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ASYMPTOTIC STABILITY OF SOLUTIONS TO THE EQUATIONS OF LINEAR ELASTICITY AND THERMOELASTICITY IN VISCOPOROUS MEDIA

Abstract. The systems of evolution equations modelling elasticity and thermoelasticity of viscoporous bounded media are considered. The existence of c_0-semigroups of contractions defining solutions to the systems is proved. The asymptotic vanishing of energies of solutions when $t \to \infty$ is explained.

1. Introduction and statement of problems

An increasing interest is observed in recent years to determine the decay behavior of the solutions of several elasticity problems. In classical thermoelasticity theory the decay effects were studied in the book [12] and in papers [10], [14], [5]. In the papers [2], [17], [18] there was studied the decay of solutions of the one-dimensional elasticity models where besides of thermal dissipation the porosity dissipation is taken into account. The similar kind of problems (indirect internal stabilization of coupled evolution equations) has recently been the focus of interest of other authors [1], [6]. Our goal in this paper is to establish the stabilization of solutions for two- and three- dimensional elasticity and thermoelasticity systems for viscoporous materials.

Let us begin from evolution equations [4], [3]

$$\begin{aligned} \rho\partial_t^2 u &= \operatorname{div} T, \\ J\partial_t^2 \phi &= \operatorname{div} h + g, \end{aligned} \tag{1.1}$$

where T denotes the stress tensor, u denotes the displacement vector, h denotes equilibrated stress vector, g denotes intrinsic equilibrated body force

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and the scalar function ϕ denotes the change in the volume fraction from the reference configuration, $(\mathrm{div}T)_i := \sum_{j=1}^{n} \partial_j T_{ij}$, $n = 2, 3$, denotes the dimension of space and u has the same dimension.

In the linear theory there are considered the following constitutive relations

$$T := \sigma(u) + b\phi I, \quad h := a\nabla\phi, \quad g := -b\sum_{l=1}^{n} e_{ll}(u) - \gamma\phi + E,$$

where $\sigma(u)$ denotes the elasticity stress tensor,

$$\sigma(u)_{ij} := \lambda \sum_{l=1}^{n} e_{ll}(u)\delta_{ij} + 2\mu e_{ij}(u), \quad e_{ij}(u) := \frac{1}{2}(\partial_j u_i + \partial_i u_j),$$

E denotes the dissipation friction and is taken to be equal $E := -r\partial_t\phi$, I denotes the $n \times n$ unit matrix.

The coefficients $a, b, \gamma, \mu > 0$, and for simplicity of the further considerations we put $\rho = 1$ and $J = 1$. After subjecting the system (1.1) with initial and boundary conditions we obtain the following system for u and ϕ.

PROBLEM 1.

$$(1.2) \qquad \begin{aligned} &\partial_t^2 u = \Delta_e u + b\nabla\phi \quad \text{in} \quad D \times R_+, \\ &\partial_t^2 \phi = a\Delta\phi - b\mathrm{div}u - \gamma\phi - r\partial_t\phi \quad \text{in} \quad D \times R_+, \\ &u = 0, \quad B\phi = 0 \quad \text{on} \quad \partial D \times R_+, \\ &u(0) = u^0, \partial_t u(0) = u^1, \phi(0) = \phi^0, \partial_t\phi(0) = \phi^1 \quad \text{in} \quad D. \end{aligned}$$

In the above $D \subset R^n$, denotes a bounded domain with boundary ∂D having regularity of class C^2, $\Delta_e := \mu\Delta I + (\mu + \lambda)\nabla\mathrm{div}$ denotes the elliptic Lamé operator, $R_+ := (0, +\infty)$, $B\phi = \phi$ or $B\phi = \partial_\nu\phi$, where ν denotes the outer unit normal vector to ∂D. Physically D is the region occupied by the body in the reference configuration.

To take into consideration also the thermal dissipation, the third equation is added to the system (1.1)

$$(1.3) \qquad \rho T_0 \partial_t \eta = \mathrm{div}q,$$

where η denotes the entropy and q the heat flux (see [11]), $T_0 > 0$ is a constant. From the classical linear theory we take the following constitutive relations for q and η:

$$q := d\nabla\theta, \quad \rho\eta = \theta + M\mathrm{div}u + M_1\phi,$$

and for g in (1.1) we take

$$g = -b\sum_{l=1}^{n} e_{ii}(u) - \gamma\phi + M_1\theta + E,$$

where θ denotes the temperature. The coefficients $d, M, M_1 > 0$, and for simplicity we put $T_0 = 1$.

After subjecting the system (1.1), (1.3) with initial and boundary conditions we obtain the following system for u, ϕ, θ:

Problem 2.

$$
\begin{aligned}
&\partial_t^2 u = \Delta_e u + b\nabla\phi - M\nabla\theta \quad \text{in} \quad D \times R_+,\\
&\partial_t^2 \phi = a\Delta\phi - b\mathrm{div} u - \gamma\phi - r\partial_t\phi + M_1\theta \quad \text{in} \quad D \times R_+,\\
&\partial_t\theta = d\Delta\theta - M\mathrm{div}\partial_t u - M_1\partial_t\phi \quad \text{in} \quad D \times R_+,\\
&u = 0, \quad B\phi = 0, \quad \theta = 0 \quad \text{on} \quad \partial D \times R_+,\\
&u(0) = u^0, \partial_t u(0) = u^1, \phi(0) = \phi^0, \partial_t\phi(0) = \phi^1, \theta(0) = \theta^0 \quad \text{in} \quad D.
\end{aligned} \tag{1.4}
$$

The term $-r\partial_t\phi$, in equation for ϕ in the systems, models the porous dissipation called viscoporosity [4], however in models derived by Iesan [11] such term does not appear. In the Appendix 1 we give explanation that the dissipative term in equations on ϕ appears naturally when one considers thermoelasticity system taking into account microtemperatures [9], and then makes the decoupling which separates the system for microtemperatures from equations for u, ϕ, θ. Usually in the literature it is considered $B\phi = \phi$ when $u = 0$ on ∂D, but there exist papers where also the operator $B\phi = \partial_\nu\phi$ is considered [2, 18]. In our paper we shall consider both possibilities for B: $B\phi = \phi$ and $B\phi = \partial_\nu\phi$. We are able to prove one of our main results only for $B\phi = \phi$.

The first topic in this paper is to establish the existence of c_0-semigroups of contractions defining solutions of Problems 1, 2. The second one is to explain when the energies of solutions asymptotically vanish when $t \to \infty$. Under the authors knowledge these both questions were not rigorously studied yet. For the 1-dimensional models some results are obtained in [2, 18]. For solutions of Problem 1 there was proved the lack of uniform stabilization (that is the energy does not tend to 0 with exponential speed when $t \to \infty$). For solutions of Problem 2 there was proved the uniform stabilization when the parameter $r > 0$. But when $r = 0$ there was proved lack of the uniform stabilization. This means that interaction between ϕ and θ weakens dissipativity effects introduced by the parabolic equation for θ, because for the classical 1-dimensional thermoelasticity we have the uniform stabilization [16]. One can observe the positive feeedback interaction between ϕ and θ in the system (1.4). In analysis of thermoelaticity systems there exists a deep difference between 1-dimensional and many-dimensional models - see Remarks 2.10 in section 2 of this paper.

The analytical difficulties involve the strong coupling of equations in the systems. Under the authors knowledge, in the mathematical literature, the problem of stabilization in the system of strong coupled hyperbolic equations, where the damping comes only from one part of the system (see system (1.2)) has not been solved.

The organization of paper is the following. In section 2 we formulate our main results. In section 3 there will be proved the existence of c_0-semigroup of contractions defining solutions of Problem 2. This same concept one can apply to Problem 1. The proof relies on Hille–Phillips theorem [7, 20]. For justification of conditions of the Hille–Phillips theorem in the context of Problem 2 we use scheme done in [10].

In section 4 we prove stabilization of solutions for Problem 1. We use the resolvent criterion done by Tomilov [21]. All results announced above will be obtained for both cases of the operator B.

Section 5 is devoted to proving the stabilization of solutions for Problem 2 when $B\phi = \phi$. Here we adopt the scheme of proof done by Dafermos [5]. Authors were not able to apply the resolvent criterion to Problem 2 successfully. Stabilization of solutions of Problem 1 for dimension $n = 1$ can be proved by the same method as we present in this paper, the verification of details we leave to the reader. As we have mentioned earlier, the 1-dimensional Problem 2 was solved in [2, 18].

2. Main results

In this paper we shall work under the following special conditions on b, λ, μ, γ.

Assumption 2.1. We require $\lambda + \mu > 0$, $(\lambda + \mu)\gamma > b^2$ when $n = 2$ and $3\lambda + 2\mu > 0$, $(3\lambda + 2\mu)\gamma > 3b^2$ when $n = 3$.

In the Appendix 2 we explain that conditions on coefficients, which we have formulated above are optimal for the linear model (1.2) to be physically realistic.

