2n + p_1(a_n + 2i + 1) \textit{ for } i = 0, 1, \ldots, 2n - 1,$$

and

$$p_2(a_n + 2i - 1) = a_n + i - 1 \textit{ for } i = 1, 2, \ldots, 2n,$$

for every positive integer $n$, where $a_n = 2n(n-1) + 1$. Moreover, set $A_1 = \bigcup_{n \in \mathbb{N}} \{a_n + 2i - 1 : \ i = 1, 2, \ldots, 2n\}$ and $A_2 = \mathbb{N} \setminus A_1$. Then the hypotheses of Lemma 3.7 are satisfied, which implies the following inclusion:

$$\sum(p_1) \cap \sum(p_2) \subseteq \sum(\sigma), \tag{3.1}$$

where $\sigma(k) := p_i(k)$ for $k \in A_i$, $i = 1, 2$.

Now define

$$b_k = \begin{cases} -n^{-1} & \textit{for } k = a_n + 2i \textit{ and for } k = a_n + 2n + 2i + 1 \\ & \textit{where } i = 0, 1, \ldots, n - 1, \\ n^{-1} & \textit{for } k = a_n + 2i - 1 \textit{ and for } k = a_n + 2n + 2(i - 1) \\ & \textit{where } i = 1, 2, \ldots, n, \end{cases}$$

for $n \in \mathbb{N}$. It is clear that the series $\sum b_k$ is convergent. Furthermore, a noncomplicated verification shows that

$$\sum_{i=1}^{2n} b_{p_1(a_n+i)} = 1 + 0.5[(-1)^n - 1]n^{-1},$$

$$\sum_{i=0}^{2n-1} b_{p_2(a_n+2n+i)} = 1 + 0.5[(-1)^{n-1} + 1]n^{-1},$$

$$\sum_{i=0}^{4n-1} b_{\sigma(a_n+i)} = 0, \quad \textit{and} \quad \left|\sum_{i\in I} b_{\sigma(i)}\right| \leq 2n^{-1}$$

for any subinterval $I$ of the interval $[a_n, a_{n+1})$ and for every $n \in \mathbb{N}$. From the above relations we deduce that the series $\sum b_{p_1(n)}$ and $\sum b_{p_2(n)}$ are both divergent but, however, the series $\sum b_{\sigma(n)}$ is convergent. Accordingly, by (3.1), we get

$$\sum(p_1) \cap \sum(p_2) \subset \sum(\sigma),$$

which is our claim.

## 4. Extending of convergence classes

The convergence class of any divergent permutation can be extended to the convergence class of some other divergent permutation. What is more, for any divergent permutation $p$ there exists a family $\mathfrak{D}(p)$ of pairwise incomparable divergent permutations having the continuum cardinality and such that the convergence class of any permutation $q \in \mathfrak{D}(p)$ is bigger than the convergence class of $p$. This is the main result of this section.

We begin with three auxiliary lemmas. First lemma is well known and comes from P. Erdős, the two other ones come from paper [10].

**Lemma 4.1.** *Let $a_k$ , $k = 1, 2, \ldots, n$, be a $1-1$ sequence of real numbers. Then $rs \geq n$ where $r$ denotes the length of the longest decreasing subsequence of $\{a_k\}$ and $s$ stands for the length of the longest increasing subsequence of $\{a_k\}$.*

**Lemma 4.2.** *Let $p \in \mathfrak{P}$. Suppose that for some interval $I$ and for some positive integer $k$ the set $p(I)$ is a union of at least $(4k^4 + 1)$ MSIs. Then there exists an increasing sequence $\mathbb{X} = \{x_n : n = 1, 2, \ldots, 2k\}$ of positive integers satisfying the following conditions:*

(1) $\mathbb{X} \subset p^{-1}([\min p(I), \max p(I)])$,
(2) *one of the two following relations holds true:*
*either $\{x_n : n = 1, 2, \ldots, k\} \subseteq I$ and $x_{k+1} > I$*
*or $x_k < I$ and $\{x_n : n = k+1, k+2, \ldots, 2k\} \subseteq I$,*