We denote

$$\sigma(u) : \epsilon(v) := \sum_{i,j=1}^{n} \sigma_{ij}(u)\epsilon_{ij}(v).$$

Recall that $\Delta_e(u)_i = \sum_{j=1}^{n} \partial_j \sigma_{ij}(u)$, $i = 1, \dots, n$.

From the Sylvester theorem concerning positively defined matrices we derive:

Proposition 2.2. *If b, λ, μ, γ satisfy Assumption 2.1 then there exist constants $c_1, c_2 > 0$ such, that*

$$\sigma(u) : \epsilon(u) \geq c_1 \sum_{i,j=1}^{n} \epsilon_{ij}^2(u),$$

$$\sigma(u) : \epsilon(u) + 2b\phi \operatorname{div} u + \gamma\phi^2 \geq c_2(\phi^2 + \sum_{i,j=1}^{n} \epsilon_{ij}^2(u)).$$

For sake of convenience we recall the classical Korn inequality.

There exist constant $c > 0$, that for each $v \in H_0^1(D)^n$

$$\int_D \sum_{i,j=1}^{n} \epsilon_{i,j}^2 \geq c \int_D |\nabla v|^2,$$

where $|\nabla v|^2 := \sum_{i=1}^n |\nabla v_i|^2$.

We shall consider $H := H_0^1(D)^n \times H_0^1(D)$ when $B\phi = \phi$ and $H := H_0^1(D)^n \times H^1(D)$ when $B\phi = \partial_\nu \phi$.

Proposition 2.3. *The bilinear form*

$$\langle \zeta_1, \zeta_2 \rangle := \int_D [\sigma(u^1) : \epsilon(u^2) + a\nabla\phi^1 \cdot \nabla\phi^2 + \gamma\phi^1\phi^2 + b\phi^1 \mathrm{div} u^2 + b\phi^2 \mathrm{div} u^1],$$

$\zeta_i := (u^i, \phi^i)$, $u^i \in H_0^1(D)^n$, $\phi^i \in H_0^1(D)$ *when* $B\phi = \phi$ *and* $\phi^i \in H^1(D)$ *when* $B\phi = \partial_\nu\phi$, $i = 1, 2$, *defines the inner product in* H.

Sketch of the proof. From the second inequality of Proposition 2.2, and the Korn inequality we derive

$$\langle \zeta, \zeta \rangle \geq c \int_D [|\nabla u|^2 + |\nabla \phi|^2 + \phi^2], \quad \zeta \in H, \tag{2.1}$$

where $c > 0$ is constant. The proof can be closed.

Let $\|\cdot\|$ denote the norm in H generated by the inner product defined in Proposition 2.3.

Now we give definitions of weak solutions for Problems 1, 2.

Definition 2.4. We say that $\zeta(\cdot) \in C(\overline{R}_+; H) \cap C^1(\overline{R}_+; L^2(D)^{n+1})$ solves Problem 1 in a weak sense when it satisfies

1. $\zeta(0) = (u^0, \phi^0)^T, \partial_t\zeta(0) = (u^1, \phi^1)^T$ in D,
2. for each $(v, \psi)^T \in H$,

$$\frac{d}{dt} \int_D (v \cdot \partial_t u(t) + \psi\partial_t\phi(t)) + \langle \zeta(t), (v, \psi)^T \rangle + r \int_D \psi\partial_t\phi(t) = 0 \quad \text{in} \quad D'(R_+).$$

Definition 2.5. We say that $(\zeta(\cdot), \theta(\cdot)), \zeta(\cdot) \in C(\overline{R}_+; H) \cap C^1(\overline{R}_+; L^2(D)^{n+1})$, $\theta(\cdot) \in C(\overline{R}_+; L^2(D))$ solves Problem 2 in a weak sense if it satisfies

1. $(\zeta(0), \theta(0)) = (u^0, \phi^0, \theta^0)^T, \partial_t\zeta(0) = (u^1, \phi^1)^T$ in D,
2. for each $(v, \psi)^T \in H$, $\chi \in H^2(D) \cap H_0^1(D)$

$$\begin{aligned} &\frac{d}{dt} \int_D (v \cdot \partial_t u(t) + \psi\partial_t\phi(t) + \chi\theta(t) + M\chi \mathrm{div} u(t)) + \langle \zeta(t), (v, \psi)^T \rangle \\ &+ \int_D [(r\psi + M_1\chi)\partial_t\phi(t) - d\theta(t)\Delta\chi \\ &\qquad -M\theta \mathrm{div} v - M_1\theta(t)\psi] = 0 \quad \text{in} \quad D'(R_+). \end{aligned} \tag{2.2}$$

To make investigations more clear we introduce the following Hilbert spaces: $H_1 := H \times L^2(D)^{n+1}$, endowed with the inner product $\langle \eta_1, \eta_2 \rangle_1 := \langle \zeta_1, \zeta_2 \rangle + \int_D \left[v^1 \cdot v^2 + \psi^1 \psi^2 \right]$, $\eta_i := (\zeta_i, (v^i, \psi^i))^T$, $\zeta_i \in H, v^i \in L^2(D)^n$, $\psi^i \in L^2(D), i = 1, 2$ and $H_2 := H_1 \times L^2(D)$ endowed with the inner product

$$\langle \zeta_1, \zeta_2 \rangle_2 := \langle \eta_1, \eta_2 \rangle_1 + \int_D \theta^1 \theta^2, \zeta_i := (\eta_i, \theta^i)^T, \eta_i \in H_1, \theta^i \in L^2(D), i = 1, 2.$$

The norm in H_i generated by the inner product $\langle \cdot, \cdot \rangle_i$ we denote by $||\cdot||_i, i = 1, 2$.

We distinguish the following dense linear subspaces $X_i \subset H_i$, $i = 1, 2$:

$X_1 := (H^2(D) \cap H_0^1(D))^n \times (H^2(D) \cap H_0^1(D)) \times H_0^1(D)^n \times H_0^1(D)$ when $B\phi = \phi$, and

$X_1 := (H^2(D) \cap H_0^1(D))^n \times (H^2(D) \cap \{\partial_\nu \phi = 0 \quad \text{on} \quad \partial D\}) \times H_0^1(D)^n \times H^1(D)$ when $B\phi = \partial_\nu \phi$,

$X_2 := X_1 \times (H^2(D) \cap H_0^1(D))$.

We formulate the main existence theorems.

THEOREM 2.6. *Let the coefficients and the domain D satisfy conditions imposed above. Then on H_1 there exist a c_0-semigroup of contractions $S_1(t), t \in \overline{R}_+$ such that whenever $(u(t), \phi(t), v(t), \psi(t))^T := S_1(t)\eta_0$, $\eta_0 := (u^0, \phi^0, u^1, \phi^1)^T$ then for $\eta_0 \in X_1, (u(\cdot), \phi(\cdot))$ is the unique strong solution of Problem 1 and $\partial_t u(\cdot) = v(\cdot)$, $\partial_t \phi(\cdot) = \psi(\cdot)$, $S_1(t)\eta_0 \in X_1$, $t \in R_+$. For $\eta_0 \in H_1$, $(u(\cdot), \phi(\cdot))$ is the unique weak solution of Problem 1.*

THEOREM 2.7. *Under the same assumptions as in Theorem 2.6 there exist on H_2 a c_0-semigroup of contractions $S_2(t), t \in \overline{R}_+$ such that if $(u(t), \phi(t), v(t), \psi(t), \theta(t))^T := S_2(t)\zeta_0$, $\zeta_0 := (u^0, \phi^0, u^1, \phi^1, \theta^0)^T$ then for $\zeta_0 \in X_2, (u(\cdot), \phi(\cdot), \theta(\cdot))$ is the unique strong solution of Problem 2 and $\partial_t u(\cdot) = v(\cdot), \partial_t \phi(\cdot) = \psi(\cdot)$, $S_2(t)\zeta_0 \in X_2$, $t \in R_+$. For $\zeta_0 \in H_2$, $(u(\cdot), \phi(\cdot), \theta(\cdot))$ is the unique weak solution of Problem 2.*

In the following by solutions of Problems 1, 2 we shall mean their strong or weak solutions. We define energies:

$$E_1(t) := \frac{1}{2}||(u(t), \phi(t))^T||^2 + \int_D \left[|\partial_t u(t)|^2 + |\partial_t \phi(t)|^2 \right]$$

for solutions of Problem 1, and

$$E_2(t) := \frac{1}{2}||(u(t), \phi(t))^T||^2 + \int_D \left[|\partial_t u(t)|^2 + |\partial_t \phi(t)|^2 + \theta^2(t) \right]$$

for solutions of Problem 2.

DEFINITION 2.8. We say that a bounded domain $D \subset R^n$, satisfies the condition (C) if for every $s > 0$, the problem

$$-\Delta v = sv \quad \text{in} \quad D, \qquad \text{div} v = 0 \quad \text{in} \quad D, \qquad v = 0 \quad \text{on} \quad \partial D,$$

has only one solution $v = 0 \in R^n$.

Now we formulate our main result about the stabilization.

THEOREM 2.9. *Let coefficients and the domain D satisfy conditions imposed above, and additionally D satisfies the condition (C). Then for energy of solutions of Problem 1 we have $\lim_{t\to\infty} E_1(t) = 0$. For boundary operator $B\phi = \phi$ for the energy of solutions of Problem 2 we have $\lim_{t\to\infty} E_2(t) = 0$.*

We place here some comments concerning the results from Theorem 2.9.

Using the classical methods based on the spectral analysis one observes that the energy of solution for the problem

$$\begin{aligned} &\partial_t^2 u = a\Delta\phi - \gamma\phi - r\partial_t\phi \quad \text{in} \quad D \times R_+, \\ &B\phi = 0 \quad \text{on} \quad \partial D \times R_+, \\ &\phi(0) = \phi^0, \partial_t\phi(0) = \phi^1 \quad \text{in} \quad D, \end{aligned}$$

uniformly decays when $t \to \infty$.

This is due to the term $-r\partial_t\phi$ in the equation. In the equation for u in the system (1.2) we have not the term of such kind. But there is the interaction between u and ϕ described by the coupling of the equations in (1.2). Theorem 2.9 says, that for domains D satisfying condition (C) this interaction causes the disappearance of the whole energy when $t \to \infty$. We have established the indirect internal stabilization of elasticity waves in the viscoporous materials.

REMARK 2.10. (i) Let a domain D does not satisfy condition (C) (see [5, 14]), and let a nontrivial u_0 solve the system from Definition 2.8 for some $s > 0$. Then $(e^{\pm i\sqrt{\mu s}t} u_0(x), 0)$, $(e^{\pm i\sqrt{\mu s}t} u_0(x), 0, 0)$ solve Problems 1, 2 suitably. These solutions have no decaying properties when $t \to \infty$.

(ii) The results of paper [14] suggest that whenever domain D satisfies additionally the appropriate geometric conditions, then the decaying energy for solutions of Problem 2 may be uniform or polynomial.

(iii) It will be very interesting to explain whether for solutions of Problem 1 the decaying of energy has logarythmic speed.

3. Well posedness of problems

The assertions of Theorems 2.6, 2.7 are true for both cases of the operator B. Because for $B\phi = \partial_\nu\phi$ the investigations are a bit more difficult we formulate proofs for such operator B. The same scheme of proofs holds when Dirichlet boundary operator is considered.

For $\zeta_i := (u^i, \phi^i), i = 1, 2$ we put

$$(\zeta_1, \zeta_2)_{L^2} := \int_D (u^1 \cdot u^2 + \phi^1\phi^2) \qquad ||\zeta||^2_{L^2} := \int_D (|u|^2 + \phi^2).$$

We define operator:

$$A := \begin{pmatrix} -\Delta_e & , & -b\nabla \\ b\mathrm{div} & , & -a\Delta + \gamma I \end{pmatrix}$$

with domain $D(A) = (H^2(D) \cap H^1_0(D))^n \times (H^2(D) \cap \{\partial_\nu = 0 \quad \text{on} \quad \partial D\})$. It is evident that $A : D(A) \to L^2(D)^{n+1}$.

Proposition 3.1. *There exists constant $k_1 > 0$ such, that*

$$||\zeta||^2 \geq k_1 ||\zeta||^2_{L^2}, \quad \zeta \in H \tag{3.1}$$

$$(A\zeta, \zeta)_{L^2} \geq k_1 ||\zeta||^2_{L^2}, \quad \zeta \in D(A). \tag{3.2}$$

Proof. The inequality (3.1) we obtain from (2.1) and the Poincaré inequality. For the completness we recall the Poincaré inequality: there exists constant $c > 0$ such, that $\int_D |\nabla v|^2 \geq c \int_D |v|^2$, $v \in (H^1_0(D))^n$.

Next we calculate $(A\zeta, \zeta)_{L^2} = ||\zeta||^2$ when $\zeta \in D(A)$ and (3.1) implies (3.2). ∎

Proof of Theorem 2.7. Let us introduce operator

$$L_2 := \begin{pmatrix} 0 & , & 0 & , & I & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & I & , & 0 \\ \Delta_e & , & b\nabla & , & 0 & , & 0 & , & -M\nabla \\ -b\mathrm{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI & , & M_1 I \\ 0 & , & 0 & , & -M\mathrm{div} & , & -M_1 I & , & d\Delta \end{pmatrix},$$

with domain $D(L_2) := X_2$. It is easy to see, that X_2 is dense in H_2 and $L_2 : X_2 \to H_2$. For $\zeta \in X_2$ we calculate

$$\langle L_2(\zeta), \zeta \rangle_2 = -r \int_D \psi^2 - d \int_D |\nabla\theta|^2 \leq 0. \tag{3.3}$$

The above inequality means that L_2 is a dissipative operator. To show that L_2 is maximally dissipative it is enough to prove (see [7]) the existence of $\lambda > 0$ such that for every $f \in H_2$ the equation

$$(\lambda I - L_2)\zeta = f \tag{3.4}$$

has a solution $\zeta \in X_2$. To show this, first we check that $\ker(L_2) = \{0\}$. So let us suppose that $L_2\zeta = 0$ for some $\zeta \in X_2$. From the formula of L_2 we deduce that $v = 0, \psi = 0$ and $\theta = 0$ (because $\Delta\theta = 0$ in D and $\theta = 0$ on ∂D) and for $\xi = (u, \phi)^T$ we get $A\xi = 0$. Because $||\xi||^2 = (A\xi, \xi)_{L^2} = 0$ we have $\xi = 0$ and injectivity is proven. Then we should prove that L_2 is surjective. We must show that for every $g \equiv (g^1, \dots, g^5)^T \in H_2$ the equation $L_2\zeta = g$ has a solution $\zeta \in X_2$. From the formula for L_2 we deduce, that $v = g^1$, $\psi = g^2$ and θ is the solution of the Dirichlet problem for Poisson equation

$$d\Delta\theta = g^5 + M\mathrm{div} g^1 + M_1 g^2 \quad in \quad D \quad \theta = 0 \quad on \quad \partial D.$$

We must only prove the existence of $\xi \in D(A)$ such, that

$$A\xi = -\begin{pmatrix} g^3 + M\nabla\theta \\ g^4 + rg^2 - M_1\theta \end{pmatrix}. \tag{3.5}$$

From Lax-Milgram theory thanks to inequality (3.2) we obtain the existence of weak solution $\xi \in H$ of problem (3.5). From the theorem about regularity of weak solutions of elliptic systems (see [19], Theorem 4.18) we deduce that $\xi \in D(A)$ and surjectivity is proven. Operator L_2^{-1} considered as operator from H_2 to H_2 is continuous. We take $\lambda \in (0, \frac{1}{||L_2^{-1}||})$ and write equation (3.4) in the form $(I - \lambda L_2^{-1})\zeta = -L_2^{-1} f$. Because $\lambda||L_2^{-1}|| < 1$ the solution of this equation is equal

$$\zeta = -L_2^{-1}\Big(\sum_{k=0}^{\infty}(\lambda L_2^{-1})^k f\Big) \in X_2,$$

and we have maximal dissipativity of L_2. From Hille-Phillips theorem [7] we obtain, that L_2 is the generator of c_0-semigroup of contractions $S_2(t); t \in \overline{R}_+$ on H_2. Moreover, when $\zeta_0 \in X_2$, we have $S_2(\cdot)\zeta_0 \in C(\overline{R}_+; X_2) \cap C^1(\overline{R}_+; H_2)$. Since $\zeta(t) := S_2(t)\zeta_0$ when $\zeta_0 \in X_2$ is the only solution of system

$$\frac{d\zeta}{dt} = L_2\zeta; \quad t > 0 \quad \zeta(0) = \zeta_0, \tag{3.6}$$

from the formula for L_2 we deduce that $\partial_t u(\cdot) = v(\cdot)$, $\partial_t\phi(\cdot) = \psi(\cdot)$ and $u(\cdot), \phi(\cdot), \theta(\cdot)$ is the strong solution solution of Problem 2. From (3.3) and (3.6) we obtain

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}||\zeta(t)||_2^2 &= \left\langle \frac{d\zeta(t)}{dt}, \zeta(t) \right\rangle_2 = \langle L_2\zeta(t), \zeta(t)\rangle_2 \\ &= -r\int_D |\partial_t\phi(t)|^2 - d\int_D |\nabla\theta(t)|^2. \end{aligned} \tag{3.7}$$

It is easy to see that $E_2(t) = \frac{1}{2}||\zeta(t)||_2^2$. From (3.7) we get

$$E_2(t) - E_2(s) = -r\int_s^t\int_D(\partial_t\phi(\tau))^2 - d\int_s^t\int_D|\nabla\theta(\tau)|^2, \quad 0 \le s \le t. \tag{3.8}$$

For $\eta_0 \in H_2$ we consider sequence $\{\zeta_0^k; k \in N\} \subset X_2$ convergent in H_2 to ζ_0. The corresponding strong solutions (u^k, ϕ^k, θ^k) with initial data given by $\zeta_0^k, k \in N$, will satisfy all conditions of Definition 2.5. From the contractivity of semigroup $S_2(\cdot)$ we deduce, that $\lim_{k\to\infty}\zeta_k(\cdot) = S_2(\cdot)\zeta_0$ in $C(\overline{R}_+; H_2)$; $\zeta_k := S_2(t)\zeta_0^k$. We see that $v = \partial_t u$, $\psi = \partial_t\phi$. Moreover after writing (2.2) for (u_k, ϕ_k, θ_k) and passing in this equality with $k \to \infty$ we claim that $(u, \phi, \theta) := \lim_{k\to\infty}(u_k, \phi_k, \theta_k)$ (the limit is considered in $H \times L^2(D)$) is the weak solution of Problem 2. This also imply that the energy of this weak solution equals $E_2(t) = \frac{1}{2}||S_2(t)\zeta_0||_2^2$. Because of the contractivity of $S_2(\cdot)$ this yields that for energy of this weak solution we have $E_2(t) \le E_2(s)$, $0 \le s \le t$. Using the technique of proof from [7] one obtains the uniqueness of weak solution. The proof is complete.