(3) *the sequences* $\{p(x_n) : n = 1, 2, \ldots, k\}$ *and* $\{p(x_n) : n = k+1, k+2, \ldots, 2k\}$ *are both strictly monotonic and simultaneously, are spliced.*

**Corollary 4.3.** *Let* $p \in \mathfrak{D}$*. Then for any* $r, s \in \mathbb{N}$ *there exists an increasing sequence* $\mathbb{S} = \{z_n : n = 1, 2, \ldots, 2r\}$ *of positive integers such that*

(1) $\mathbb{S} \cup p(\mathbb{S}) > s$,
(2) *the sequences* $\{p(z_n) : n = 1, 2, \ldots, r\}$ *and* $\{p(z_n) : n = r+1, r+2, \ldots, 2r\}$ *are both strictly monotonic and are spliced.*

**Lemma 4.4.** *Let* $p, q \in \mathfrak{P}$*. Assume that there exist increasing sequences* $\mathbb{X}_r = \{x_n^{(r)} : n = 1, 2, \ldots, 2r\}$*,* $r \in \mathbb{N}$ *of positive integers such that for every* $r \in \mathbb{N}$

(1) $\mathbb{X}_r \cup p(\mathbb{X}_r) \cup q^{-1}p(\mathbb{X}_r) < \mathbb{X}_{r+1} \cup p(\mathbb{X}_{r+1}) \cup q^{-1}p(\mathbb{X}_{r+1})$,
(2) *the sequences* $\mathbb{X}_r^{(1)}$ *and* $\mathbb{X}_r^{(2)}$ *are spliced where* $\mathbb{X}_r^{(1)} := \{p(x_n^{(r)}) : n = 1, 2, \ldots, r\}$ *and* $\mathbb{X}_r^{(2)} := \{p(x_n^{(r)}) : n = r+1, r+2, \ldots, 2r\}$ *and, at the same time,*
(3) *the sequences* $\{q^{-1}p(x_n^{(r)}) : n = 1, 2, \ldots, r\}$ *and* $\{q^{-1}p(x_n^{(r)}) : n = r+1, r+2, \ldots, 2r\}$ *are spliced.*

*Then we have* $\sum(q) \setminus \sum(p) \neq \emptyset$*.*

**Proof.** Put

$$a_n = \begin{cases} r^{-1} \text{ for } n \in \mathbb{X}_r^{(1)}, \\ -r^{-1} \text{ for } n \in \mathbb{X}_r^{(2)}, \end{cases}$$

for every $r \in \mathbb{N}$. For the remaining indices $n \in \mathbb{N}$ we set $a_n = 0$.

Obviously, by the assumptions (1) and (2), the series $\sum a_n$ is convergent. By the condition (3) the following two relations hold true for every $r \in \mathbb{N}$:

$$\Big|\sum_{n \in I} a_{q(n)}\Big| \leq r^{-1} \tag{4.1}$$

for any subinterval $I$ of the interval $J_r := [\min q^{-1}p(\mathbb{X}_r), \max q^{-1}p(\mathbb{X}_r)]$ and

$$\sum_{n \in J_r} a_{q(n)} = 0. \tag{4.2}$$

By the condition (1) this means that the series $\sum a_{q_n}$ is also convergent. On the other hand, we have

$$\sum_{n \in [x_1^{(r)}, x_r^{(r)}]} a_{p(n)} = \sum_{n \in \mathbb{X}_r^{(1)}} a_n = 1$$

for every $r \in \mathbb{N}$ which yields the divergence of the series $\sum a_{p(n)}$. Therefore the relation $\sum(q) \setminus \sum(p) \neq \emptyset$ holds as claimed. ∎

**THEOREM 4.5.** *For every permutation $p \in \mathfrak{D}$ there exists a family $\Phi(p)$ of divergent permutations such that*

(1) $\operatorname{card}\Phi(p) = \mathfrak{c}$,
(2) $\sum(p) \subset \sum(q)$ *for every permutation* $q \in \Phi(p)$,
(3) *any two different permutations* $q_1, q_2 \in \Phi(p)$ *are incomparable.*