About the proof of Theorem 2.6. The concept of the proof of Theorem 2.6 is the same as the one for Theorem 2.7.

The generator L_1 of c_0-semigroup of contractions $S_1(t), t \in \overline{R}_+$ on H_1 will be equal

$$L_1 := \begin{pmatrix} 0 & , & 0 & , & I & , & 0 \\ 0 & , & 0 & , & 0 & , & I \\ \Delta_e & , & b\nabla & , & 0 & , & 0 \\ -b\mathrm{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI \end{pmatrix},$$

his domain $D(L_1) = X_1$. The energy of solutions of Problem 1 will be equal $E_1(t) = \frac{1}{2}||S_1(t)\eta_0||_1^2$. We omit the details of the proof.

4. Stabilization of solutions of Problem 1

In this section we give proof for the first part of Theorem 2.9. We prove results only for $B\phi = \partial_\nu\phi$. The case of the Dirichlet boundary operator one can handle by the same scheme of proof.

We begin this section with proving results concerning the operator A (defined in Section 3). Let $k \in (0, k_1)$ and consider the operator $\widetilde{A} := A - kI$ (the number $k_1 > 0$ was defined in Section 3). From (3.2) we derive $(\widetilde{A}\xi, \xi)_{L^2} \ge (k_1 - k)||\xi||_{L^2}^2$.

The operator L_1 is closed because it is the generator of c_0-semigroup of contractions [5, 19]. This implies that A and $\widetilde{A}$ are closed as well. Because $A, \widetilde{A}$ are symmetric this yields that they are selfadjoint.

The inequality written above and Lax-Milgram theory imply that there exists $\widetilde{A}^{-1} : L^2(D)^{n+1} \to D(A)$, which as the operator in $L^2(D)^{n+1}$ is compact and selfadjoint. This yields that $\sigma(\widetilde{A}) = \sigma_p(\widetilde{A}) \subset [k_1 - k, \infty)$, and all eigenvalues have finite multiplicities. The different eigenvalues we enumerate according to the natural order: $k_1 - k \le r_1 < r_2 < \dots$, $\lim_{l\to\infty} r_l = \infty$. Let P_l denote the orthogonal projection operator in $L^2(D)^{n+1}$ onto the eigensubspace of $\widetilde{A}$ corresponding to $r_l, l \in N$. We can write $\widetilde{A} = \sum_{l=1}^{\infty} r_l P_l$ and for any $s \in R, A+(s-r_j)I = \sum_{l=1}^{\infty}(r_l - r_j + s + k)P_l$. Because $r_l - r_j, j, l \in N$ are independent of k we can assume

$$k \notin \bigcup_{j=2}^{\infty} \bigcup_{l=1}^{j-1} \{r_j - r_l\}. \tag{4.1}$$

Let us define

$$m_j(s) := \inf\{(s + k + r_l - r_j)^2 : l \in N\}, \qquad j \in N.$$

We have inequality

$$||(A + (s - r_j)I)\xi||_{L^2}^2 \ge m_j(s)||\xi||_{L^2}^2, \qquad \xi \in D(A).$$

This implies that whenever $m_j(s) > 0$ then there exists the resolvent operator $R(r_j - s; A)$ and

$$||R(r_j - s; A)|| \le \frac{1}{\sqrt{m_j(s)}}, \tag{4.2}$$

where we understand $R(r_j - s; A)$ as the operator in $L^2(D)^{n+1}$.

PROPOSITION 4.1. *Let $k \in (0; k_1)$ satisfy the condition (4.1) Then for each $j \in N$ there exists $s_j > 0$ such that $m_j(s_j) > 0$ and $\frac{s_j}{\sqrt{m_j(s_j)}} < 1$.*

Proof. We take arbitrary $j \in N$ and denote $a_l = (k + r_l - r_j)$. We have chosen k such that $0 \notin \{a_l\}$. It is clear that $a_l \to \infty$ when $l \to \infty$ and $\sqrt{m_j(s)} = \inf_{l\in N} |a_l + s|$. We consider three cases.

Case 1. $a_l > 0$ for each $l \in N$

We take $s_j = \frac{a_1}{2} > 0$. Because $\sqrt{m_j(s_j)} = \inf_{l\in N} |a_l + s_j| = \frac{3}{2}a_1$ we have

$$\frac{s_j}{\sqrt{m_j(s_j)}} = \frac{\frac{a_1}{2}}{\frac{3}{2}a_1} = \frac{1}{3} < 1.$$

This ends case 1.

When $a_l < 0$ for some $l \in N$ then we can choose $n \in N$ such, that $0 \in (a_n, a_{n+1})$.

Case 2a. $-a_n \le a_{n+1}$.

We take $s_j = -\frac{a_n}{3} > 0$. We have $0 \in (a_n + s_j, a_{n+1} + s_j) = (\frac{2}{3}a_n, a_{n+1} - \frac{a_n}{3})$ so $\sqrt{m_j(s_j)} = \inf_{l \in N} |a_l + s_j| = -\frac{2}{3}a_n$. We check that

$$\frac{s_j}{\sqrt{m_j(s_j)}} = \frac{-\frac{a_n}{3}}{-\frac{2}{3}a_n} = \frac{1}{2} < 1.$$

Case 2b. $-a_n > a_{n+1}$.

In this case we take $s_j = -\frac{a_n + a_{n+1}}{2} > 0$. We have $0 \in (a_n + s_j, a_{n+1} + s_j) = (-\frac{a_{n+1} - a_n}{2}, \frac{a_{n+1} - a_n}{2})$ and $\sqrt{m_j(s_j)} = \inf_{l \in N} |a_l + s_j| = \frac{a_{n+1} - a_n}{2}$. We calculate

$$\frac{s_j}{\sqrt{m_j(s_j)}} = \frac{-\frac{a_n + a_{n+1}}{2}}{\frac{a_{n+1} - a_n}{2}} = \frac{-a_{n+1} - a_n}{a_{n+1} - a_n} = 1 - \frac{2a_{n+1}}{a_{n+1} - a_n} < 1. \blacksquare$$

From [20] (page 11) we recall:

Proposition 4.2. *Let L be the generator of c_0-semigroup of contractions in a Hilbert space. Then $\{\lambda \in C : re\lambda > 0\} \subset \rho(L)$ and $||R(\lambda; L)|| \leq \frac{1}{re\lambda}$ for each λ with $re\lambda > 0$.*

From [21] we quote

Theorem 4.3. *Let L be the generator of c_0-semigroup of contractions $T(t), t \in \overline{R}_+$ in a Hilbert space Z. Whenever there exists a dense subset $M \subset Z$, such that*

$$\lim_{\alpha \to 0_+} \sqrt{\alpha} R(\alpha + i\beta; L) y = 0, \qquad y \in M,\ \beta \in R, \tag{4.3}$$

then $\lim_{t\to\infty} ||T(t)x||_Z = 0$ for each $x \in Z$.

Proof of Theorem 2.9 for $E_1(\cdot)$. From Section 3 we know that $E_1(t) = \frac{1}{2}||S_1(t)\eta_0||_1$, $\eta_0 \in H_1$. Because of Theorem 4.3 to prove $\lim_{t\to\infty} E_1(t) = 0$ it is enough to verify that (4.3) is satisfied for $L \equiv L_1$, $Z \equiv H_1$, $M \equiv H_1$. We begin this verification.