**Proof.** Fix $p \in \mathfrak{D}$. By Corollary 4.3 we can choose a countable family of increasing sequences $\mathbb{S}_r = \{z_n^{(r)} : n = 1, 2, \ldots, 2r\}$, $r \in \mathbb{N}$, of positive integers such that for every $r \in \mathbb{N}$

$$\mathbb{S}_r \cup p(\mathbb{S}_r) < \mathbb{S}_{r+1} \cup p(\mathbb{S}_{r+1}), \tag{4.3}$$

and the sequences $\mathbb{S}_r^{(1)}$ and $\mathbb{S}_r^{(2)}$ are both strictly monotonic and simultaneously are spliced, where

$$\begin{aligned} \mathbb{S}_r^{(1)} &:= \{p(z_n^{(r)}) :\ n = 1, 2, \ldots, r\}, \\ \mathbb{S}_r^{(2)} &:= \{p(z_n^{(r)}) :\ n = r+1, r+2, \ldots, 2r\}. \end{aligned} \tag{4.4}$$

We shall use the following notation:

$$z_0^{(r)} = \min\{\mathbb{S}_r \cup p(\mathbb{S}_r)\}, \quad z_{2r+1}^{(r)} = \max\{\mathbb{S}_r \cup p(\mathbb{S}_r)\},$$

$$I_i^{(r)} = [z_i^{(r)}, z_{i+1}^{(r)}), \quad i = 0, 1, \ldots, 2r-1,$$

$$I_{2r}^{(r)} = [z_{2r}^{(r)}, z_{2r+1}^{(r)}], \text{ for every } r \in \mathbb{N}.$$

It is obvious that the intervals $I_i^{(r)}$, $i = 0, 1, \ldots, 2r$, form a partition of the interval $\mathbb{K}_r := [z_0^{(r)}, z_{2r+1}^{(r)}]$ and, by (4.3), we have

$$\mathbb{K}_r < \mathbb{K}_{r+1} \ r \in \mathbb{N}. \tag{4.5}$$

Notice that it may happen that some of the sets $I_0^{(r)}$, $r \in \mathbb{N}$, are empty.

Our next goal is to define a new partition of the interval $\mathbb{K}_r$ having the form $\{J_i^{(r)} : i = 0, 1, \ldots, 2r\}$ for every $r \in \mathbb{N}$. The desired partition of the interval $\mathbb{K}_r$ will be uniquely determined by the conditions:

$$J_0^{(r)} = I_0^{(r)} < J_i^{(r)} < J_{i+1}^{(r)},$$

for $i = 1, 2, \ldots, 2r-1$, and

$$\operatorname{card} J_{2i-1}^{(r)} = \operatorname{card} I_i^{(r)} \quad \text{and} \quad \operatorname{card} J_{2i}^{(r)} = \operatorname{card} I_{i+r}^{(r)},$$

for $i = 1, 2, \ldots, r$.

We will denote by $\gamma_{2i-1}^{(r)}$ the increasing mapping of the interval $J_{2i-1}^{(r)}$ onto the interval $I_i^{(r)}$ and by $\gamma_{2i}^{(r)}$ the increasing mapping of the interval $J_{2i}^{(r)}$ onto the interval $I_{i+r}^{(r)}$, for every $i = 1, 2, \ldots, r$ and for every $r \in \mathbb{N}$. Moreover, let

$\gamma_0^{(r)}$ be the identity function of the interval $I_0^{(r)}$ whenever $I_0^{(r)} \neq \emptyset$. In the case when $I_0^{(r)} = \emptyset$ we define $\gamma_0^{(r)}$ as the empty function.

Let $S$ denote a *Sierpiński's family* of increasing sequences of positive integers. With each sequence $s \in S$, $s = \{s(n)\}$, we associate some permutation $q_s$ acting as follows:

$$q_s(n) = p\gamma_i^{(s(r))}(n),$$

for $n \in J_i^{(s(r))}$, $i = 0, 1, \ldots, 2s(r)$ and $r \in \mathbb{N}$. For the remaining positive integers $n$ we set $q_s(n) = p(n)$.

Put $\mathfrak{D}(p) = \{q_s : s \in S\}$. By (4.4) and by the definition of $q_s$ the set $q_s\left(\left[z_1^{(r)}, z_r^{(r)}\right]\right)$ is a union of at least $r$ MSIs for every $r \in \mathbb{N} \setminus s$. Since the set $\mathbb{N} \setminus s$ is infinite we conclude that the permutation $q_s$ is divergent for each $s \in S$. Hence $\mathfrak{D}(p) \subset \mathfrak{D}$.

Fix an $s \in S$, $s = \{s(n)\}$. Then, from the definition of $q_s$ we deduce that for any $n \in \mathbb{N}$ and for any subinterval $U$ of the interval $\mathbb{K}_{s(n)}$ there exist two subintervals $V_1$ and $V_2$ of the intervals $\left[z_0^{(s(n))}, z_{s(n)+1}^{(s(n))}\right)$ and $\left[z_{s(n)+1}^{(s(n))}, z_{2s(n)+1}^{(s(n))}\right]$ respectively, such that

$$q_s(U) = p(V_1) \cup p(V_2).$$

Simultaneously we have

$$q_s(\mathbb{K}_r) = p(\mathbb{K}_r)$$

for every $r \in \mathbb{N}$, and

$$q_s(n) = p(n)$$

for all positive integers $n \in \left(\mathbb{N} \setminus \bigcup_{n\in\mathbb{N}} \mathbb{K}_{s(n)}\right)$.

From the last three relations and from (4.5) it can be easily concluded that the weak inclusion $\sum(p) \subseteq \sum(q_s)$ holds true. Furthermore, if we set $\mathbb{X}_r = \mathbb{S}_{s(r)}$, $r \in \mathbb{N}$, then the hypotheses of Lemma 4.4 are fulfilled with $q_s$ instead of the permutation $q$, and consequently $\sum(q_s) \setminus \sum(p) \neq \emptyset$. Hence and from the previous relation we get $\sum(p) \subset \sum(q_s)$.

Take $s, t \in S$, $s \neq t$, $s = \{s(n)\}$, $t = \{t(n)\}$. Then for almost all indices $n \in \mathbb{N}$ we have

$$q_s(i) = p(i), \quad i \in \mathbb{K}_{t(n)}, \tag{4.6}$$

and

$$q_t(i) = p(i), \quad i \in \mathbb{K}_{s(n)}. \tag{4.7}$$

Set

$$a_n = \begin{cases} r^{-1} \text{ for } n \in \mathbb{S}^{(1)}_{s(r)}, \\ -r^{-1} \text{ for } n \in \mathbb{S}^{(2)}_{s(r)}, \end{cases}$$

and

$$b_n = \begin{cases} r^{-1} \text{ for } n \in \mathbb{S}^{(1)}_{t(r)}, \\ -r^{-1} \text{ for } n \in \mathbb{S}^{(2)}_{t(r)}, \end{cases}$$

for every $r \in \mathbb{N}$. Immediately from the definitions of the permutations $q_s$ and $q_t$ it follows that the series $\sum a_{q_{s(n)}}$ and $\sum b_{q_{t(n)}}$ are both convergent. At the same time, from (4.6) and (4.7) we get that the series $\sum a_{q_{t(n)}}$ and $\sum b_{q_{s(n)}}$ are both divergent. This means that the permutations $q_s$ and $q_t$ are incomparable, and hence the proof is completed. ∎

**Remark 4.6.** If $p \in \mathfrak{DD}$ then we may assume that for every $r \in \mathbb{N}$ there exists an interval $I$ such that

$$z^{(r)}_{2r+1} < I \cup p^{-1}(I) < z^{(r+1)}_0$$

and the set $p^{-1}(I)$ is a union of at least $r$ MSIs. Then from the definition of $q_s$, we see that each permutation $q_s$, $s \in S$, belongs to $\mathfrak{DD}$. Thus $\mathfrak{D}(p) \subset \mathfrak{DD}$.