From Proposition 4.2 for each $f \in H_1$ and $\alpha > 0$ there exists $\eta \equiv \eta(\alpha, \beta) \in X_1$ solving the system $((\alpha + i\beta)I - L_1)\eta = f$. We rewrite this system in a more exact form. Let $f \equiv (f^1, \ldots, f^n)^T$, $\eta \equiv (\xi, v, \psi)^T$, $\xi \equiv (u, \phi)^T$, $\lambda := \alpha + i\beta$. Then we obtain: $v = \lambda u - f^1$, $\psi = \lambda\phi - f^2$ and claim that $\xi \in D(A)$ and solves the equation

$$\lambda^2 \xi + A\xi + \lambda B\xi = F(\lambda), \tag{4.4}$$

where $B := \begin{pmatrix} 0 & , & 0 \\ 0 & , & rI \end{pmatrix}$, $F(\lambda) := (f^3 + \lambda f^1, f^4 + (\lambda + r)f^2)$.

Let us denote $\xi_R := re\xi$, $F_R(\lambda) := reF(\lambda)$, $\xi_I := im\xi$, $F_I(\lambda) := imF(\lambda)$.

From (4.4) we derive

$$(4.5)\qquad \begin{aligned} (A-\beta^2)\xi_R + (\alpha^2+\alpha\beta)\xi_R - (2\alpha\beta+\beta B)\xi_I &= F_R(\lambda),\\ (A-\beta^2)\xi_I + (\alpha^2+\alpha\beta)\xi_I + (2\alpha\beta+\beta B)\xi_R &= F_I(\lambda). \end{aligned}$$

Without loss of generality we can assume that $\alpha \in (0,1)$. Now we derive estimations for norms $||\sqrt{\alpha}\xi||$ and $||\sqrt{\alpha}\xi||_{L^2}$, which will be essential for proving (4.3).

We multiply the first equation in (4.5) by $\alpha\xi_R$, the second equation by $\alpha\xi_I$, integrate over D, and make the summation by parts. This gives

$$(4.6)\qquad ||\sqrt{\alpha}\xi||^2 \le \beta^2||\sqrt{\alpha}\xi||^2_{L^2} + \alpha I,$$

where $I := |\int_D(F_R(\alpha)\cdot\xi_R + F_I(\alpha)\cdot\xi_I)|$.

We estimate

$$\begin{aligned} I &\le ||F_R(\alpha)||_{L_2}||\xi_R||_{L_2} + ||F_I(\alpha)||_{L_2}||\xi_I||_{L_2} \le\\ &\le C(\beta)||f||_1||\xi||_{L_2} \le C_1(\beta)||f||_1||\xi|| \le C_1(\beta)||f||_1||\eta||_1. \end{aligned}$$

From Proposition 4.2 : $\eta = R(\lambda; L_1)f$, $||\eta||_1 \le ||R(\lambda;L_1)||\cdot||f||_1 \le \alpha^{-1}||f||_1$, which gives

$$(4.7)\qquad I \le \alpha^{-1}C_1(\beta)||f||_1^2.$$

From this and (4.6) we have

$$(4.8)\qquad ||\sqrt{\alpha}\xi||^2 \le \beta^2||\sqrt{\alpha}\xi||^2_{L^2} + C_1(\beta)||f||_1^2.$$

To obtain the estimation of $||\sqrt{\alpha}\xi||^2_{L^2}$ we derive first the inequality:

$$(4.9)\qquad r|\beta|\int_D|\sqrt{\alpha}\phi|^2 \le C(\beta)||f||_1^2.$$

We multiply the first equation in (4.5) by $-\alpha\xi_I$ and the second equation by $\alpha\xi_R$. Then we integrate over D, and make the summation by parts. After proving the similar estimations as above for ineqaulity (4.7) we obtain (4.9).

Now consider first case $\beta^2 \notin \sigma(A)$. We rewrite the system (4.5) in the form

$$(4.10)\qquad \begin{aligned} \sqrt{\alpha}\xi_R &= R(\beta^2;A)\left[(\alpha^2+\alpha\beta)\sqrt{(\alpha)}\xi_R - 2\alpha\beta\sqrt{\alpha}\xi_I - \sqrt{\alpha}F_R(\lambda)\right] -\\ &\qquad -r\beta R(\beta^2;A)\begin{pmatrix} 0\\ \sqrt{\alpha}\phi \end{pmatrix},\\ \sqrt{\alpha}\xi_I &= R(\beta^2;A)\left[(\alpha^2+\alpha\beta)\sqrt{(\alpha)}\xi_I + 2\alpha\beta\sqrt{\alpha}\xi_R - \sqrt{\alpha}F_I(\lambda)\right] +\\ &\qquad +r\beta R(\beta^2;A)\begin{pmatrix} 0\\ \sqrt{\alpha}\phi \end{pmatrix}. \end{aligned}$$

Taking into account (4.9) we can derive from (4.10) the estimation

$$||\sqrt{\alpha}\xi||_{L^2} \le C(\beta)||f||_1, \tag{4.11}$$

when $\alpha \in (0,\alpha_0), \alpha_0 \equiv \alpha_0(\beta) > 0$ is chosen sufficiently small.

When $\beta^2 \in \sigma(A)$ we have $\beta^2 = r_j$ for some $j \in N$. By Proposition 4.1 we can choose $s_j > 0$ such, that $m_j(s_j) > 0$. The latter means that $s_j - r_j \in \rho(A)$. This makes it possible to rewrite (4.5) in the following form:

$$\begin{aligned} \sqrt{\alpha}\xi_R - s_j R(r_j - s_j; A)\sqrt{\alpha}\xi_R &= R(r_j - s_j; A) \cdot \left[(\alpha^2 + \alpha\beta)\sqrt{\alpha}\xi_R - \right. \\ &\left. -2\alpha\beta\sqrt{\alpha}\xi_I - \sqrt{\alpha}F_R(\lambda)\right] - r\beta R(r_j - s_j; A)\begin{pmatrix} 0 \\ \sqrt{\alpha}\phi \end{pmatrix}, \\ \sqrt{\alpha}\xi_I - s_j R(r_j - s_j; A)\sqrt{\alpha}\xi_I &= R(r_j - s_j; A) \cdot \left[(\alpha^2 + \alpha\beta)\sqrt{\alpha}\xi_I + \right. \\ &\left. +2\alpha\beta\sqrt{\alpha}\xi_R - \sqrt{\alpha}F_I(\lambda)\right] - r\beta R(r_j - s_j; A)\begin{pmatrix} 0 \\ \sqrt{\alpha}\phi \end{pmatrix}. \end{aligned} \tag{4.12}$$

From (4.2) and Proposition 4.1 we get

$$||s_j R(r_j - s_j; A)|| \le \frac{s_j}{\sqrt{m_j(s_j)}} < 1. \tag{4.13}$$

This allows us to use the Neumann series for $(I - s_j R(r_j - s_j; A))^{-1}$, which together with (4.9) from (4.12) gives (4.11) for sufficiently small $\alpha_0 > 0$.

Arguing by contradiction, let us suppose that there exist $\epsilon > 0$, $\beta \in R$, $f \in H_1$ and a sequence $\{\alpha^k; k \in N\} \subset (0, \min\{1, \alpha_0\})$ such, that $\lim_{k\to\infty} \alpha^k = 0$, and $||\sqrt{\alpha^k}\eta(\alpha^k, \beta)||_1 \ge \epsilon, k \in N$. Let us denote shortly $\eta_k \equiv \eta(\alpha^k, \beta)$, $\lambda_k \equiv \alpha^k + i\beta$, and appropriately $\xi_k, u_k, \phi_k, v_k, \psi_k, k \in N$. Estimations (4.8) and (4.11) allow us to conclude that there exists a subsequnce of $\left\{\sqrt{\alpha^k}\xi^k, k \in N\right\}$ which is convergent weakly in H and strongly in $L^2(D)^{n+1}$. The limit of such subsequence we denote by ξ^0. For the simplicity of notation we assume that the whole sequence $\left\{\sqrt{\alpha^k}\xi^k, k \in N\right\}$ is convergent to ξ^0. We substitude $\sqrt{\alpha^k}\xi^k$ into (4.4) instead of ξ and after passing $k \to \infty$ we obtain

$$A\xi^0 - \beta^2\xi^0 + i\beta B\xi^0 = 0, \tag{4.14}$$

in the weak sense. When $\beta = 0$ this gives $A\xi^0 = 0$ in the weak sense which yields $\xi = 0$.

Now we consider case $\beta \ne 0$. The same arguments that have been used to derive (4.9) give us $r|\beta|||\phi^0||_{L^2} = 0$, and hence $\phi^0 = 0$. From (4.14) this allows to infer that u^0 satisfies

$$-\mu\Delta u^0 = \beta^2 u^0 \quad \text{in} \quad D, \qquad \text{div} u^0 = 0 \quad \text{in} \quad D, \qquad u^0 = 0 \quad \text{on} \quad \partial D.$$

Because of the condition (C) we have $u^0 = 0$ and hence $\xi^0 = 0$.