**Remark 4.7.** Let $\mathfrak{R}$ be the set of all positive integers $r$ for which only one of the sequences $\mathbb{S}^{(1)}_r$ or $\mathbb{S}^{(2)}_r$ is increasing. Suppose that the definitions of the intervals $J^{(r)}_i$, $i = 1, 2, \ldots, 2r-1$, and the mappings $\gamma^{(r)}_{2i}$, $i = 1, 2, \ldots, r$, are replaced for every $r \in \mathfrak{R}$ by the following ones. The intervals $J^{(r)}_i \subseteq \mathbb{K}$ are uniquely determined by the conditions:

$$J^{(r)}_0 = I^{(r)}_0 < J^{(r)}_i = J^{(r)}_{i+1}$$

for $i = 1, 2, \ldots, 2r-1$, and

$$\operatorname{card} J^{(r)}_{2i-1} = \operatorname{card} I^{(r)}_i \quad \text{and} \quad \operatorname{card} J^{(r)}_{2i} = \operatorname{card} I^{(r)}_{2r-i+1},$$

for $i = 1, 2, \ldots, r$. Next $\gamma^{(r)}_{2i}$ is defined to be the increasing mapping of the interval $J^{(r)}_{2i}$ onto the interval $I^{(r)}_{2r-i+1}$, for every $i = 1, 2, \ldots, r$.

Take $s, t \in S$, $s \neq t$, $s = \{s(n)\}$, $t = \{t(n)\}$. Then it can be easily concluded that

$$q_s \in \mathfrak{C}\Big(q_s^{-1}\Big(\bigcup_{n \in \mathbb{N}} p(\mathbb{S}_{s(n)})\Big)\Big) \quad \text{and} \quad q_t \in \mathfrak{D}\Big(q_t^{-1}\Big(\bigcup_{n \in \mathbb{N}} p(\mathbb{S}_{s(n)})\Big)\Big),$$

and

$$q_s \in \mathfrak{D}\Big(q_s^{-1}\Big(\bigcup_{n \in \mathbb{N}} p(\mathbb{S}_{t(n)})\Big)\Big) \quad \text{and} \quad q_t \in \mathfrak{C}\Big(q_t^{-1}\Big(\bigcup_{n \in \mathbb{N}} p(\mathbb{S}_{t(n)})\Big)\Big).$$

This implies, by Corollary 3.4, that the permutations $q_s$ and $q_t$ are incomparable.

**Example 4.8.** Let $\{I_n\}$ be an increasing sequence of intervals which form a partition of $\mathbb{N}$ and such that $\operatorname{card} I_n \to \infty$ as $n \to \infty$.

Let $p, q \in \mathfrak{D}$ be permutations such that the restriction to the set $\mathbb{E}$ of $p$ is the identity function of $\mathbb{E}$ and the restriction to $\mathbb{O}$ of $q$ is the identity function of $\mathbb{O}$, where

$$\mathbb{E} := \bigcup_{n\in\mathbb{N}} I_{2n} \quad \text{and} \quad \mathbb{O} := \bigcup_{n\in\mathbb{N}} I_{2n-1}.$$

Moreover, assume that

$$p(I_n) = q(I_n) = I_n, \quad n \in \mathbb{N}. \tag{4.8}$$

Then, by Remark 3.8, the following equality is fulfilled:

$$\sum(p) \cap \sum(q) = \sum(\sigma),$$

where $\sigma(i) := p(i)$ for $i \in \mathbb{O}$ and $\sigma(i) := q(i)$ for $i \in \mathbb{E}$. On the other hand, the set $\sum(p) \cup \sum(q)$ is not a subset of the convergence class of any divergent permutation since the following lemma holds true:

**Lemma 4.9.** *If $\sigma \in \mathfrak{P}$ and $\sum(p) \cup \sum(q) \subseteq \sum(\sigma)$ then $\sigma \in \mathfrak{C}$.*

**Proof.** Suppose, contrary to our claim, that there exists a permutation $\sigma \in \mathfrak{D}$ such that $\sum(p) \cup \sum(q) \subseteq \sum(\sigma)$.