Let us consider the system (4.4) for $\sqrt{\alpha}\xi_k$ once more. After taking the inner product in $L^2(D)^{n+1}$ we derive

$$\begin{aligned}||\sqrt{\alpha^k}\xi_k||^2 &= (A\sqrt{\alpha^k}\xi_k, \sqrt{\alpha^k}\xi_k)_{L^2}\\ &= (\sqrt{\alpha^k}F(\lambda_k) + \lambda_k^2\sqrt{\alpha^k}\xi_k - \lambda_k B\sqrt{\alpha^k}\xi_k, \sqrt{\alpha^k}\xi_k)_{L^2}.\end{aligned}$$

This gives $\lim_{k\to\infty}\sqrt{\alpha^k}\xi_k = 0$ in H, and consequently $\lim_{k\to\infty}\sqrt{\alpha^k}\eta_k = 0$ in H_1. We have obtained the contradiction. The proof is closed. ∎

5. Stabilization of solutions for Problem 2

In this section we give proof for the second part of Theorem 2.9. Everywhere in this section we have the boundary condition $\phi = 0$ on ∂D.

We begin with making the following auxillary observations. According to the theory of elliptic systems [19] we can define:

Definition 5.1. Let $v \in H_0^1(D)^n$, $w \in L^2(D)^n$, $R, h, \psi \in L^2(D)$ be given and u, ϕ, θ solve the following problem

$$\begin{aligned}\Delta_e u + b\nabla\phi - M\nabla\theta &= w,\\ a\Delta\phi - b\mathrm{div}u - \gamma\phi + M_1\theta &= r\psi + h \quad \text{in} \quad D,\\ d\Delta\theta &= M\mathrm{div}v + M_1\psi + R,\\ u = 0, \phi = 0, \theta &= 0 \quad \text{on} \quad \partial D.\end{aligned}$$

We define the operator $P : D(L_2^{j-1}) \to D(L_2^j)$, $j \in N$ by the formula $P(v, w, \psi, h, R) := (u, v, \phi, \psi, \theta)$. We put $D(L_2^0) \equiv H_2$.

Proposition 5.2. *Let (u, ϕ, θ) solve the Problem 2, and*

$$u_1(t) := \int_0^t u(s) + u_1^0, \quad \phi_1(t) := \int_0^t \phi(s) + \phi_1^0, \quad \theta_1 := \int_0^t \theta(s) + \theta_1^0,$$

where $(u_1^0, u^0, \phi_1^0, \phi^0, \theta_1^0) = P(u^0, u^1, \phi^0, \phi^1, \theta^0)$, and $(u^0, u^1, \phi^0, \phi^1, \theta^0) \in X_2$ is the initial data for (u, ϕ, θ). Then (u_1, ϕ_1, θ_1) solves the Problem 2 with initial conditions $u_1(0) = u_1^0$, $\partial_t u_1(0) = u^0$, $\phi_1(0) = \phi_1^0$, $\partial_t\phi_1(0) = \phi^0$, $\theta_1(0) = \theta_1^0$, and $(u_1(t), u(t), \phi_1(t), \phi(t), \theta_1(t)) = P(u(t), \partial_t u(t), \phi(t), \partial_t\phi(t), \theta(t))$, $t \in R_+$.

The proof is based on straightforward calaculations, the details are omitted.

Proof of Theorem 2.9 for $E_2(\cdot)$. We adopt the scheme of proof done by Dafermos in [5]. We shall consider separately the terms standing in $E_2(\cdot)$ and prove that they tend to 0 when $t \to \infty$.

The principal considerations will be proved under the assumption that the initial data belongs to $D(L_2^{3+J})$, $J \in \{0\} \cup N$. The case when the initial data belong to H_2 will be investigated at the end of the proof. When the initial data belong to $D(L_2^{3+J})$ then $(\partial_t^i u, \partial_t^i \phi, \partial_t^i \theta), i = 1, \dots, 2+J$ also solve the Problem 2 in a strong sense with the appriopriate initial conditions. Their energies are denoted by $E_2^{(i)}(t)$, $i = 1, \dots, 2+J$, $t \in \overline{R}_+$. The equality:

$$E_2^{(i)} + d\int_0^t\int_D |\nabla \partial_s^i \theta(s)|^2 + r\int_0^t\int_D |\nabla \partial_s^{i+1} \phi(s)|^2 = E_2^{(i)}(0), \tag{5.1}$$
$$\text{where} \qquad t > 0, i = 1, \dots, 2+J$$

still holds. Our further considerations will be separated into parts and formulated as propositions.

PROPOSITION 5.3.

$$\lim_{t\to\infty} \partial_t^i \phi(t) = 0 \quad in \quad L^2(D), i = 1, \dots, 2+J.$$

Proof of Proposition 5.3. Let us denote $f_i(t) := \int_D |\partial_t^i \phi(t)|^2$, $i = 1, 2, \dots,$ $2+J$.. From inequalities $E_2(t) \leq E_2(0)$, $E_2^{(j)}(t) \leq E_2^{(j)}(0)$, $t > 0$, $j = 1, \dots, 2+J$ we get $f_1(t) \leq E_2(0)$, $f_{j+1}(t) \leq E_2^{(j)}(0)$, $t > 0, j = 1, \dots, 2+J$ and from (3.8), (5.1) we claim that $f_i \in L^1([0, \infty))$, $i = 1, 2, \dots, 3+J$. The Cauchy inequality and the above estimations allow us to write

$$\left|\frac{d}{dt} f_i(t)\right| \leq 2\sqrt{f_i(t)}\sqrt{f_{i+1}(t)} \leq C < \infty, \quad t > 0, i = 1, \dots, 2+J.$$

This yields $\lim_{t\to\infty} f_i(t) = 0$, $i = 1, \dots, 2+J$ because f_i, $i = 1, \dots, 2+J$ are uniformly continuous and integrable on $(0, \infty)$. Proof of Proposition 5.3 is finished. ∎

PROPOSITION 5.4.

$$\lim_{t\to\infty} \partial_i^j \partial_t^k \theta(t) = 0 \quad in \quad L^2(D), \qquad i = 1, \dots, n, \; j = 0, 1, \; k = 0, \dots, 1+J.$$

Proof of Proposition 5.4. Denote $g_l(t) := \int_D \left|\nabla \partial_t^l \theta(t)\right|^2$, $l = 1, \dots, 1+J$, $t \in R_+$. From (3.8), (5.1) we infer $g_l \in L^1([0, \infty))$, $l \in 0, \dots, 1+J$. Again from inequalities for energies (used in the proof of Proposition 5.3) together with the Poincaré inequality we derive

$$\int_D \left|\partial_t^{l+1} \theta(t)\right|^2 \leq E_2^{(l+1)}(0), \quad \int_D \left|\mathrm{div} \partial_t^{l+1} u(t)\right|^2 \leq c E_2^{l+1}(0),$$

$l = 0, \dots, 1+J$, $c > 0$ is constant.

Then using the equation for θ in the system (1.4) we get

$$\frac{d}{dt}g_l(t) = -2\int_D \left[\frac{1}{d}\left|\partial_t^{l+1}\theta(t)\right|^2 + \frac{M}{d}\partial_t^{n+1}\theta(t)\mathrm{div}\partial_t^{l+1}u(t)\right. \tag{5.2}$$

$$\left.+\frac{M_1}{d}\partial_t^{l+1}\theta(t)\partial_t^{l+1}\phi(t)\right], \quad l = 0, \ldots, 1+J,\ t > 0. \tag{5.3}$$

From this, the estimations written above, and known estimations for $f_l, l = 1, \ldots, 2+J$ we obtain that $\left|\frac{d}{dt}g_l(t)\right| \leq c_1 < \infty$, $l = 0, \ldots, 1+J$, $t \in R_+$, $c_1 > 0$ is a constant. This implies $\lim_{t\to\infty} g_l(t) = 0$, $l = 0, \ldots, 1+J$.