Since $p \in \mathfrak{C}(p^{-1}(\mathbb{E}))$ and $q \in \mathfrak{C}(q^{-1}(\mathbb{O}))$, from Corollary 3.5 we get $\sigma \in \mathfrak{C}(\sigma^{-1}(\mathbb{E}))$ and $\sigma \in \mathfrak{C}(\sigma^{-1}(\mathbb{O}))$. The following positive integer $k$ is therefore well defined:

$$k = \max\left\{c\left(\sigma \mid \sigma^{-1}(\mathbb{E})\right), c\left(\sigma \mid \sigma^{-1}(\mathbb{O})\right)\right\}.$$

Now we choose a sequence $\{J_n\}$ of intervals such that for every $n \in \mathbb{N}$

$$\text{the set } \sigma(J_n) \text{ is a union of at least } 2(n+k) \text{ MSIs} \tag{4.9}$$

and

$$\sigma(J_n) < \sigma(J_{n+1}). \tag{4.10}$$

Let $\left\{L_i^{(n)} : i = 1, 2, \ldots, v_n\right\}$ and $\left\{K_i^{(n)} : i = 1, 2, \ldots, v_n - 1\right\}$ be the increasing MSIs-partitions of the sets

$$\sigma(J_n) \quad \text{and} \quad [\min \sigma(J_n), \max \sigma(J_n)] \setminus \sigma(J_n), \tag{4.11}$$

respectively, for every $n \in \mathbb{N}$. There is no loss of generality in assuming that

$$\lambda_n := \operatorname{card}\{i : \ L_i^n \cap \mathbb{E} \neq \emptyset\} \geq \operatorname{card}\{i : \ L_i^n \cap \mathbb{O} \neq \emptyset\} \tag{4.12}$$

for every $n \in \mathbb{N}$. By (4.9) we get

$$\lambda_n \geq n + k, \quad n \in \mathbb{N}. \tag{4.13}$$

Since the set $J_n \cap \sigma^{-1}(\mathbb{E})$ is an interval of the set $\sigma^{-1}(\mathbb{E})$ it follows from the definition of $k$ that the set $\sigma(J_n) \cap \mathbb{E}$ is a union of at most $k$ MSIs of $\mathbb{E}$. On the other hand, if a subset $I$ of the set $\sigma(J_n) \cap \mathbb{E}$ is simultaneously an interval of $\mathbb{E}$ and we have $I \cap L_i^{(n)} \neq \emptyset$ and $I \cap L_j^{(n)} \neq \emptyset$ for some indices $i, j \in \mathbb{N}$ such that $i < j \leq v_n$, then $K_s^{(n)} \subset \mathbb{O}$ for every index $s$ such that $L_i^{(n)} < K_s^{(n)} < L_j^{(n)}$. The two last remarks and the conditions (4.12) and (4.13) assure us that we can choose increasing sequences $\{s(i) : i = 1, \dots, n\}$ and $\{t(i) : i = 1, \dots, n\}$ of positive integers such that

$$(4.14) \qquad \begin{cases} s(n), t(n) < v_n, \\ L_{s(i)}^{(n)} < K_{t(i)}^{(n)} < L_{s(i+1)}^{(n)} < K_{t(n)}^{(n)}, \quad i < n, \\ L_{s(i)}^{(n)} \cap \mathbb{E} \neq \emptyset \quad \text{and} \quad K_{t(i)}^{(n)} \subset \mathbb{O}, \end{cases}$$

for every index $i = 1, \dots, n$.

Let $\psi$ be a choice function of the following family:

$$\left\{U : \text{either } U = L_{s(i)}^{(n)} \cap \mathbb{E} \text{ or } U = K_{t(i)}^{(n)} \text{ for some } i, n \in \mathbb{N}, i \leq n\right\}.$$

Define

$$a_n = \begin{cases} -w^{-1} \text{ for } n \in \{\psi(K_{t(i)}^{(w)}) : w \in \mathbb{N} \text{ and } i = 1, \dots, w\}, \\ w^{-1} \text{ for } n \in \{\psi(L_{s(i)}^{(w)} \cap \mathbb{E}) : w \in \mathbb{N} \text{ and } i = 1, \dots, w\}, \\ 0 \text{ for the remaining indices } n \in \mathbb{N}. \end{cases}$$

We notice that by (4.10) this definition is correct.

From (4.8), (4.10), (4.14) and from the definition of $\psi$ we easily deduce that all three series $\sum a_n$, $\sum a_{p(n)}$ and $\sum a_{q(n)}$ are convergent to zero. On the other hand, we have $\sum_{i \in J_n} a_{\sigma(i)} = 1$ for every $n \in \mathbb{N}$, which is equivalent to the divergence of the series $\sum a_{\sigma(n)}$. This contradicts our assumption. ∎

## 5. The family $\Omega$

We denote by $\Omega$ the family of all divergent permutations $p$ for which there exist an increasing sequence $\{I_n(p)\}$ of intervals and a positive integer $k(p)$ with the following properties:

(i) $p^{-1}(I_n(p)) < p^{-1}(I_{n+1}(p))$,

(ii) any set $p^{-1}(I_n(p))$ is a union of at most $k(p)$ MSIs,

(iii) $\lim_{n \to \infty} \gamma_n(p) = \infty$ where

$$\gamma_n(p) := \max\left\{c(p \mid J) : J \text{ is an interval and } J \subseteq p^{-1}(I_n(p))\right\},$$

for every $n \in \mathbb{N}$. We notice that $\mathfrak{DC} \subset \Omega$. Furthermore, if $p \in \mathfrak{DC}$ then we may assume that $k(p) = c(p)$.

**THEOREM 5.1.** *For any permutation $p \in \Omega$ there exists a subset $\Omega(p)$ of $\Omega$ having the power of the continuum such that $\sum(p) \subset \sum(\phi)$ for each permutation $\phi \in \Omega(p)$ and any two different permutations $\phi$ and $\psi$ from $\Omega(p)$ are incomparable. Moreover, it can be assumed that*

$$\Omega(p) \subset \mathfrak{DC} \quad \textit{and} \quad c(\phi^{-1}) \leq 4c(p^{-1}) + 1, \quad \textit{for every} \quad \phi \in \Omega(p),$$

*whenever $p \in \mathfrak{DC}$.*

**Proof.** Let us fix a permutation $p \in \Omega$. Take an increasing sequence $\{I_n\}$ of intervals and a positive integer $k$ satisfying the conditions (i)–(iii) above. Additionally, by passing to a subsequence if necessary, we may suppose that

$$I_n \cup p^{-1}(I_n) < I_{n+1} \cup p^{-1}(I_{n+1}) \quad \text{for every } n \in \mathbb{N}. \tag{5.1}$$

Let $S$ be a *Sierpiński's* family of increasing sequences of positive integers with the following property: for any two different sequences $s, t \in S$, $s = \{s(n)\}$, $t = \{t(n)\}$ the inequallty $t(i) > s(i)$ for some index $i \in \mathbb{N}$ implies that

$$s(n+1) > t(n) > s(n) \quad \text{for large enough } n \in \mathbb{N}. \tag{5.2}$$

With each sequence $s \in S$, $s = \{s(n)\}$, we associate the permutation $\phi_s$ of $\mathbb{N}$ which transforms the elements of the set $\mathbb{I}(s) := \bigcup_{n \in \mathbb{N}} p^{-1}\left(I_{s(n)}\right)$ as follows: $\phi_s(i) = p(i)$ for $i \in \mathbb{I}(s)$ and such that $\phi_s$ is the increasing map of the complement of the set $\mathbb{I}(s)$ onto the complement of the set $p(\mathbb{I}(s))$.

Put $\Omega(p) = \{\phi_s : s \in S\}$. We remark that for each $s \in S$, $s = \{s(n)\}$, the conditions (i)–(iii) are satisfied with the permutation $\phi_s$ instead of $p$ and with the intervals $I_{s(n)}$ instead of $I_n$, $n \in \mathbb{N}$. Thus $\Omega(p) \subset \Omega$.

Set $J_n = \left[\min\left(I_n \cup p^{-1}(I_n)\right), \max\left(I_n \cup p^{-1}(I_n)\right)\right]$, $n \in \mathbb{N}$. Then, from (5.1) and from the definition of the permutations $\phi_s$, $s \in S$, we deduce that for every $s \in S$ and $n \in \mathbb{M}$ the following conditions hold true:

(1) $\phi_s(J_{s(n)}) = J_{s(n)}$,
(2) the restriction to the open interval $\left(\max J_{s(n)}, \min J_{s(n+1)}\right)$ of $\phi_s$ is either the identity function or the empty function,
(3) and if $p \in \mathfrak{DC}$ then the set $\phi_s^{-1}(K)$ is a union of at most $2c(p^{-1})$ MSIs, for any subinterval $K$ of the interval $J_{s(n)}$.