Since $\partial_t^l\theta(t) \in H_0^1(D)$, $l = 0, \ldots, 1+J$ from the Poincaré inequality we obtain the assertion of Proposition corresponding to $j = 0$. Proof of Proposition 5.4 is finished. ∎

Proposition 5.5.

$$\lim_{t\to\infty} \mathrm{div}u(t) = 0 \quad \textit{weakly in} \quad H_0^1(D) \quad \textit{and strongly in} \quad L^2(D).$$

Proof of Proposition 5.5. Consider $(u_1(\cdot), \phi_1(\cdot), \theta_1(\cdot))$ defined in Proposition 5.2. Taking into account the equation for θ_1 in (1.4) and $\partial_t u_1 = u$, we immediately show that $\mathrm{div}u(t) \in H_0^1(D)^n$, $t \geq 0$. From the same equation for θ_1 we derive for $\chi \in C_0^\infty(D)$:

$$\int_D [\partial_t\theta_1(t)\chi + d\nabla\theta_1(t)\nabla\chi + M_1\partial_t\phi_1(t)\chi] = -M\int_D \mathrm{div}u(t)\chi,$$

and

$$\int_D [\partial_t\partial_j\theta_1(t)\chi - d\nabla\theta_1(t)\nabla\partial_j\chi - M_1\partial_t\phi_1(t)\partial_j\chi] = -M\int_D \partial_j\mathrm{div}u(t)\chi,$$

$$j = 1 \ldots n.$$

From Propositions 5.3, 5.4 applied to $\theta_1(\cdot), \phi_1(\cdot)$ we conclude that the left hand sides in the equalities written above tend to 0 when $t \to \infty$. This gives the weak convergence of $\mathrm{div}u(t) \to 0$ in $H_0^1(D)$ when $t \to \infty$. Since the inclusion $H_0^1(D) \subset L^2(D)$ is compact we have second assertion. Proof of Proposition 5.5 is finished. ∎

Proposition 5.6.

$$\lim_{t\to\infty} \phi(t) = 0 \quad \textit{in} \quad H_0^1(D).$$

Proof of Proposition 5.6. Using the equation for ϕ in (1.4) we get

$$(5.4)\qquad a\int_D |\nabla\phi(t)|^2 = \int_D \big(\big[M_1\theta(t) - \partial_t^2\phi(t) - b\mathrm{div}u(t) - r\partial_t\phi(t)\big]\,\phi(t)\big) - \gamma\|\phi(t)\|^2_{L^2(D)} \le \int_D \big(\big[M_1\theta(t) - \partial_t^2\phi(t) - b\mathrm{div}u(t) - r\partial_t\phi(t)\big]\,\phi(t)\big).$$

From inequality $E_2(t) \le E_2(0), t > 0$ we get $\|\phi(t)\|^2_{L^2(D)} \le \gamma^{-1}E_2(0)$.

From Propositions 5.3–5.5 we observe that the right hand side in (5.4) has the limit 0 when $t \to \infty$. This yields that $\lim_{t\to\infty}\nabla\phi(t) = 0$ in $L^2(D)^n$. Proof of Proposition 5.6 is finished. ∎

We define $w(t), t \in R_+$ as the solution of the problem :

$$(5.5)\qquad \Delta_e w(t) = -b\nabla\phi(t) + M\nabla\theta(t) \quad \text{in} \quad D \quad w(t) = 0 \quad \text{on} \quad \partial D.$$

Proposition 5.7.

$$(5.6)\qquad \lim_{t\to\infty} w(t) = 0 \quad \textit{in} \quad H_0^1(D)^n, \quad \textit{and}$$

$$(5.7)\qquad \lim_{t\to\infty} \partial_t^l w(t) = 0 \quad \textit{weakly in} \quad H_0^1(D)^n \quad \textit{and strongly in} \quad L^2(D)^n, \quad \textit{for} \quad l = 1, \dots, 1 + J.$$

Proof of Proposition 5.7. From Proposition 2.2 and Korn inequality we infer that the bilinear form $\int_D \sigma(f) : \epsilon(g), f, g \in H_0^1(D)^n$, defines the inner product in $H_0^1(D)^n$.

From (5.5) we derive

$$\int_D \sigma(\partial_t^l w(t)) : \epsilon(\chi) = \int_d \Big[-b\partial_t^l\phi(t) + M\partial_t^l\theta(t)\Big]\,\mathrm{div}\chi, \quad \chi \in H_0^1(D)^n, \quad \text{for} \quad l = 0, \dots, 1 + j.$$

Because of Propositions 5.3, 5.4 this yields $\lim_{t\to\infty}\partial_t^l w(t) = 0$ weakly in $H_0^1(D)^n$ (hence strongly in $L^2(D)^n$), $l = 0, \dots, 1+J$. Using (5.5) once more we get

$$\int_D \sigma(w(t)) : \epsilon(w(t)) = -b\int_D \phi(t)\mathrm{div}w(t) - M\int_D \nabla\theta(t)w(t).$$

From Propositions 5.4, 5.6 and the convergence proved above we claim that the right hand side in this equality tends to 0 when $t \to \infty$. This establishes that $\lim_{t\to\infty} w(t) = 0$ in $H_0^1(D)^n$. Proof of Proposition 5.7 is finished. ∎

Now we define $v := u - w$. From the equation for u in 1.4 we claim that v solves the problem

$$\partial_t^2 v = \Delta_e v - \partial_t^2 w \quad \text{in} \quad D \times R_+, \qquad v = 0 \quad \text{on} \quad D \times R_+,$$
$$v(0) = u^0 - w(0), \partial_t v(0) = u^1 - \partial_t w(0) \quad \text{in} \quad D.$$

Here we are in the same position as in Lemma 5.4 in [5]. Arguing in the same way as in the proof of Lemma 5.4 in [5] we obtain $\lim_{t\to\infty} \partial_t v(t) = 0$ in $L^2(D)^n$, $\lim_{t\to\infty} v(t) = 0$ in $H_0^1(D)^n$.

The proof is rather long, so we address the reader to paper [5]. In this proof the condition (C) on the domain D is essential.

We have proved that $\lim_{t\to\infty} E_2(t) = 0$ when initial data belongs to $D(L_2^{3+J})$, $J \geq 1$. Consider now the initial data $\xi_0 \in H_2$ and take $\widetilde{\xi_0} \in D(L_2^{3+J})$. The solution with initial data $\widetilde{\xi_0}$ we denote by $\widetilde{\xi}(\cdot)$, its energy by $\widetilde{E_2}(\cdot)$ and let $E_2^1(t)$ denote the energy of solution with initial data $\xi_0 - \widetilde{\xi_0}$. We have the inequality

$$E_2(t) \leq 2\left[\widetilde{E_2}(t) + E_2^1(t)\right].$$

Because of the inequality $E_2^1(t) \leq E_2^1(0)$ this gives

$$E_2(t) \leq 2\widetilde{E_2}(t) + 2E_2^1(0). \tag{5.8}$$

Since $D(L_2^{3+J})$ is dense in H_2 (see [19]), for each $\epsilon > 0$ we can choose $\widetilde{\xi_0} \in D(L_2^{3+J})$ such, that $E_2^1(0) < \frac{\epsilon}{2}$.

Since $\lim_{t\to\infty} \widetilde{E_2}(t) = 0$ we get from (5.8) that $\limsup_{t\to\infty} E_2(t) \leq \epsilon$ for every $\epsilon > 0$. The proof is finished. ∎

Remark 5.8. In the formula for energy $E_1(t)$ we have the term $2b\int_D \phi(t)\mathrm{div}u(t)$. In the proof given above we have estimated first $\mathrm{div}u(t)$ from equation for $\theta(\cdot)$ and then $\int_D |\phi(t)|^2$ was estimated.

In system (1.2) θ does not appear. So this scheme of proof does not work for Problem 1. The application of the resolvent criterion for Problem 2 remains as the open problem. Solving it will give the unified solution for both Problem 1 and Problem 2.

6. Final remarks

From the form of systems (1.1), (1.3) we deduce that besides the boundary conditions considered in this paper also the following (so called free boundary condition) can be considered:

$$T \cdot \nu = f, \quad q\nu = f_0 \quad \text{on} \quad \Gamma_N \times R_+,$$

where $(T \cdot \nu)_l := \sum_{k=1}^n T_{lk}\nu_k$, $q\nu := \sum_{l=1}^n q_l \nu_l$, $\Gamma_N \subset \partial D$, f denotes the vector of external force and f_0 the external heat supply.

There is no problem for proving the existence of c_0-semigroups describing the solutions (see [8]). But under the knowledge of the authors there does not exist works in which the stabilization is proved when there are taken into considerations $f = 0$ and $f_0 = 0$.

In our paper [8] we solved the problem, such as Problem 1 but with the boundary conditions $T \cdot \nu = -\partial_t u$ on $\Gamma_N \times R_+$ (the so called feedback stabilization) and $u = 0$ on $\Gamma_D \times R_+$, $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial D$, $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$, $\overline{\Gamma}_D \neq \emptyset$, $\overline{\Gamma}_N \neq \emptyset$ and $\partial_\nu \phi = 0$ on $\partial D \times R_+$.

We have proved the uniform stabilization of the energy of solution. The methods which we have used are quite different from the ones presented in this paper.

For the ideas useful for solving stabilization problems with the Neumann type boundary conditions one should look into literature cited in our paper [8].

Appendix 1

In the model proposed by Grot [9] the termoelasticity system describes the evolution of quantities (u, ϕ, θ, w) where u, ϕ, θ are the same as in (1.4) and $w \in R^n$ represents microtemperatures. The governing equations of the system are following:

$$\begin{aligned}
\partial_t^2 u &= \Delta_e u + b\nabla\phi - M\nabla\theta \\
\partial_t^2 \phi &= a\Delta\phi - b\mathrm{div}u - \gamma\phi + M_1\theta - \delta\mathrm{div}w \\
\partial_t \theta &= d\Delta\theta - M\mathrm{div}\partial_t u - M_1\partial_t\phi + k_1\mathrm{div}w \\
\partial_t w &= \widetilde{\Delta}_e w - k_3\nabla\theta - \delta\nabla\partial_t\phi - k_2 w
\end{aligned} \tag{A.1}$$

where $\widetilde{\Delta}_e$ is the elliptic Lamé operator, $k_1, k_2, k_3, \delta > 0$.