Hence, we get $c(\phi_s^{-1}) \leq 4c(p^{-1}) + 1$ for all $s \in S$ whenever $p \in \mathfrak{DC}$. This yields, by (iii), that $\Omega(p) \subset \mathfrak{DC}$ whenever $p \in \mathfrak{DC}$.

Let $s, t \in S$, $s = \{s(n)\}$, $t = \{t(n)\}$ and $s \neq t$. We show that the permutations $\phi_s$ and $\phi_t$ are incomparable.

By the conditions (5.2) and (2) there exists $m \in \mathbb{N}$ such that the restriction to the set $\bigcup_{n=m}^{\infty} I_{t(n)}$ of $\phi_s$ is the identity function and that the restriction to the set $\bigcup_{n=m}^{\infty} I_{s(n)}$ of $\phi_t$, is also the identity function. On the other hand, by (iii), for any sufficiently large $n \in \mathbb{N}$ there exist two intervals $L_n \subset p^{-1}(I_{s(n)})$ and $K_n \subset p^{-1}(I_{t(n)})$ and two increasing sequences of positive integers

$$a_n = \left\{a_i^{(n)} : i = 1, 2, \ldots, 2(\gamma_{s(n)} - 1)\right\} \subset I_{s(n)}$$

and

$$b_n = \left\{b_i^{(n)} : i = 1, 2, \ldots, 2(\gamma_{t(n)} - 1)\right\} \subset I_{t(n)}$$

such that

$$\phi_s(L_n) \cap a_n = \left\{a_{2i}^{(n)} : i = 1, 2, \ldots, \gamma_{s(n)} - 1\right\}$$

and

$$\phi_t(K_n) \cap b_n = \left\{b_{2i}^{(n)} : i = 1, 2, \ldots, \gamma_{t(n)} - 1\right\}$$

for every $n \in \mathbb{N}$. Putting these observations together we can easy conclude that

$$\phi_s \in \mathfrak{D}\Big(\phi_s^{-1}\Big(\bigcup_{n=w}^{\infty} a_n\Big)\Big) \quad \text{and} \quad \phi_t \in \mathfrak{C}\Big(\phi_t^{-1}\Big(\bigcup_{n=w}^{\infty} a_n\Big)\Big),$$

and

$$\phi_s \in \mathfrak{C}\Big(\phi_s^{-1}\Big(\bigcup_{n=w}^{\infty} b_n\Big)\Big) \quad \text{and} \quad \phi_t \in \mathfrak{D}\Big(\phi_t^{-1}\Big(\bigcup_{n=w}^{\infty} b_n\Big)\Big),$$

for any $w \geq m$ chosen such that $\gamma_{s(n)} > 1$ and $\gamma_{t(n)} > 1$ for $n \geq w$. Hence, by Corollary 3.4, we obtain

$$\sum(\phi_s) \setminus \sum(\phi_t) \neq \emptyset \quad \text{and} \quad \sum(\phi_t) \setminus \sum(\phi_s) \neq \emptyset.$$

This means that the permutations $\phi_s$ and $\phi_t$ are incomparable as required.

Now let the series $\sum a_n$ and $\sum a_{p(n)}$ be simultaneously convergent. Then using the definition of permutations $\phi_s$, $s \in S$, and the conditions (ii), (5.1), (1) and (2) we deduce that any series of the form $\sum a_{\phi_s(n)}$, where $s \in S$, satisfies the Cauchy condition i.e. it is also convergent. Hence we get $\sum(p) \subseteq \sum(\phi_s)$ for each index $s \in S$.

Finally, since for each $s \in S$ there exist infinite many positive integers $n \in \mathbb{N}$ such that the restriction to the interval $I_n$ of $\phi_s$ is the identity function, the strict inclusion $\sum(p) \subset \sum(\phi_s)$ follows immediately from the condition (iii) and from Corollary 3.4. ∎

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