As in Problem 2 the system is considered in the domain $(t, x) \in (0, \infty) \times D$ and is subjected by boundary and initial conditions. For u, ϕ, θ the boundary conditions are the same as in Problem 2 and for w it is considered $w = 0$ on $(0, \infty) \times \partial D$. In the system (A.1) there is no dissipation term $-r\partial_t\phi$ in the equation for ϕ similarly as in models proposed by Iesan [11]. Taking the idea from [10] we propose the following decoupling, separating the equation for w from the system for (u, ϕ, θ). Let $\widetilde{w}$ be the solution of the following elliptic problem:

$$\widetilde{\Delta}_e w - k_2 w = \delta\nabla\partial_t\phi \quad \text{in} \quad D, \qquad w = 0 \quad \text{on} \quad \partial D.$$

We put the solution $\widetilde{w} = \delta(\widetilde{\Delta}_e - k_2)_D^{-1}\nabla\partial_t\phi$ into the system for u, ϕ, θ instead of w, denote $O = \mathrm{div}(\widetilde{\Delta}_e - k_2)_D^{-1}\nabla$, and write the decoupled system:

$$\begin{aligned}
\partial_t^2 \overline{u} &= \Delta_e \overline{u} + b\nabla\overline{\phi} - M\nabla\overline{\theta} \\
\partial_t^2 \overline{\phi} &= a\Delta\overline{\phi} - b\mathrm{div}\overline{u} - \gamma\overline{\phi} - \delta^2 O\partial_t\overline{\phi} + M_1\overline{\theta} \\
\partial_t \overline{\theta} &= d\Delta\overline{\theta} - M\mathrm{div}\partial_t\overline{u} - M_1\partial_t\overline{\phi} + k_1\delta O\partial_t\overline{\phi} \\
\partial_t \overline{w} &= \widetilde{\Delta}_e \overline{w} - k_3\nabla\overline{\theta} - \delta\nabla\partial_t\overline{\phi} - k_2\overline{w}.
\end{aligned} \tag{A.2}$$

This system is subjected with the same boundary and initial conditions as in the system (A.1).

For $f \in H_0^1(D)$ we immediately show

$$\int_D fOf = \int_D ((\widetilde{\Delta}_e - k_2)_D^{-1} \nabla f) \cdot \nabla f \geq 0.$$

The boundedness of O in $H_0^1(D)$ is clear. We see that the term $\delta^2 O \partial_t \overline{\phi}$ introduces dissipation into the equation for $\overline{\phi}$. Let $\overline{T}(t)$, $t \in R$, denote the semigroup for system (A.2) and $T(t)$ for system (A.1). Our conjecture is: the operator $T(t) - \overline{T}(t)$ is compact on $C([0,T], \widetilde{H})$ for each $T > 0$; $\widetilde{H}$ is a suitable Hilbert space. The work on this problem is in progress. Our aim in this Appendix is only to explain that the dissipation in equation on ϕ appears when one considers the full thermoelasticity system with microtemperatures.

Appendix 2

Let us notice first, that Assumption 2.1 is necessary to construct semigroups $S_i(\cdot)$, $i = 1, 2$ and then state and solve problems about stabilization. The first conditions $\lambda + \mu > 0$ when $n = 2$ and $3\lambda + 2\mu > 0$ when $n = 3$ are generally assumed in linear hyperelasticity theory. So we investigate only what would happen when the second condition $(\lambda + \mu)\gamma > b^2$ when $n = 2$ and $(3\lambda + 2\mu)\gamma > 3b^2$ when $n = 3$ were not satisfied. We focus on the system (1.2). The well posedness of Problem 1 follows from [13]. From the paper [15] for solutions of Problem 1 we obtain: when $E_1(0) < 0$ then there exist $c_1, c_2 > 0$, dependent of initial data, such that

$$\int_D [|\partial_t u(t)|^2 + |\partial_t \phi(t)|^2] + r \int_0^t \int_D |\partial_t \phi(s)|^2 dx ds \geq C_1 e^{C_2 t}, \quad t \geq 0.$$

We establish that initial data for which $E_1(0) < 0$ can exist when the second condition in Asssumption 2.1 is not satisfied. Let us consider $n = 2$, $(\lambda + \mu)\gamma < b^2$ and take the initial data: $u^1 = 0, \phi^1 = 0, u^0 = \nabla v, \phi^0 = s \mathrm{div} u^0$, $v \in C_0^\infty(D), s \in R$. After calculations we get

$$\begin{aligned} &\sigma(u^0) : \epsilon(u^0) + 2b\phi^0 \mathrm{div} u^0 + \gamma(\phi^0)^2 + a|\nabla \phi^0| \\ &\quad = 2\mu \sum_{i=1}^{2} (\partial_i^2 v)^2 + 4\mu(\partial_1 \partial_2 v)^2 + (\lambda + 2sb + s^2\gamma)(\Delta v)^2 + a|\nabla \Delta v|^2. \end{aligned}$$

Observe first that for $v \in C_0^\infty(D)$, $\int_D (\partial_1 \partial_2 v)^2 = \int_D \partial_1^2 v \partial_2^2 v$. Then we put $s = -\frac{b}{\gamma}$. For $E_1(0)$ we obtain:

$$E_1(0) = \left(\mu + \frac{1}{\gamma} \left((\mu + \lambda)\gamma - b^2 \right) \right) \int_D (\Delta v)^2 + a \int_D |\Delta \nabla v|^2.$$

If $(\mu + \lambda)\gamma - b^2 < 0$ we see that $E_1(0) \to -\infty$ when $\gamma \to 0_+$, therefore obtaining negative values of $E_1(0)$ is possible. In the light of the result from [15], cited above, this means that the system (1.2) is physically not realistic when second condition in Asssumption 2.1 is not satisfied.

References

[1] F. Alabau, P. Cannarsa, V. Komornik, *Indirect internal stabilization of weakly coupled evolution equations*, J. Evolution Equations (2) 2002, 127–150.

[2] P. S. Casas, R. Quintanilla, *Exponential decay in one-dimensional porous-thermo-elasticity*, Mech. Res. Comm. (32) 2005, 652–658.

[3] M. Ciarletta, D. Iesan, *Non-classical Elastic Solids*, Pitman Research Notes in Mathematics Series, vol 293, Longman Scientific and Technical, Harlow; John Wiley&Sons, New York, 1993

[4] S. C. Cowin, J. W. Nunziato, *Linear elastic materials with voids*, J. Elasticity 13 (1983), 125–147.

[5] C. M. Dafermos, *On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity*, Arch. Rational Mech. Anal. 29 (1968), 241–271.

[6] R. Dager, E. Zuazua, *Wave Propagation, Observation and Control in 1-d Flexible Multi-Structures*, Math. Appl. 50, 2006.

[7] R. Dautray, J. L. Lions, *Mathematical Analysis and Numerical Methods for Sciences and Technology*, v5, Evolution Problems I, Springer-Verlag, Berlin, 1992.

[8] P. Głowiński, A. Łada, *Stabilization of elasticity-viscoporosity system by linear boundary feedback*, Math. Methods Appl. Sci. 32 (2009), 702–722.

[9] R. Grot, *Thermodynamics of continuum with microstructure*, Internat. J. Engrg. Sci. 7 (1969), 801-814

[10] D. Henry, O. Lopes, A. Perissinitto, *On the essential spectrum of a semigroup of thermoelasticity*, Nonlinear Anal. 21 (1993), 65–75.

[11] D. Iesan, *Thermoelastic Models of Continua*, Springer, Berlin, 2004.

[12] S. Jiang, R. Racke, *Evolution Equations in Thermoelasticity*, Chapman and Hall ICRC, Boca Raton, 2000.

[13] H. Koch, *Mixed problems for fully nonlinear hyperbolic equations*, Math. Z. 214 (1993), 9-42.

[14] G. Lebeau, E. Zuazua, *Decay rates for the three-dimensional linear system of thermoelasticity*, ARMA 148 (1999), 179–231.

[15] M. C. Leseduarte, R. Quintanilla, *Instability, nonexistence and uniqueness in elasticity with porous dissipation*, Differential Equations and Nonlinear Mechanics (2006), 1–14.

[16] K. Liu, *Locally distributed control and damping for the conservative systems*, SIAM J. Control Optim. 35(5) (1997), 1547–1590.

[17] A. Magana, R. Quintanilla, *On the exponential decay of solutions in one-dimensional generalized of porous-thermo-elasticity*, Asymptotic Anal. 49 (2006), 173–187.

[18] A. Magana, R. Quintanilla, *On the time decay of solutions in one-dimensional theories of porous materials*, Internat. J. Solids Structures 43 (2006), 3414–3427.

[19] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer-Verlag, 1983.

[21] Y. Tomilov, *A resolvent approach to stability of operator semigroups*, J. Operator Theory 46 (2001), 63–98.

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